On a lemma of P. Erdős and A. Rényi about random graphs

By E. GODEHARDT (Köln) and J. STEINEBACH (Marburg)

In 1959 P. ERDŐS and A. RÉNYI stated some theorems on random graphs using — as they called it — 'a rather surprising lemma' which is given below.

Lemma 1 *(Erdős—Rényi). Let Γ_{n,N_c} be a 'random graph' with n possible (labelled) vertices and N_c edges, i.e. all $\binom{\binom{n}{2}}{N_c}$ possible graphs are equiprobable. Further let $P(\bar{A}, n, N_c)$ denote the probability that Γ_{n,N_c} does not consist of one connected graph having $n-k$ effective vertices and k isolated points $(k=0, 1, 2, \dots)$. Then for $N_c=\left[\frac{1}{2} n \log n + cn\right]$ it holds true that $\lim_{n\to\infty} P(\bar{A}, n, N_c)=0$.*

PROOF See [2], 292—295.

This basic lemma has recently been used by WRIGHT (1976) to obtain further results about random graphs (see [3]). The first part of Erdős's and Rényi's proof was mainly based on the inequalities

$$a_n(s) = \binom{n}{s} \frac{\binom{\binom{n}{2}-s(n-s)}{N_c}}{\binom{\binom{n}{2}}{N_c}} \leqq \frac{e^{(3-2c)s}}{s!} \quad \text{for} \quad s \leqq \frac{n}{2}, \tag{1}$$

and

$$a_n(s) = \binom{n}{s} \frac{\binom{\binom{n}{2}-s(n-s)}{N_c}}{\binom{\binom{n}{2}}{N_c}} \leqq \frac{e^{(3-2c)(n-s)}}{(n-s)!} \quad \text{for} \quad s \geqq \frac{n}{2}, \tag{2}$$

which were pointed out to hold true for all $n \geqq n_0$, where n_0 may depend on c but not on s. Here (2) follows from (1) by symmetry.

But the following arguments indicate that the uniform estimates in (1) and (2) cannot be valid:

Let $t \in \left(0, \frac{1}{2}\right]$ be fixed, and consider the N_c-th root of

$$b_n(t) \equiv a_n([nt]) \Big/ \frac{e^{(3-2c)[nt]}}{[nt]!}.$$

Setting $a = \binom{n}{2}$, $d = [nt](n-[nt])$, $N_c = \left[\frac{1}{2} n \log n + cn\right]$, and using Stirling's formula $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$, we get

$$b_n(t) = \frac{n!}{(n-[nt])!} \frac{(a-d)!}{(a-d-N_c)!} \frac{(a-N_c)!}{a!} e^{-(3-2c)[nt]} \sim$$

$$\sim \frac{n^{n+\frac{1}{2}} e^{-n} (a-d)^{a-d+\frac{1}{2}} e^{-a+d} (a-N_c)^{a-N_c+\frac{1}{2}} e^{-a+N_c} e^{-(3-2c)[nt]}}{(n-[nt])^{n-[nt]+\frac{1}{2}} e^{-n+[nt]} (a-d-N_c)^{a-d-N_c+\frac{1}{2}} e^{-a+d+N_c} a^{a+\frac{1}{2}} e^{-a}} =$$

$$= n^{[nt]} \left(\frac{1}{1-\frac{[nt]}{n}} \right)^{n-[nt]+\frac{1}{2}} \frac{\left(1-\frac{d}{a}\right)^{N_c} \left(1-\frac{N_c}{a}\right)^{a-N_c+\frac{1}{2}} e^{(2c-4)[nt]}}{\left(1-\frac{N_c}{a-d}\right)^{a-d-N_c+\frac{1}{2}}}.$$

From this

$$\lim_{n\to\infty} b_n(t) = e^{2t} \cdot 1 \cdot \frac{(1-2t(1-t))e^{-1} \cdot 1}{e^{-1}} = (1-2t(1-t))e^{2t} \equiv b(t)$$

is an immediate consequence. Since $b(0)=1$ and $b'(t)=4t^2e^{2t}>0$ $(t\neq 0)$, we have $b(t)>1$ for $t \in \left(0, \frac{1}{2}\right]$, which contradicts (1), i.e. (1) cannot hold true for such s with $s=[nt]$ (t as above).

But though the inequalities (1) and (2) are false, Lemma 1 remains true since it suffices to find another sequence $\{a(s)\}_{s=0,1,2,\dots}$ with $\sum_{s=0}^{\infty} a(s) < \infty$ such that (1) holds true if $e^{(3-2c)s}/s!$ is replaced by $a(s)$ (cf. [2], (16)). Such a sequence will be given in the following lemma.

Lemma 2. *There is an integer $n_0(c)$ such that for all $n \geqq n_0(c)$*

(1a) $$a_n(s) \leqq \frac{e^{(3-2c)s}}{s!} \quad \text{if} \quad s \leqq \frac{n}{\log n},$$

and

(1b) $$a_n(s) \leqq q^s \quad \text{if} \quad \frac{n}{\log n} \leqq s \leqq \frac{n}{2},$$

where $0<q<1$.

Proof. Using an inequality in Bollobás and Sauer ([1], p. 1340) we find that

(3) $$a_n(s) \leqq \frac{n^s}{s!} e^{-s(n-s)N_c/\binom{n}{2}}.$$

Underestimating $n-s$ by $n-\frac{n}{\log n}$ in the first case it follows immediately that (1a) holds.

For $\frac{n}{\log n} \leqq s \leqq \frac{n}{2}$ and n sufficient large the inequality

$$\frac{s(n-s)N_c}{\binom{n}{2}} \geqq \frac{1}{3} s \log n \tag{4}$$

follows. Furthermore Stirling's formula yields

$$s! \geqq s^s e^{-s}. \tag{5}$$

Taking (3), (4), and (5) together we have

$$a_n(s) \leqq \left(\frac{e \log n}{\sqrt[3]{n}}\right)^s$$

(for that range of s named above) from which (1b) is obvious.

Now, to retain Erdős's and Rényi's lemma, inequality (16) in [2] can be replaced by

$$P(\bar{E}_M, n, N_c) \leqq \sum_{s > \frac{2N_c}{n}} a(s) + \sum_{s > M} a(s) \quad \text{for} \quad n \geqq n_0,$$

where

$$a(s) = \frac{e^{(3-2c)s}}{s!} + q^s, \quad s = 0, 1, 2, \ldots.$$

Then

$$\lim_{n \to \infty} P(\bar{E}_{\log \log n}, n, N_c) = 0$$

still holds, and the lemma remains true.

References

[1] B. Bollobás, N. Sauer, Uniquely colourable graphs with large girth. *Canad. J. Math.* **28** (1976), 1340—1344.

[2] P. Erdős, A. Rényi, On random graphs I. *Publ. Math. (Debrecen)* **6** (1959), 290—297.

[3] E. M. Wright, The evolution of unlabelled graphs. *J. London Math. Soc., Ser. 2*, **14** (1976), 554—558.

ERHARD GODEHARDT
INST. F. MED. DOKUM. U. STATIST.
UNIVERSITÄT KÖLN
JOSEPH — STELZMANN — STR. 9
D—5000 KÖLN 41

JOSEF STEINEBACH
FACHBEREICH MATHEMATIK
UNIVERSITÄT MARBURG
LAHNBERGE
D—3550 MARBURG

(Received August 3, 1978.)