A generalized Pexider equation

By JOHN A. BAKER (Waterloo, Canada)

In [2] the general solution of the functional equation

$$F(x)+G(y) = P(x+y)+Q(x/y), \quad x, y > 0,$$

was found among functions F, G, P and Q mapping the positive real numbers into an additive abelian group H in which division by 2 is uniquely defined. By this we mean that for every $x \in H$ there exists a unique $y \in H$ such that $x=2y$, in which case we write $y=x/2$. In this paper we study the functional equation

$$f(x)+g(y) = p(x+y)+q(xy) \tag{1}$$

which is a generalized Pexider equation (see [1]). Using different techniques than those employed in [2] we will prove

Theorem 1. *Let* $(H, +)$ *be an abelian group in which division by 2 is uniquely defined and suppose* $f, g, p, q\colon (0, \infty) \to H$. *Then* (1) *holds for all* $x, y > 0$ *if and only if* $f(x)=\alpha(x^2)+a(x)+m(x)+b$, $g(x)=\alpha(x^2)+a(x)+m(x)+c$, $p(x)=\alpha(x^2)+a(x)+\beta$ *and* $q(x)=-2\alpha(x)+m(x)+\gamma$ *for all* $x>0$ *where* b, c, β *and* γ *are constants in* H *with* $b+c=\beta+\gamma$ *and where* $\alpha, a, m\colon (0, \infty) \to H$ *such that for all* $x, y > 0$, $\alpha(x+y)=\alpha(x)+\alpha(y)$, $a(x+y)=a(x)+a(y)$ *and* $m(xy)=m(x)+m(y)$.

If p and q satisfy the Pexider equations $p(x+y)=p_1(x)+p_2(y)$, $q(xy)=q_1(x)+q_2(y)$ for $x, y>0$ then clearly $p(x+y)+q(xy)=f(x)+g(y)$ where $f=p_1+q_1$ and $g=p_2+q_2$. But Theorem 1 shows that not every solution of (1) can be obtained in this way. Thus (1) may be thought of as a generalized Pexider equation.

If (1) is assumed to hold for all *real* x and y, then putting $y=0$ shows that f and p differ by a constant. Putting $x=0$ shows that g and p differ by a constant. Thus, for an appropriate Q, $p(x)+p(y)=p(x+y)+Q(xy)$ for all real x and y. This equation was studied by ECSEDI and HOSSZÚ [5].

Equation $\left(\frac{x^2}{4}\right)$s also equivalent to what might be called a generalized Jensen equation (see [For if (1) holds for $x, y>0$ and if we replace x by $x/2$, y by $y/2$ and let $R(x)=q\left(\frac{x^2}{4}\right)$ we find that

$$p\left(\frac{x+y}{2}\right)+R(\sqrt{xy}) = \varphi(x)+\psi(y)$$

where $\varphi(x)=f(x/2)$ and $\psi(y)=g(y/2)$. Interchanging x and y we see that φ and ψ differ by a constant and so we have

$$p\left(\frac{x+y}{2}\right)+R\left(\sqrt{xy}\right) = T(x)+T(y).$$

Putting $x=y$ we find $2T(x)=p(x)+R(x)$ so that

$$2p\left(\frac{x+y}{2}\right)+2R\left(\sqrt{xy}\right) = p(x)+p(y)+R(x)+R(y)$$

which is a generalized Jensen equation.

In order to get some feeling for theorem 1 and its proof let us consider the "regular" solutions of (1). Suppose f, g, p and q are real valued, twice differentiable on $(0, \infty)$ and satisfy (1) for all $x, y>0$. Differentiating (1) with respect to x and then differentiating the resulting equation with respect to y we conclude that

$$p''(x+y)+q'(xy)+xyq''(xy) = 0 \quad \text{for all} \quad x, y > 0.$$

From Lemma 1 below we deduce that p'' is constant and hence $p(x)=\alpha x^2+ax+\beta$ for all $x>0$ where α, a and β are real constants. Similarly, since $q'(x)+xq''(x)= -2\alpha$ for $x>0$, we find that $q(x)=-2\alpha x+k\log x+\gamma$, $x>0$, where k and γ are real constants. Having found p and q it is easy to determine f and g.

To prove Theorem 1 we determine p using results of Djoković [3] concerning a difference analogue of the differential equation $p'''(x)=0$. We then use a theorem of Daróczy, Lajkó and Székelyhidi [4] to find q.

Lemma 1. *If S is a set and $F, G\colon (0, \infty)\to S$ such that $F(x+y)=G(xy)$ for all $x, y>0$ then F and G are constant.*

Proof. Fix $k>0$ arbitrarily. Then $F\left(x+\frac{k}{x}\right)=G(k)$ for all $x>0$. If $t\geqq 2\sqrt{k}$ then there is an $x>0$ such that $x+\frac{k}{x}=t$. Hence $F(t)=G(k)$ for $t\geqq 2\sqrt{k}$, i.e. F is constant on $(2\sqrt{k}, \infty)$ for any $k>0$. Thus F is constant on $(0, \infty)$ and hence so is G.

We will also need

Lemma 2. *If S is a set and $F, G\colon (0, \infty)\to S$ such that $F(2x+y)=G(x+2y)$ for all $x, y>0$ then F and G are constant.*

Proof. For $0<\frac{v}{2}<u<2v$ put $x=\frac{2u-v}{3}>0$ and $y=\frac{2v-u}{3}>0$ so that $2x+y=u$ and $x+2y=v$. Thus $F(u)=G(v)$ for $\frac{v}{2}<u<2v$ and hence F is constant on $\left(\frac{v}{2}, 2v\right)$ for any $v>0$. It follows that F is constant on $(0, \infty)$ and hence G is as well.

Notation. $(H, +)$ denotes an abelian group in which division by 2 is uniquely defined. If $f\colon (0, \infty)\to H$ and $t>0$ we let $\Delta_t f(x)=f(x+t)-f(x)$ and $\delta_t f(x)=f(tx)-f(x)$ for all $x>0$.

PROOF OF THEOREM 1. Suppose $f, g, p, q\colon (0, \infty)\to H$ and (1) holds for all $x, y>0$. Then

$$p(2x+y)+q(2xy) = f(2x)+g(y)$$

and

$$p(x+2y)+q(2xy) = f(x)+g(2y)$$

so that

$$p(2x+y)-p(x+2y) = \varphi(x)+\psi(y) \quad \text{for all} \quad x, y>0 \tag{2}$$

where $\varphi(x)=f(2x)-f(x)$ and $\psi(y)=g(y)-g(2y)$ for all $x, y>0$.

Replace x by $x+r$ in (2) and subtract (2) from the resulting equation to get

$$\Delta_{2r}p(2x+y)-\Delta_r p(x+2y) = \Delta_r\varphi(x) \quad \text{for} \quad x, y, r>0. \tag{3}$$

Replace y by $y+s$ in (3) and subtract to get

$$\Delta_s\Delta_{2r}p(2x+y) = \Delta_{2s}\Delta_r p(x+2y) \quad \text{for} \quad x, y, r, s>0. \tag{4}$$

Thinking of r and s as parameters and using Lemma 2 we conclude from (4) that $\Delta_s\Delta_{2r}p(2x+y)$ depends only on s and r. Hence we may write $\Delta_s\Delta_r p(x)=C(r, s)$ for $x, r, s>0$, and then conclude that

$$\Delta_t\Delta_s\Delta_r p(x) = 0 \quad \text{for all} \quad x, r, s, t>0. \tag{5}$$

Now from [3] it follows that

$$p(x) = A(x, x)+a(x)+\beta, \quad x>0 \tag{6}$$

where $A\colon (0, \infty)\times(0, \infty)\to H$ is symmetric and biadditive, $a\colon (0, \infty)\to H$ is additive and β is a constant in H.

Using (1), (6), the symmetry and biadditivity of A and the additivity of a we find that

$$q(tx) = f(t)+f(x)-A(t, t)-2A(x, t)-A(x, x)-a(t)-a(x)-\beta$$

so that

$$\delta_t q(x) = q(tx)-q(x) = \varphi(t)+B(x, t) \quad \text{for} \quad x, t>0 \tag{7}$$

where $\varphi(t)=f(t)-f(1)-A(t, t)+A(1, 1)-a(t)+a(1)$ and $B(x, t)=-2A(x, t)+2A(x, 1)$ for all $x, t>0$. Notice that $B(x+y, t)=B(x, t)+B(y, t)$ for all $x, y, t>0$ and so from (7) we conclude that

$$2\delta_t q\left(\frac{x+y}{2}\right) = \delta_t q(x)+\delta_t q(y) \quad \text{for all} \quad x, y, t>0. \tag{8}$$

It follows from theorem 1 of [4] that, because of (8),

$$q(x) = m(x)+J(x) \quad \text{for} \quad x>0 \tag{9}$$

where $m, J\colon (0, \infty) \to H$ such that

$$m(xy) = m(x) + m(y) \tag{10}$$

and

$$2J\left(\frac{x+y}{2}\right) = J(x) + J(y) \quad \text{for all} \quad x, y > 0. \tag{11}$$

Now it is well known that since J satisfies (11) there must exist an additive function $a_1\colon (0, \infty) \to R$ and a constant $\gamma \in H$ such that $J(x) = a_1(x) + \gamma$ for all $x > 0$. It is notationally convenient to let $a_1(x) = -2\alpha(x)$ so that (9) may be written

$$q(x) = m(x) - 2\alpha(x) + \gamma, \quad x > 0, \tag{12}$$

where $\alpha\colon (0, \infty) \to H$ such that

$$\alpha(x+y) = \alpha(x) + \alpha(y) \quad \text{for all} \quad x, y > 0. \tag{13}$$

It remains to show that $A(x, x) = \alpha(x^2)$ for $x > 0$ and to determine f and g. Interchange x and y in (1) and deduce that

$$p(x+y) + q(xy) = h(x) + h(y), \quad x, y > 0 \tag{14}$$

where

$$h(x) = \frac{f(x) + g(x)}{2} \quad \text{for} \quad x > 0. \tag{15}$$

From (14), (6), (10), (12), (13), the biadditivity and symmetry of A and the additivity of a we find that

$$A(x,x) + 2A(x,y) + A(y,y) + a(x) + a(y) + \beta + m(x) + m(y) - 2\alpha(xy) + \gamma = h(x) + h(y)$$

or

$$A(x, y) - \alpha(xy) = k(x) + k(y) \quad \text{for all} \quad x, y > 0 \tag{16}$$

where

$$2k(x) = h(x) - A(x, x) - a(x) - m(x) - \frac{\beta+\gamma}{2} \quad \text{for} \quad x > 0. \tag{17}$$

From (16) it follows that

$$A(x, 1) - \alpha(x) = k(x) + k(1) \tag{18}$$

and

$$A(x, x) - \alpha(x^2) = 2k(x) \quad \text{for} \quad x > 0. \tag{19}$$

From (18) and (19) we find

$$A(x, x) - \alpha(x^2) = 2A(x, 1) - 2\alpha(x) - 2k(1), \quad x > 0. \tag{20}$$

Replacing x by $2x$ in (20), using additivity and dividing by 2 we find

$$2A(x, x)-2\alpha(x^2) = 2A(x, 1)-2\alpha(x)-k(1). \tag{21}$$

Subtracting (20) from (21) gives

$$A(x, x)-\alpha(x^2) = k(1). \tag{22}$$

From (19) and (22) it follows that $2k(x)=k(1)$ for all $x \geqq 0$ and hence

$$k(x) = 0 \quad \text{for all} \quad x > 0. \tag{23}$$

Now (19) and (23) imply $A(x, x)=\alpha(x^2)$ so, from (6),

$$p(x) = \alpha(x^2)+a(x)+\beta \quad \text{for} \quad x > 0. \tag{24}$$

From (17) and (23) it follows that

$$h(x) = \alpha(x^2)+a(x)+m(x)+\frac{\beta+\gamma}{2} \quad \text{for} \quad x > 0. \tag{25}$$

But from (1) and (14) we find

$$f(x)+g(y) = h(x)+h(y) \quad \text{for} \quad x, y > 0$$

from which it follows that f (and g) differ from h by a constant. Thus we have

$$f(x) = \alpha(x^2)+a(x)+m(x)+b \tag{26}$$

and

$$g(x) = \alpha(x^2)+a(x)+m(x)+c \quad \text{for} \quad x > 0 \tag{27}$$

where b and c are constants.

From (1), (12), (24), (26) and (27) it follows that $\beta+\gamma=b+c$.

An easy calculation shows that the converse is true. This completes the proof of Theorem 1.

Theorem 2. *Suppose f, g, p, q: $(0, \infty)\to R$ (the real numbers) such that* (1) *holds for all $x, y>0$. If there exist Lebesgue measurable subsets S and T of $(0, \infty)$ of positive Lebesgue measure such that p is bounded on S and q is bounded on T then there exist real constants $b, c, \beta, \gamma, \varrho, \sigma$ and τ such that $f(x)=\varrho x^2+\sigma x+\tau \log x+b$, $g(x)=\varrho x^2+\sigma x+\tau \log x+c$, $p(x)=\varrho x^2+\sigma x+\beta$ and $q(x)=-2\varrho x+\tau \log x+\gamma$ for all $x>0$.*

Proof. From (5) we have $\Delta_t^3 p(x)=0$ for all $x, t>0$ and p is bounded on a set of positive measure so, from a theorem of Kemperman [6], it follows that

$$p(x) = \varrho x^2+\sigma x+\beta \quad \text{for all} \quad x > 0$$

where ϱ, σ and β are real constants. Hence we conclude from (12) that m is bounded on a set of positive measure and satisfies (10). It follows that $m(x)=\tau \log x$ for all $x>0$ where τ is a real constant. To see this let $M(t)=m(e^t)$ for t real so that $M(s+t)=M(s)+M(t)$ for all real s and t and M is bounded on a set of positive measure. Thus (see [6]) $M(t)=\tau t$ for all real t where τ is a real constant. Therefore $m(x)=M(\log x)=\tau \log x$ for all $x>0$. This completes the proof.

References

[1] J. Aczél, Lectures on functional equations and their applications, *Academic Press, New York—London,* 1966.

[2] John A. Baker, On the functional equation $f(x)g(y)=p(x+y)q(x/y)$, *Aequationes Math.*, **14** (1976), 493—506.

[3] D. Ž. Djoković, A representation theorem for $(X_1-1)(X_2-1)\dots(X_n-1)$ and its applications, *Ann. Polon. Math.* **22** (1969), 189—198.

[4] Z. Daróczy, K. Lajkó and L. Székelyhidi, Functional Equations on Ordered Fields, *Publ. Math. (Debrecen)* **24** (1977), 173—179.

[5] I. Ecsedi and M. Hosszú, Remarks on the functional equation $f(x+y)-f(x)-f(y)=F(x, y)$. *Aequationes Math.*, **7** (1971), 254—255.

[6] J. H. B. Kemperman, A general functional equation, *Trans. Amer. Math. Soc.* **86** (1957), 28—56.

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF WATERLOO
WATERLOO, ONTARIO, CANADA

(Received July 20, 1978.)