An extension theorem for a functional equation

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In this note we deal with extension theorems concerning a functional equation. This problem and the first results are due to Z. DARÓCZY and L. LOSONCZI [1] concerning Cauchy's functional equation. Since then several results and applications have been found ([2], [3], [4]). The present paper contains a new extension theorem for the functional equation $\Delta_y^{n+1} f(x)=0$. In what follows $\mathbf{R}$ denotes the set of real numbers. If $f\colon D\subseteqq \mathbf{R}\to\mathbf{R}$ is a function, n is a positive integer and $x, y\in D$, then let for $x, x+y, \ldots, x+(n+1)y\in D$

$$\Delta_y^{n+1} f(x) = \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k f[x+(n+1-k)y].$$

If f is a function then $\operatorname{dom} f$ and $\operatorname{rg} f$ denote the domain and the range of f, respectively.

Theorem 1. *Let* $r>0$ *and n be a positive integer. Let* $f\colon [0, r)\to\mathbf{R}$ *be a function such that*

$$\Delta_y^{n+1} f(x) = 0$$

for $x, y\geqq 0, x+(n+1)y<r$. *Then there exists a function* $F\colon [0, \infty)\to\mathbf{R}$ *such that*

(i) $\Delta_y^{n+1} F(x) = 0$ *for* $x, y \geqq 0$,

(ii) $f\subseteqq F$.

PROOF. The proof is based on Zorn's lemma. Let $\mathscr{F}$ denote the set of all functions φ with the following properties:

a) $f\subseteqq \varphi$,

b) $\operatorname{dom}\varphi = [0, p)$ for some $p>0$,

c) $\operatorname{rg}\varphi \subseteqq \mathbf{R}$,

d) $\Delta_y^{n+1}\varphi(x) = 0$ for $x, y\geqq 0$ and $x+(n+1)y<p$.

As $f\in\mathscr{F}$, so $\mathscr{F}\neq\emptyset$. The set $\mathscr{F}$ is partially ordered with the obvious inclusion of functions. Moreover, if $\mathscr{C}\subseteqq\mathscr{F}$, $\mathscr{C}\neq\emptyset$ and $\mathscr{C}$ is an arbitrary chain, then for $h=\bigcup\mathscr{C}$ we get $h\in\mathscr{F}$. Thus by Zorn's lemma there exists a maximal element $F\in\mathscr{F}$. Let $\operatorname{dom} F=[0, R)$ and suppose that $R<+\infty$. Let $z\in\left[0, \frac{n+1}{n}R\right)$

and

$$\bar{F}(z) = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} F\left[(n+1-k)\frac{z}{n+1}\right].$$

If $z \in \left[0, \frac{n+1}{n} R\right)$ then for $k = 1, 2, \ldots, n+1$

$$0 \leqq \frac{n+1-k}{n+1} z \leqq \frac{n}{n+1} z < R$$

hence $\bar{F}$ is defined at the point z. Further if $z \in [0, R)$ then

$$\bar{F}(z) = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} F\left[(n+1-k)\frac{z}{n+1}\right] =$$

$$= F(z) - \sum_{k=0}^{n+1} (-1)^{k} \binom{n+1}{k} F\left[(n+1-k)\frac{z}{n+1}\right] = F(z) - \Delta_{\frac{z}{n+1}}^{n+1} F(0) = F(z)$$

that is $F \subseteqq \bar{F}$. Let $x, y \geqq 0, x+(n+1)y < \frac{n+1}{n} R$, then by the definition of $\bar{F}$ and by $F \in \mathscr{F}$ we have

$$\sum_{j=1}^{n+1} (-1)^{j+1} \binom{n+1}{j} \bar{F}[x+(n+1-j)y] =$$

$$= \sum_{j=1}^{n+1} (-1)^{j+1} \binom{n+1}{j} \left[\sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} F\left[(n+1-k)\frac{x+(n+1-j)k}{n+1}\right]\right] =$$

$$= \sum_{j=1}^{n+1} \sum_{k=1}^{n+1} (-1)^{j+1} (-1)^{k+1} \binom{n+1}{j} \binom{n+1}{k} F\left[\frac{n+1-k}{n+1} x + (n+1-j)\frac{n+1-k}{n+1} y\right] =$$

$$= \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} F\left[\frac{n+1-k}{n+1} x + (n+1-k) y\right] =$$

$$= \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} F\left[(n+1-k)\frac{x+(n+1)y}{n+1}\right] = \bar{F}[x+(n+1)y]$$

which implies

$$0 = \sum_{j=1}^{n+1} (-1)^{j} \binom{n+1}{j} \bar{F}[x+(n+1-j)y] + \bar{F}[x+(n+1)y] = \Delta_y^{n+1} \bar{F}(x).$$

It follows that $\bar{F} \in \mathscr{F}$ and by $\frac{n+1}{n} R > R$ this is a contradiction.

Theorem 2. *Let n be a positive integer and let f: $[0, \infty) \to \mathbf{R}$ be a function such that $\Delta_y^{n+1} f(x) = 0$ for $x, y \geqq 0$. Then there exists a function F: $\mathbf{R} \to \mathbf{R}$ such that*

(i) $\Delta_y^{n+1} F(x) = 0$ *for* $x, y \in \mathbf{R}$,

(ii) $f \subseteqq F$.

PROOF. Let $F_0=f$, m be a nonnegative integer and suppose that we have defined the function F_m: $[-m, \infty)\to\mathbf{R}$ such that for $-m\leqq x+(n+1-i)y$ $(i=0, 1, \ldots, n+1)$ the equality

$$\Delta_y^{n+1}F_m(x)=0$$

holds and $F_m\supseteqq F_{m-1}$. Let $x\geqq -(m+1)$ and

$$F_{m+1}(x)=\sum_{k=0}^{n}(-1)^{k+1}\binom{n+1}{k}F_m(x+n-k+1).$$

Then obviously $F_m\supseteqq F_{m-1}$.

If $-(m+1)\leqq x+(n+1-i)y$ $(i=0, 1, \ldots, n+1)$ then

$$\sum_{j=0}^{n}(-1)^{j+1}\binom{n+1}{j}F_{m+1}[x+(n+1-j)y]=$$

$$=\sum_{k=0}^{n}(-1)^{k+1}\binom{n+1}{k}\left[\sum_{j=0}^{n}(-1)^{j+1}\binom{n+1}{j}F_m[x+n+1-k+(n+1-j)y]\right]=F_{m+1}(x)$$

thus $\Delta_y^{n+1}F_{m+1}(x)=0$.

Finally we define $F=\bigcup_{m=0}^{\infty}F_m$, it is obvious that F is a function fulfilling the required conditions.

Theorem 3. *Let $r>0$ and n be a positive integer. Let f: $\left(-r\sqrt{1+(n+1)^2},\ r\sqrt{1+(n+1)^2}\right)\to\mathbf{R}$ be a function such that*

$$\Delta_y^{n+1}f(x)=0 \quad \textit{for} \quad x^2+y^2<r^2.$$

Then there exists a function F: $\mathbf{R}\to\mathbf{R}$ such that

(i) $\Delta_y^{n+1}F(x)=0$ *for* $x, y\in\mathbf{R}$,
(ii) $f\subseteqq F$.

PROOF. Let $r_1=\dfrac{r}{\sqrt{1+n^2}}$. If $x^2+y^2<r_1^2$ and $x, y\geqq 0$ then $x+iy\in\left[0, r_1\sqrt{1+n^2}\right)=$ $=[0, r)$ $(i=0, 1, \ldots, n)$. It is obvious that

$$\Delta_y^{n+1}f(x)=0 \quad \text{for} \quad x, y\geqq 0, \quad x+(n+1)y<r$$

thus, by theorems 1 and 2 there exists a function F: $\mathbf{R}\to\mathbf{R}$ such that (i) holds and $F(x)=f(x)$ for $x\in[0, r)$. If $x^2+y^2<r_1^2$ and $x, y\geqq 0$ then

$$f[x+(n+1)y]=\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}f[x+(n+1-k)y]=$$

$$=\sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}F[x+(n+1-k)y]=F[x+(n+1)y],$$

that is $F(x)=f(x)$ for $x\in\left[0, r\frac{\sqrt{1+(n+1)^2}}{\sqrt{1+n^2}}\right]$. Now let $r_2=r\frac{\sqrt{1+(n+1)^2}}{\sqrt{1+n^2}}$ then by a similar argument we get that

$$F(x)=f(x) \quad \text{for} \quad x\in\left[0, r\left(\frac{\sqrt{1+(n+1)^2}}{\sqrt{1+n^2}}\right)^2\right]$$

As $\frac{\sqrt{1+(n+1)^2}}{\sqrt{1+n^2}}>1$ continuing this process we arrive at

$$F(x)=f(x) \quad \text{for} \quad x\in[0, r\sqrt{1+(n+1)^2}).$$

If $x\in(-r\sqrt{1+(n+1)^2}, 0]$ then by a similar method we get

$$F(x)=f(x).$$

Thus $f\subseteqq F$.

Definition. If $D\subseteqq\mathbf{R}^2$ and n is a positive integer then let for $k=0, 1, 2, \dots, n+1$

$$D_k=\{x+(n+1-k)y\colon (x, y)\in D\}.$$

Theorem 4. *Let $D\subseteqq\mathbf{R}^2$ be an open and connected set with $(0, 0)\in D$. Let n be a positive integer and $f\colon \bigcup_{k=0}^{n+1} D_k\to\mathbf{R}$ be a function such that*

$$\Delta_y^{n+1}f(x)=0 \quad \text{for} \quad (x, y)\in D.$$

Then there exists a function $F\colon \mathbf{R}\to\mathbf{R}$ such that

(i) $\Delta_y^{n+1}F(x)=0$ *for* $(x, y)\in\mathbf{R}^2$,

(ii) $f\subseteqq F$.

Proof. Since D is open and connected, and $(0, 0)\in D$, it is obvious that $\bigcup_{k=0}^{n+1} D_k$ is an open interval containing 0, for instance $\bigcup_{k=0}^{n+1} D_k=(-a, b)$ where $a, b>0$. As $(0, 0)\in D$ there exists an $r>0$ such that $\{(x, y)\colon x^2+y^2<r^2\}\subseteqq D$ and

$$\{(x, y)\colon x, y\geqq 0,\ x+y<r\}\subseteqq D.$$

By theorem 3 there exists a function $F\colon \mathbf{R}\to\mathbf{R}$ for which (i) holds and $F(x)=f(x)$ for $x\in(-r, r)$. Let

$$p_0=\sup\{p\colon F(x)=f(x) \text{ for } x\in[0, p)\}.$$

Obviously $r\leqq p_0\leqq b$. Assume that $p_0<b$. As $\bigcup_{k=0}^{n+1} D_k=(-a, b)$ for every $t\in(p_0, b)$

there exists $(x, y)\in D$ such that

$$x, y \geqq 0,\ x, x+y, \ldots, x+ny < p_0, \quad x+(n+1)y = t > p_0.$$

Then

$$f[x+(n+1)y] = \sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}f[x+(n+1-k)y] =$$

$$= \sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}F[x+(n+1-k)y] = F[x+(n+1)y]$$

which contradicts the definition of p_0. Consequently $p_0=b$. Similarly we obtain

$$-a = \inf\{q\colon F(x)=f(x) \quad \text{for} \quad x\in(q, b)\}.$$

References

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