On the conditional laws of large numbers I

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1. Introduction

In 1954 A. RÉNYI [3] has given a new axiomatic theory of probability in which the conditional probability is the fundamental concept. He has also shown that his theory has the same relation with reality as Kolmogorov's one, that is the most important laws of large numbers hold true.

Our interest in this note will focus only the weak laws of large numbers*). The proofs are generally based on an inequality (see Lemma 3.2) which is a generalization of the Markov's inequality to conditional probability field. To study the weak laws related to weakly dependent random variables we shall define the conditional correlation-coefficient of two random variables.

2. Basic concepts and notations

We now define the basic concepts and give a list of symbols and conventions frequently used in this paper without further explanation.

We shall denote the *conditional probability field* by $[\mathscr{H}, T_1, T_2, p]$, where $\mathscr{H}$ is a non empty set, T_1 is a σ-algebra of subsets of $\mathscr{H}$, $T_2 \subseteq T_1$ is non empty, and finally, the set function $p(A|B)$ of two set variables is defined for every $A \in T_1$, $B \in T_2$. Furthermore, it is supposed that the set function $p(A|B)$ satisfies the following three axioms:

(i) $p(A|B) \geqq 0, p(B|B)=1$ if $A \in T_1, B \in T_2$.
(ii) For any fixed $B \in T_2$ $p(A|B)$ is a σ-additive set function of $A \in T_1$.
(iii) $p(A|BC)p(B|C)=p(AB|C)$, provided that all expressions involved exist. Here AB denotes the product of A and B.

Let ω denote an arbitrary element of $\mathscr{H}$ and let $\xi=\xi(\omega)$ be a T_1-measurable real valued function on $\mathscr{H}$. ξ is called *random variable*.

If $C \in T_2$ is fixed then $[\mathscr{H}, T_1, p(\cdot|C)]$ is trivially a probability field and ξ is a random variable in ordinary sence. This allows one to define the *conditional distribution function* $F(x|C)$, *conditional expectation* $E(\xi|C)$, *conditional variance* $D(\xi|C)$, *conditional independence*, etc. with respect to C, as the distribution function, expectation, variance, etc. in $[\mathscr{H}, T_1, p(\cdot|C)]$.

*) In Part II. we want to be concerned with strong laws of large numbers.

For example

$$E(\xi|C) = \int_{\mathscr{H}} \xi(\omega)\, dp(A|C),$$

where on the right-hand side is the abstract Lebesque integral of ξ with respect to the measure $p(\cdot|C)$.

Let $\xi_1, \xi_2, \ldots$ be a sequence of random variables. We say that ξ_n *converges* to ξ *weakly* with respect to $C \in T_2$ if for all $\varepsilon > 0$

$$\lim_{n\to\infty} p(|\xi_n - \xi| \geqq \varepsilon | C) = 0.$$

For this kind of convergence the symbol

$$\xi_n \underset{C}{\Rightarrow} \xi$$

is used. We say that ξ_n *converges* to ξ *with probability 1* with respect to C if $\xi_n(\omega) \to \xi(\omega)$ except on a set of $p(\cdot|C)$-measure 0. We write

$$\xi_n \underset{C}{\rightarrow} \xi.$$

For convenience we employ some constant notation: Let $\xi_1, \xi_2, \ldots$ be random variables on the conditional probability field $[\mathscr{H}, T_1, T_2, p]$ and let Q be a Borel set of the real line. Let $C \in T_2$ and assume that

$$\xi_n^{-1}(Q) \in T_2, \quad \xi_n^{-1}(Q) \subseteq C.$$

We put

$$B_n = \xi_n^{-1}(Q), \quad p_n = p(B_n|C), \quad E_n = E(\xi_n|B_n), \quad D_n = D(\xi_n|B_n),$$

provided that $E(\xi_n|B_n)$, $D(\xi_n|B_n)$ exist. At last let

$$\varepsilon_i = \begin{cases} 1, & \xi_i \in Q \\ 0, & \xi_i \notin Q \end{cases} \quad (i = 1, 2, \ldots)$$

$$\tau_n = \sum_{i=1}^{n} \varepsilon_i \quad \text{and} \quad v_n = \sum_{i=1}^{n} \varepsilon_i(\xi_i - E_i).$$

3. Weak laws of large numbers

One of the most important problem to investigate some kind of limit of the mean value of those observations (with respect to $\xi_1, \xi_2, \ldots$) which gave such result that the conditionals $B_1, B_2, \ldots$ occured respectively. We can express this conditional mean in terms of the following formula:

$$\frac{\sum_{i=1}^{n} \varepsilon_i \xi_i}{\tau_n}.$$

Rényi has already studied weak laws in [3]. From one of his strongest theorem ([3], Theorem 4.c.) follows that if $\xi_1, \xi_2, \ldots$ are identically distributed and mutually independent with respect to $C \in T_2$ with finite expectation and variance E, D and

$\sum_{k=1}^{\infty} p_n = \infty$ then

$$\frac{\sum_{i=1}^{n} \varepsilon_i \xi_i}{\tau_n} \underset{C}{\Rightarrow} E$$

and what is more

$$\frac{\sum_{i=1}^{n} \varepsilon_i \xi_i}{\tau_n} \underset{C}{\rightarrow} E.$$

As Rényi also remarked $\sum_{i=1}^{\infty} p_i = \infty$ is a natural condition, because by Borel—Cantelli lemma if $\sum_{i=1}^{\infty} p_i < \infty$ then with probability 1 (with respect to the measure $p(\cdot\,|C)$) only finitely many events of $\{B_n, n=1, 2, \ldots\}$ may occur.

First of all we prove two lemmas for later use.

Lemma 3.1. *If any k of $\{\xi_1, \xi_2, \ldots, \xi_n\}$ is independent with respect to C and $E(\tau_n^k|C)$ exists (k is a fixed natural number) then*

$$E(\tau_n^k|C) \geqq \left(\sum_{i=1}^{n} p_i\right)^k. \tag{3.1}$$

Proof.

$$E(\tau_n^k|C) = E\left(\left(\sum_{i=1}^{n} \varepsilon_i\right)^k \middle| C\right) =$$

$$= \sum_{m_1+\ldots+m_n=k} \frac{k!}{m_1!\,m_2!\,\ldots\,m_n!} E(\varepsilon_1^{m_1} \varepsilon_2^{m_2} \ldots \varepsilon_n^{m_n}|C) =$$

$$= \sum_{m_1+\ldots+m_n=k} \frac{k!}{m_1!\,m_2!\,\ldots\,m_n!} E(\varepsilon_1^{m_1}|C) \ldots E(\varepsilon_n^{m_n}|C) \geqq$$

$$\geqq \sum_{m_1+\ldots+m_n} \frac{k!}{m_1!\,m_2!\,\ldots\,m_n!} \left(E(\varepsilon_1|C)\right)^{m_1} \ldots \left(E(\varepsilon_n|C)\right)^{m_n} = (p_1+p_2+\ldots+p_n)^k.$$

Q.e.d.

Lemma 3.2. *Let α be a positive real number and let ξ, η be two random variables such that there exist $E(\xi^\alpha|C), E(\eta^\alpha|C) > 0$ and $\xi^\alpha \geqq 0, |\eta| > 0, \eta^\alpha > 0$. Then for all $\varepsilon > 0$*

$$p\left(\frac{|\xi|}{|\eta|} \geqq \varepsilon | C\right) \leqq \frac{1}{\varepsilon^\alpha} \frac{E(\xi^\alpha|C)}{E(\eta^\alpha|C)}. \tag{3.2}$$

Proof. Let $A = \{|\xi| \geqq \varepsilon |\eta|\}$ and $\chi = \frac{\xi^\alpha}{E(\eta^\alpha|C)}$. Then $E(\chi|C) = \frac{E(\xi^\alpha|C)}{E(\eta^\alpha|C)}$, where

$$E(\chi|C) = E(\chi|AC)\,p(A|C) + E(\chi|\bar{A}C)\,p(\bar{A}|C) \geqq$$

$$\geqq E(\chi|AC)\,p(A|C).$$

Thus,

$$E(\chi|C) \geqq \frac{E(\xi^\alpha|AC)}{E(\eta^\alpha|C)}\, p(A|C) \geqq \varepsilon^\alpha p(A|C).$$

Q.e.d.

We now give some sufficient conditions for the existence of a real constant E for which

$$\frac{\sum_{i=1}^{n} \varepsilon_i E_i}{\tau_n} \underset{C}{\Rightarrow} E \quad \text{and} \quad \frac{\sum_{i=1}^{n} \varepsilon_i \xi_i}{\tau_n} \underset{C}{\Rightarrow} E.$$

Our first result is a simple generalization of the well-known classical result of Markov.

Theorem 3.3. *Let $\xi_1, \xi_2, \dots$ be pairwise independent with respect to C such that there exist E_n, D_n $(n=1, 2, \dots)$. Assume that*

(3.3) $$\frac{S_n}{\sum_{i=1}^{n} p_i} \to 0 \;(n \to \infty), \quad \textit{where} \quad S_n^2 = \sum_{i=1}^{n} p_i D_i^2 \quad (n = 1, 2, \dots),$$

(3.4) $$\frac{\sum_{i=1}^{n} p_i |E_i - E|}{\sum_{i=2}^{n} p_i} \to 0 \quad (n \to \infty).$$

Then

$$\frac{\sum_{i=1}^{n} \varepsilon_i E_i}{\tau_n} \underset{C}{\Rightarrow} E \quad \text{and} \quad \frac{\sum_{i=1}^{n} \varepsilon_i \xi_i}{\tau_n} \underset{C}{\Rightarrow} E.$$

PROOF. Firstly we prove that our conditions implies

(3.5) $$\frac{\sum_{i=1}^{n} \varepsilon_i E_i}{\tau_n} \underset{C}{\Rightarrow} E.$$

Using Lemma 3.1 and Lemma 3.2 we have

$$p\left(\left|\frac{\sum_{i=1}^{n} \varepsilon_i (E_i - E)}{\tau_n}\right| \geqq \varepsilon \Big| C\right) \leqq \frac{E\left(\left|\sum_{i=1}^{n} \varepsilon_i (E_i - E)\right| \Big| C\right)}{\varepsilon \sum_{i=1}^{n} p_i} \leqq$$

$$\leqq \frac{\sum_{i=1}^{n} E(\varepsilon_i |E_i - E| \,|C)}{\varepsilon \sum_{i=1}^{n} p_i} = \frac{\sum_{i=1}^{n} p_i |E_i - E|}{\varepsilon \sum_{i=1}^{n} p_i} \to 0 \quad (n \to \infty).$$

This proves (3.5). On the other hand by Lemma 3.2

$$p\left(\left|\frac{\sum_{i=1}^{n}\varepsilon_i(\xi_i-E_i)}{\tau_n}\right|\geqq\varepsilon|C\right)\leqq\frac{1}{\varepsilon^2}\frac{E\left(\left(\sum_{i=1}^{n}\varepsilon_i(\xi_i-E_i)\right)^2\Big|C\right)}{E(\tau_n^2|C)}=$$

$$=\frac{1}{\varepsilon^2}\frac{\sum_{i=1}^{n}E\left(\varepsilon_i^2(\xi_i-E_i)^2|C\right)}{E(\tau_n^2|C)}=\frac{1}{\varepsilon^2}\frac{\sum_{i=1}^{n}p_iE\left((\xi_i-E_i)^2|B_i\right)}{E(\tau_n^2|C)}=\frac{1}{\varepsilon^2}\frac{\sum_{i=1}^{n}p_iD_i^2}{E(\tau_n^2|C)}.$$

By Lemma 3.1 $E(\tau_n^2|C)\geqq\left(\sum_{i=1}^{n}p_i\right)^2$ so

$$p\left(\left|\frac{\sum_{i=1}^{n}\varepsilon_i(\xi_i-E_i)}{\tau_n}\right|\geqq\varepsilon|C\right)\leqq\frac{1}{\varepsilon^2}\frac{\sum_{i=1}^{n}p_iD_i^2}{\left(\sum_{i=1}^{n}p_i\right)^2}=\frac{1}{\varepsilon^2}\frac{S_n^2}{\left(\sum_{i=1}^{n}p_i\right)^2}\to 0\quad(n\to\infty).$$

This means that

$$\frac{\sum_{i=1}^{n}\varepsilon_i(\xi_i-E_i)}{\tau_n}\underset{C}{\Rightarrow}0. \tag{3.6}$$

Furthermore,

$$\frac{\sum_{i=1}^{n}\varepsilon_i\xi_i}{\tau_n}=\frac{\sum_{i=1}^{n}\varepsilon_i(\xi_i-E_i)}{\tau_n}+\frac{\sum_{i=1}^{n}\varepsilon_iE_i}{\tau_n},$$

where the right-hand side tends weakly to E with respect to the measure $p(\cdot|C)$, so the theorem is proved.

Remark. Let us observe that (3.6) holds without (3.4) too, thus, under the hypotheses of Theorem 3.3 except (3.4) it follows that

$$\frac{\sum_{i=1}^{n}\varepsilon_i(\xi_i-E_i)}{\tau_n}\underset{C}{\Rightarrow}0.$$

In Theorem 3.3 the condition of independence can be considerably weakened if we exchange condition (3.3) for a stronger one. To do this we have to give the notion of conditional correlation-coefficient of two random variables.

Assume that for the random variables ξ_1, ξ_2 there exist $E(\xi_i|B_i), D(\xi_i|B_i)$ $(i=1, 2)$. Then $E\left(\varepsilon_1(\xi_1-E_1)\varepsilon_2(\xi_2-E_2)|C\right)$ also exists. Suppose that $p_1, p_2, D_1, D_2>0$ and let

$$R(\xi_1, \xi_2, Q, C)=\frac{E\left(\varepsilon_1(\xi_1-E_1)\varepsilon_2(\xi_2-E_2)|C\right)}{\sqrt{p_1p_2}D_1D_2}. \tag{3.7}$$

The quantity $R(\xi_1, \xi_2, Q, C)$ which is dependent on Q and C too is called the *conditional correlation-coefficient* of ξ_1, ξ_2 with respect to Q and C.

To the conditional correlation-coefficient we make some simple remarks.

a) If $Q=R^1$ then $C=\mathscr{H}$ and in this case $R(\xi_1, \xi_2, Q, C)$ may be regarded as the ordinary correlation-coefficient of ξ_1, ξ_2 in the probability field $[\mathscr{H}, T_1, p(\cdot|\mathscr{H})]$.

b) If ξ_1, ξ_2 are independent with respect to C then $R(\xi_1, \xi_2, Q, C)=0$, that is ξ_1, ξ_2 are *uncorrelated.*

c) $|R(\xi_1, \xi_2, Q, C)|\leqq 1$ for all ξ_1, ξ_2.

This last property can be proved in the following simple way by the Cauchy inequality.

Let $\xi_i^*=\varepsilon_i(\xi_i-E_i)$ $(i=1, 2)$. Then

$$\big(E(\varepsilon_1(\xi_1-E_1)\varepsilon_2(\xi_2-E_2)|C)\big)^2 \leqq \int_{B_1} \xi_1^{*2}\, dp(A|C) \int_{B_2} \xi_2^{*2}\, dp\,(A|C) =$$

$$= p_1 p_2 \int_{B_1} \xi_1^{*2}\, dp(A|B_1) \int_{B_2} \xi_2^{*2}\, dp(A|B_2) = p_1 p_2 D_1^2 D_2^2.$$

Now, the theorem in question can be stated as follows:

Theorem 3.4. *Let us assume that there exist* $E(\xi_i|B_i), D(\xi_i|B_i)$ $(i=1, 2, \dots)$ *and*

$$\frac{\sum_{i=1}^{n} p_i|E_i-E|}{\sum_{i=1}^{n} p_i} \to 0 \quad (n \to \infty), \tag{3.8}$$

$$S_n^2 \leqq K \sum_{i=1}^{n} p_i, \quad \textit{where} \quad S_n^2 = \sum_{i=1}^{n} p_i D_i^2 \quad \textit{and} \quad K \textit{ is a real constant,} \tag{3.9}$$

$$R(\xi_i, \xi_j, Q, C) \leqq R(|i-j|), \tag{3.10}$$

where $R(k)$ *is a non-negative function on* $\{0, 1, 2, \dots\}$ *such that* $R(0)=1$ *and*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} R(i)}{\sum_{i=1}^{n} p_i} = 0.$$

Then

$$\frac{\sum_{i=1}^{n} \varepsilon_i \xi_i}{\tau_n} \underset{C}{\Rightarrow} E.$$

Proof. We have seen in the proof of theorem 3.3 that (3.8) implies.

$$\frac{\sum_{i=1}^{n} \varepsilon_i E_i}{\tau_n} \underset{C}{\Rightarrow} E.$$

Thus, it will be sufficient to show that

$$\frac{\sum_{i=1}^{n} \varepsilon_i(\xi_i - E_i)}{\tau_n} - \frac{\nu_n}{\tau_n} \underset{C}{\Rightarrow} 0.$$

However, by Lemma 3.2

$$p\left(\left|\frac{\nu_n}{\tau_n}\right| \geqq \varepsilon | C\right) \leqq \frac{1}{\varepsilon^2} \frac{E(\nu_n^2|C)}{E(\tau_n^2|C)} =$$

$$= \frac{1}{\varepsilon^2} \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} E\big(\varepsilon_i(\xi_i - E_i)\varepsilon_j(\xi_j - E_j|C)}{E(\tau_n^2|C)}.$$

Here

$$\sum_{i=1}^{n} \sum_{j=1}^{n} E\big(\varepsilon_i(\xi_i - E_i)\varepsilon_j(\xi_j - E_j)|C\big) = \sum_{i=1}^{n} \sum_{j=1}^{n} R(\xi_i, \xi_j, Q, C) \sqrt{p_i p_j} D_i D_j \leqq$$

$$\leqq \sum_{i=1}^{n} \sum_{j=1}^{n} R(|i-j|) \sqrt{p_i p_j} D_i D_j = S_n^2 + 2 \sum_{k=1}^{n-1} R(k) \sum_{i=1}^{n-k} \sqrt{p_i p_{i+k}} D_i D_{i+k} \leqq$$

$$\leqq S_n^2 + 2 S_n^2 \sum_{k=1}^{n-1} R(k)$$

due the following computations

$$2 \sqrt{p_i p_{i+k}} D_i D_{i+k} \leqq p_i D_i^2 + p_{i+k} D_{i+k}^2$$

and consequently

$$2 \sum_{i=1}^{n-k} \sqrt{p_i p_{i+k}} D_i D_{i+k} \leqq \sum_{i=1}^{n-k} (p_i D_i^2 + p_{i+k} D_{i+k}^2) \leqq 2 \sum_{i=1}^{n} p_i D_i^2 = 2 S_n^2.$$

Hence,

$$p\left(\left|\frac{\nu_n}{\tau_n}\right| \geqq \varepsilon | C\right) \leqq \frac{1}{\varepsilon^2} \frac{S_n^2 + 2 S_n^2 \sum_{i=1}^{n} R(i)}{E(\tau_n^2|C)} \leqq$$

$$\leqq \frac{1}{\varepsilon^2} \frac{S_n^2 + 2 S_n^2 \sum_{i=1}^{n} R(i)}{\left(\sum_{i=1}^{n} p_i\right)^2} \leqq \frac{K}{\sum_{i=1}^{n} p_i} + \frac{2K \sum_{i=1}^{n} R(i)}{\sum_{i=1}^{n} p_i} \to 0 \quad (n \to \infty).$$

Thus, $\frac{\nu_n}{\tau_n} \underset{C}{\Rightarrow} 0$ and Theorem 3.4 is established.

In classical theory of probability it is known (see [1]) that a sequence $\eta_1, \eta_2, \dots$ of random variables having first moment converges weakly to 0 if and only if

$$\lim_{n \to \infty} E\left(\frac{\eta_n^2}{1 + \eta_n^2}\right) = 0$$

that is the Khinchin's condition holds. By using this fact for

$$\eta_n = \frac{v_n}{\tau_n} \quad \text{and} \quad [\mathscr{H}, T_1, p(\cdot|C)]$$

we can deduce that

$$\eta_n = \frac{v_n}{\tau_n} \underset{C}{\Rightarrow} 0 \quad \text{if and only if} \quad E\left(\frac{v_n^2}{\tau_n^2+v_n^2}\middle| C\right) \to 0 \quad (n \to \infty). \tag{3.11}$$

From the above relation it follows a useful sufficient condition for $\frac{v_n}{\tau_n} \underset{C}{\Rightarrow} 0$, furthermore, an other necessary one too.

Theorem 3.5. *If* $\frac{E(v_n^2|C)}{E(\tau_n^2+v_n^2|C)} \to 0 \ (n \to \infty)$ *then* $\frac{v_n}{\tau_n} \underset{C}{\Rightarrow} 0$.

Proof. Let $G = \left\{\omega \in \mathscr{H} \middle| \frac{v_n^2}{\tau_n^2+v_n^2} \geqq \varepsilon^2\right\}$, where $\varepsilon > 0$. Then with the help of Lemma 3.2 we get

$$E\left(\frac{v_n^2}{\tau_n^2+v_n^2}\middle| C\right) \leqq E\left(\frac{|v_n|}{\sqrt{\tau_n^2+v_n^2}}\middle| C\right) = \int_G \frac{|v_n|}{\sqrt{\tau_n^2+v_n^2}}\, dp(A|C) +$$

$$+ \int_{\mathscr{H}\setminus G} \frac{|v_n|}{\sqrt{\tau_n^2+v_n^2}}\, dp(A|C) \leqq p(G|C)+\varepsilon \leqq \frac{1}{\varepsilon^2}\frac{E(v_n^2|C)}{E(\tau_n^2+v_n^2|C)} + \varepsilon.$$

This implies

$$E\left(\frac{v_n^2}{\tau_n^2+v_n^2}\middle| C\right) \leqq \frac{1}{\varepsilon^2}\frac{E(v_n^2|C)}{E(\tau_n^2+v_n^2|C)} + \varepsilon. \tag{3.12}$$

By combining (3.11) and (3.12) we obtain the statement of Theorem 3.5.

Remark. It follows simply by Lemma 3.2 that

$$\frac{E(v_n^2|C)}{E(\tau_n^2|C)} \to 0 \quad \text{implies} \quad \frac{v_n}{\tau_n} \underset{C}{\Rightarrow} 0.$$

By the preceding theorem in order that

$$\frac{v_n}{\tau_n} \underset{C}{\Rightarrow} 0 \quad \text{if is sufficient also} \quad \frac{E(v_n^2|C)}{E(\tau_n^2+v_n^2|C)} \to 0 \quad (n \to \infty).$$

Theorem 3.6. *If* $\frac{v_n}{\tau_n} \underset{C}{\Rightarrow} 0$ *then* $\frac{E^2(v_n^2|C)}{E((\tau_n^2+v_n^2)^2|C)} \to 0 \ (n \to \infty)$.

Proof. By the Cauchy's inequality

$$E^2(v_n^2|C) \leqq E((\tau_n^2+v_n^2)^2|C)\, E\left(\frac{v_n^4}{(\tau_n^2+v_n^2)^2}\middle| C\right),$$

so

$$\frac{E^2(v_n^2|C)}{E((\tau_n^2+v_n^2)^2|C)} \leq E\left(\frac{v_n^4}{(\tau_n^2+v_n^2)^2}\middle|C\right) \leq E\left(\frac{v_n^2}{\tau_n^2+v_n^2}\middle|C\right).$$

Hence, applying (3.11) the proof is complete.

Let now $\xi_1, \xi_2, \ldots$ and $\xi_1', \xi_2', \ldots$ two sequences of random variables on $[\mathscr{H}, T_1, T_2, p]$. The two sequences are called *equivalent* (in the sence of Khinchin) with respect to C if

$$\sum_{i=1}^{\infty} p(\xi_i \neq \xi_i'|C) < \infty.$$

We shall prove that two equivalent sequences obey essentially the same weak laws. In addition to the notation already adopted let us introduce some new one: $B_n' = \xi_n'^{-1}(a)$, $p_n' = p(B_n'|C)$, where $\xi_n'^{-1}(Q) \in T_2$ and $\xi_n'^{-1}(Q) \subseteq C$ $(n=1, 2, \ldots)$ are assumed. Furthermore, let

$$\varepsilon_i' = \begin{cases} 1, & \xi_i' \in Q \\ 0, & \xi_i' \notin Q \end{cases}, \quad \tau_n' = \sum_{i=1}^{n} \varepsilon_i' \quad (n = 1, 2, \ldots).$$

Theorem 3.7. *Let us assume that the sequences $\xi_1, \xi_2, \ldots, \xi_1', \xi_2', \ldots$ are equivalent with respect to C and along with the random variables ϱ, ϱ_n $(n=1, 2, \ldots)$ satisfy the following condition*

$$\frac{\sum_{i=1}^{n} \varepsilon_i \xi_i}{\tau_n} - \varrho_n \underset{C}{\Rightarrow} \varrho.$$

Then, if $p\left(\sum_{i=1}^{\infty} \varepsilon_i = \infty | C\right) = 1$ is also fulfilled,

$$\frac{\sum_{i=1}^{n} \varepsilon_i' \xi_i'}{\tau_n'} - \varrho_n' \underset{C}{\Rightarrow} \varrho, \quad \textit{where} \quad \varrho_n' = \varrho_n \frac{\tau_n}{\tau_n'}.$$

PROOF. Let

$$\mathscr{H} \setminus D = \limsup_{n \to \infty} \{\xi_n \neq \xi_n'\},$$

$$E = \left\{\omega \in \mathscr{H} \middle| \sum_{i=1}^{\infty} \varepsilon_i(\omega) = \infty\right\}.$$

Since $p(E|C)=1$ and $\sum_{i=1}^{\infty} p(\xi_i \neq \xi_i'|C) < \infty$, so by the Borel—Cantelli lemma $p(D|C)=1$.

If $\omega \in D$ then $\xi_i(\omega) = \xi_i'(\omega)$ implies $\varepsilon_i(\omega) = \varepsilon_i'(\omega)$ and only finitely many $\xi_i(\omega)$ may be equal to $\xi_i'(\omega)$. Thus, if $\omega \in E \cap D$ then

$$\frac{\tau_n}{\tau_n'} \underset{C}{\to} 1. \tag{3.13}$$

Put $\varphi_n = \sum_{i=1}^{n} \varepsilon_i \xi_i$, $\varphi_n' = \sum_{i=1}^{n} \varepsilon_i \xi_i'$ and let us consider the following transformations:

$$\frac{\varphi_n'}{\tau_n'} - \varrho_n \frac{\tau_n}{\tau_n'} = \frac{\tau_n}{\tau_n'} \left(\frac{\varphi_n'}{\varphi_n} - \varrho_n \right) = \frac{\tau_n}{\tau_n'} \left(\frac{\varphi_n + (\varphi_n' - \varphi_n)}{\tau_n} - \varrho_n \right) =$$

$$= \frac{\tau_n}{\tau_n'} \left(\frac{\varphi_n}{\tau_n} - \varrho_n + \frac{\varphi_n' - \varphi_n}{\tau_n} \right).$$

If $\omega \in E \cap D$ then there exists a constant $K = K(\omega)$ depended only on ω so that for sufficiently large n we have

$$|\varphi_n' - \varphi_n| < K.$$

For this ω $\tau_n(\omega) \to \infty$ $(n \to \infty)$, so

$$\frac{\varphi_n'(\omega) - \varphi_n(\omega)}{\tau_n(\omega)} \to 0 \quad (n \to \infty).$$

Now, applying $p(E \cap D | C) = 1$ we have

$$\frac{\varphi_n' - \varphi_n}{\tau_n} \underset{C}{\to} 0. \tag{3.14}$$

Thus, by using (3.13) and (3.14)

$$\frac{\varphi_n'}{\tau_n'} - \varrho_n \frac{\tau_n}{\tau_n'} = \frac{\varphi_n'}{\tau_n'} - \varrho_n' \underset{C}{\Rightarrow} \varrho.$$

Q.e.d.

Remarks.

a) If in theorem 3.7 $\varrho_n = \frac{\sum_{i=1}^{n} \alpha_i \varepsilon_i}{\tau_n}$, where $\alpha_1, \alpha_2, \dots$ are real constants then ϱ_n' may appear as $\varrho_n' = \frac{\sum_{i=1}^{n} \alpha_i \varepsilon_i'}{\tau_n'}$.

b) If the sequence $\xi_1, \xi_2, \dots$ is bounded with probability 1 with respect to C then

$$\frac{\varphi_n'}{\tau_n'} - \varrho_n \underset{C}{\Rightarrow} \varrho$$

is also fulfilled.

In the following two theorems we give sufficient conditions for the existence of such real constants $c_1, c_2, \dots$ that

$$\frac{\sum_{i=1}^{n} \varepsilon_i (\xi_i - c_i)}{\tau_n} \underset{C}{\Rightarrow} 0, \tag{3.15}$$

and for the existence of real constants c_{in} $(i, n=1, 2, \dots)$ such that

$$\frac{\sum_{i=1}^{n} \varepsilon_i(\xi_i - c_{in})}{\tau_n} \underset{C}{\Rightarrow} 0. \tag{3.16}$$

Theorem 3.8. *Let $\xi_1, \xi_2, \dots$ be mutually independent random variables with respect to C. Let us assume that there exists a real sequence $a_1, a_2, \dots$ so that*

$$\sum_{k=1}^{\infty} p(\varepsilon_k|\xi_k - a_k| \geqq k|C) < \infty, \tag{3.17}$$

(3.18)

$$\left(\sum_{k=1}^{n} p_k\right)^{-2} \sum_{k=1}^{n} p_k \int_{|x| \leqq k} x^2\, dF_k(x+a_k) \to 0 \quad (n \to \infty), \quad \text{where} \quad F_k(x) = p(\xi_k < x|B_k),$$

$$\sum_{k=1}^{\infty} p(\xi_k = a_k|C) < \infty. \tag{3.19}$$

Moreover, let Q' is the set of non zero real numbers x for which there exist $y \in Q$ and $i \geqq 1$ such that $x = y - a_i$. Assume that

$$B_k^* = \xi_k^{*-1}(Q') \in T_2 \quad \text{and} \quad B_k^* \subseteqq C \quad (k = 1, 2, \dots),$$

where

$$\xi_k^* = \begin{cases} \varepsilon_k(\xi_k - a_k), & \varepsilon_k|\xi_k - a_k| < k \\ 0 \quad , & \varepsilon_k|\xi_k - a_k| \geqq k. \end{cases}$$

Under such conditions

$$\sum_{k=1}^{\infty} p_k = \infty \tag{3.20}$$

is also fulfilled and there exists a sequence $c_1, c_2, \dots$ of real numbers such that

$$\frac{\sum_{i=1}^{n} \varepsilon_i(\xi_i - c_i)}{\tau_n} \underset{C}{\Rightarrow} 0.$$

PROOF. Let us introduce the following notations:

$$\varepsilon_k^* = \begin{cases} 1, & \omega \in B_k^* \\ 0, & \omega \notin B_k^* \end{cases}$$

$$p_k^* = p(B_k^*|C), \quad D_k^* = D(\xi_k^*|B_k).$$

Firstly we shall prove that there exists a real sequence $b_1, b_2, \dots$ such that

$$\frac{\sum_{k=1}^{n} \varepsilon_k^*(\xi_k^* - b_k)}{\sum_{k=1}^{n} \varepsilon_k^*} \underset{C}{\Rightarrow} 0. \tag{3.21}$$

By virtue of the remark made after Theorem 3.3 to prove (3.21) it is sufficient that

$$\frac{\sum_{i=1}^{n} p_i^* D_i^{*2}}{\left(\sum_{i=1}^{n} p_i^*\right)^2} \to 0 \quad (n \to \infty). \tag{3.22}$$

To show (3.22) we could write

$$p_k^* D_k^{*2} \leqq p_k^* E(\xi_k^{*2}|B_k) = p_k^* \int\limits_{\varepsilon_k|\xi_k - a_k| < k} \varepsilon_k(\xi_k - a_k)^2 \, dp(A|B_k^*) =$$

$$= \int\limits_{\varepsilon_k|\xi_k - a_k| < k} \varepsilon_k(\xi_k - a_k)^2 \, dp(A|C) = p_k \int\limits_{\varepsilon_k|\xi_k - a_k| < k} (\xi_k - a_k)^2 \, dp(A|B_k) \geqq$$

$$\leqq \int\limits_{|\xi_k - a_k| < k} (\xi_k - a_k)^2 \, dp(A|B_k) = \int\limits_{|x| < h} x^2 \, dF_k(x + a_k).$$

Hence

$$\frac{\sum_{k=1}^{n} p_k^* D_k^{*2}}{\left(\sum_{k=1}^{n} p_k^*\right)^2} \leqq \left(\sum_{k=1}^{n} p_k^*\right)^{-2} \sum_{k=1}^{n} p_k \int\limits_{|x| < k} x^2 \, dF_k(x + a_k). \tag{3.23}$$

Let now $\xi_k' = \varepsilon_k(\xi_k - a_k)$ and

$$\varepsilon_k' = \begin{cases} 1, & \xi_k' \in Q' \\ 0, & \xi_k' \notin Q'. \end{cases}$$

Then

$$\xi_k^* = \begin{cases} \xi_k', & \varepsilon_k|\xi_k - a_k| < k \\ 0, & \varepsilon_k|\xi_k - a_k| \geqq k \end{cases}$$

and trivially hold the following

$$\begin{aligned} &\text{If} \quad \varepsilon_k = 0 \quad \text{then} \quad \varepsilon_k' = 0. \\ &\text{If} \quad \varepsilon_k = 1 \quad \text{and} \quad \xi_k - a_k \neq 0 \quad \text{then} \quad \varepsilon_k' = 1. \\ &\text{If} \quad \varepsilon_k = 1 \quad \text{and} \quad \xi_k - a_k = 0 \quad \text{then} \quad \varepsilon_k' = 0. \end{aligned} \tag{3.24}$$

According to (3.24) we have

$$\sum_{k=1}^{\infty} |p_k^* - p_k| = \sum_{k=1}^{\infty} \left| p(B_k|C) - p(B_k^*|C) \right| =$$

$$= \sum_{k=1}^{\infty} \left| p(B_k|C) - p(B_k \cap \{\xi_k - a_k \neq 0\}|C) \right| \leqq \sum_{k=1}^{\infty} p(\xi_k - a_k = 0|C) < \infty.$$

Consequently,

$$\frac{\sum_{k=1}^{n} p_k^*}{\sum_{k=1}^{n} p_k} \to 1 \quad (n \to \infty). \tag{3.25}$$

Now by using (3.23) and (3.25) it is obvious that (3.22) and consequently (3.21) hold.

On the other hand from the inequality

$$\sum_{k=1}^{\infty} p(\xi_k^* \neq \xi_k' | C) \leqq \sum_{k=1}^{\infty} p(\varepsilon_k |\xi_k - a_k| \geqq k | C) < \infty$$

we infer that the sequences $\{\xi_n^*\}$ and $\{\xi_n'\}$ are equivalent. Consequently, by remark b) mentioned after Theorem 3.7 we have

$$\delta_n = \frac{\sum_{k=1}^{n} \varepsilon_k' \xi_k' - \sum_{k=1}^{n} \varepsilon_k' b_k}{\sum_{k=1}^{n} \varepsilon_k'} \underset{C}{\Rightarrow} 0. \tag{3.26}$$

With the help of the Borel—Cantelli lemma we get

$$\frac{\sum_{k=1}^{n} \varepsilon_k'}{\tau_n} \underset{C}{\rightarrow} 1.$$

Hence, if $\delta_n' = \frac{\sum_{k=1}^{n} \varepsilon_k'}{\tau_n} \delta_n$ then

$$\delta_n' \underset{C}{\Rightarrow} 0. \tag{3.27}$$

We can write

$$\delta_n' = \frac{\sum_{k=1}^{n} \varepsilon_k (\xi_k - a_k) - \sum_{k=1}^{n} \varepsilon_k' b_k}{\tau_n} = \frac{\sum_{k=1}^{n} \varepsilon_k \xi_k}{\tau_n} - \frac{\sum_{k=1}^{n} (a_k + b_k)}{\tau_n} + \frac{\sum_{k=1}^{n} (\varepsilon_k - \varepsilon_k') b_k}{\tau_n}.$$

Since $\sum_{k=1}^{\infty} p(\varepsilon_k \neq \varepsilon_k' | C) < \infty$ we conclude that

$$\frac{\sum_{k=1}^{n} (\varepsilon_k - \varepsilon_k') b_k}{\tau_n} \underset{C}{\rightarrow} 0.$$

Thus, according to (3.27)

$$\frac{\sum_{k=1}^{n} \varepsilon_k \xi_k}{\tau_n} - \frac{\sum_{k=1}^{n} \varepsilon_k (a_k + b_k)}{\tau_n} \underset{C}{\Rightarrow} 0.$$

Let now $c_k = a_k + b_k$ $(k = 1, 2, \ldots)$. Then from the above convergence

$$\frac{\sum_{k=1}^{n} \varepsilon_k (\xi_k - c_k)}{\tau_n} \underset{C}{\Rightarrow} 0.$$

Q.e.d.

Remark. If $\xi_1, \xi_2, \ldots$ are continuously distributed random variables with respect to $p(\cdot|C)$ then it is obvious that in the preceding theorem condition (3.19) holds. Moreover, it may be omitted if instead of condition (3.20) a stronger one is fulfilled. For example, the statement of Theorem 3.8 is true if besides of the independence, conditions (3.17), (3.18) and

$$\sum_{k=1}^{\infty} p(\mathscr{H} \setminus B_k|C) < \infty \tag{3.28}$$

are fulfilled too.

By this remark one can observe that Theorem 3.8 is a generalization of [2], Theorem 2.5.1 for conditional probability field.

Likewise, the following theorem may be compare with [2], Theorem 2.5.2. To be more precise it generalizes a part of [2], Theorem 2.5.2.

Theorem 3.9. *If $\xi_1, \xi_2, \ldots$ are independent with respect to C and there exists a real sequence $a_1, a_2, \ldots$ such that*

$$\sum_{k=1}^{n} p(\varepsilon_k|\xi_k - a_k| \geqq n|C) \to 0 \quad (n \to \infty) \tag{3.29}$$

$$\left(\sum_{k=1}^{n} p_k\right)^{-2} \sum_{k=1}^{n} p_k \int_{|x|<n} x^2 \, dF_k(x+a_k) \to 0 \quad (n \to \infty). \tag{3.30}$$

Then there exist real numbers c_{kn} $(k, n=1, 2, \ldots)$ such that

$$\frac{\sum_{k=1}^{n} \varepsilon_k(\xi_k - c_{kn})}{\tau_n} \underset{C}{\Rightarrow} 0,$$

where

$$c_{kn} = a_k + b_{kn}, \quad b_{kn} = \int_{|x|<n} x \, dF_k(x+a_k), \quad F_k(x) = p(\xi_k < x|B_k).$$

PROOF. Let

$$\xi_k' = \varepsilon_k(\xi_k - a_k), \quad \xi_{kn}^* = \begin{cases} \xi_{kn}', & |\xi_k'| < n \\ 0, & |\xi_k'| \geqq n \end{cases},$$

$$d_n = \frac{\sum_{k=1}^{n} \varepsilon_k c_{kn}}{\tau_n}, \quad G(n) = \bigcap_{k=1}^{n} \{\omega | \xi_{kn}^*(\omega) = \xi_k'(\omega)\}.$$

Then

$$p\left(\left|\frac{\sum_{k=1}^{n} \varepsilon_k \xi_k}{\tau_n} - d_n\right| \geqq \varepsilon|C\right) \leqq p\left(\left\{\left|\frac{\sum_{k=1}^{n} \varepsilon_k \xi_k}{\tau_n} - d_n\right| \geqq \varepsilon\right\} \cap G(n)|C\right) + p(\mathscr{H} \setminus G(n)|C). \tag{3.31}$$

Since $p(\mathscr{H} \setminus G(n)|C) \leqq \sum_{k=1}^{n} p(\xi_k' \neq \xi_{kn}^*|C) \leqq \sum_{k=1}^{n} p(|\xi_k'| \geqq n|C)$, it follows

$$p(\mathscr{H} \setminus G(n)|C) \to 0 \quad (n \to \infty). \tag{3.32}$$

On the other hand

$$p\left(\left\{\left|\frac{\sum_{k=1}^{n}\varepsilon_k\xi_k}{\tau_n}-d_n\right|\geqq\varepsilon\right\}\cap G(n)|C\right)\geqq p\left(\left|\frac{\sum_{k=1}^{n}\varepsilon_k(\xi_{kn}^*-b_{kn})}{\tau_n}\right|\geqq\varepsilon|C\right)\leqq$$

$$\leqq\frac{1}{\varepsilon^2}\frac{E\left(\left(\sum_{k=1}^{n}\varepsilon_k(\xi_{kn}^*-b_{kn})\right)^2\middle|C\right)}{E(\tau_n^2|C)}.$$

It is easy to see that $b_{kn}=E(\xi_{kn}^*|B_k)$ and consequently,

$$E\big(\varepsilon_k(\xi_{kn}^*-b_{kn})|C\big)=\int_{\mathscr{H}}\varepsilon_k(\xi_{kn}^*-b_{kn})\,dp(A|C)=\int_{B_k}(\xi_{kn}^*-b_{kn})\,dp(A|C)=$$

$$=p_k\int_{B_k}(\xi_{kn}^*-b_{kn})\,dp(A|B_k)=p_k\int_{\mathscr{H}}(\xi_{kn}^*-b_{kn})\,dp(A|B_k)=0.$$

Hence, by the independence of $\xi_1,\xi_2,\dots$

$$E\big(\varepsilon_k(\xi_{kn}^*-b_{kn})\varepsilon_l(\xi_{ln}^*-b_{ln})\big)=0\quad(k\neq l).\tag{3.33}$$

Since

$$D^2(\xi_{kn}^*|B_k)\leqq E(\xi_{kn}^*|B_k)=\int_{|\xi_k'|<n}\xi_k'^2\,dp(A|B_k)=\int_{|x|<n}x^2\,dF_k(x+a_k),$$

with the help of (3.33)

$$p\left(\left\{\left|\frac{\sum_{k=1}^{n}\varepsilon_k\xi_k}{\tau_n}-d_n\right|\geqq\varepsilon\right\}\cap G(n)|C\right)\leqq\frac{\sum_{k=1}^{n}E\big(\varepsilon_k(\xi_{kn}^*-b_{kn})^2|C\big)}{\varepsilon^2\left(\sum_{k=1}^{n}p_k\right)^2}\leqq$$

$$\leqq\frac{\sum_{k=1}^{n}p_kE\big((\xi_{kn}^*-b_{kn})^2|B_k\big)}{\varepsilon^2\left(\sum_{k=1}^{n}p_k\right)^2}=\frac{\sum_{k=1}^{n}p_kD^2(\xi_{kn}^*|B_k)}{\varepsilon^2\left(\sum_{k=1}^{n}p_k\right)^2}\leqq\frac{\sum_{k=1}^{n}p_k\int_{|x|<n}x^2\,dF_k(x+a_k)}{\varepsilon^2\left(\sum_{k=1}^{n}p_k\right)^2}.$$

From this inequality and (3.30) it follows that

$$p\left(\left\{\left|\frac{\sum_{k=1}^{n}\varepsilon_k\xi_k}{\tau_n}-d_n\right|\geqq\varepsilon\right\}\cap G(n)|C\right)\to 0\quad(n\to\infty).\tag{3.34}$$

By (3.31), (3.32), (3.34) we have

$$\frac{\sum_{i=1}^{n}\varepsilon_i\xi_i}{\tau_n}-d_n\underset{C}{\Rightarrow}0.$$

Q.e.d.

We mention some simple consequences of Theorem 3.9.

Corollary 3.10. *Under the conditions of Theorem* 3.9 *and if*

$$\left(\sum_{k=1}^{n} p_k\right)^{-1} \sum_{k=1}^{n} p_k\big(E(\xi_k|B_k)-a_k-b_{kn}\big) \to 0 \quad (n \to \infty), \tag{3.35}$$

then

$$\frac{\sum_{k=1}^{n} \varepsilon_k\big(\xi_k - E(\xi_k|B_k)\big)}{\tau_n} \underset{C}{\Rightarrow} 0.$$

Proof. It follows from the inequality

$$p\left(\left|\frac{\sum_{k=1}^{n} \varepsilon_k\big(E(\xi_k|B_k)-a_k-b_{kn}\big)}{\tau_n}\right| \geqq \varepsilon \middle| C\right) \leqq \frac{\sum_{k=1}^{n} p_k\big(E(\xi_k|B_k)-a_k-b_{kn}\big)}{\varepsilon^2 \sum_{k=1}^{n} p_k}.$$

Corollary 3.11. *In Theorem* 3.9 *condition* (3.30) *can be exchanged for*

$$\left(\sum_{k=1}^{n} p_k\right)^{-2} \sum_{k=1}^{n} p_k\left\{\int_{|x|<n} x^2\, dF_k(x+a_k) - \Big(\int_{|x|<n} x\, dF_k(x+a_k)\Big)^2\right\} \to 0 \quad (n \to \infty). \tag{3.36}$$

Proof. It is a direct consequence of equality

$$D^2(\xi_{kn}^*|B_k) = \int_{|x|<n} x^2\, dF_k(x+a_k) - \Big(\int_{|x|<n} x\, dF_k(x+a_k)\Big)^2. \tag{3.37}$$

Corollary 3.12. *If* $\xi_1, \xi_2, \ldots$ *are independent with respect to* C *having finite* $E_1, E_2, \ldots$ *and satisfying the following convergences*

$$\sum_{k=1}^{n} p_k \int_{|x|\geqq n} dF_k(x+E_k) \to 0 \quad (n \to \infty), \tag{3.38}$$

$$\left(\sum_{k=1}^{n} p_k\right)^{-2} \sum_{k=1}^{n} p_k\left\{\int_{|x|<n} x^2\, dF_k(x+E_k) - \Big(\int_{|x|<n} x\, dF_k(x+E_k)\Big)^2\right\} \to 0 \quad (n \to \infty), \tag{3.39}$$

$$\left(\sum_{k=1}^{n} p_k\right)^{-1} \sum_{k=1}^{n} p_k \int_{|x|<n} x\, dF_k(x+E_k) \to 0 \quad (n \to \infty), \tag{3.40}$$

then

$$\frac{\sum_{k=1}^{n} \varepsilon_k(\xi_k - E_k)}{\tau_n} \underset{C}{\Rightarrow} 0.$$

PROOF. By Corollary 3.11 we have

$$\frac{\sum_{k=1}^{n} \varepsilon_k \xi_k}{\tau_n} - d_n \underset{C}{\Rightarrow} 0,$$

where

$$d_n = \frac{\sum_{k=1}^{n} \varepsilon_k (E_k - b_{kn})}{\tau_n}, \quad b_{kn} = \int_{|x|<n} x\, dF_k(x+E_k), \quad F_k(x) = p(\xi_k < x|B_k).$$

It is sufficient to show that

$$\frac{\sum_{k=1}^{n} \varepsilon_k b_{kn}}{\tau_n} \underset{C}{\Rightarrow} 0. \tag{3.41}$$

We can write

$$p\left(\left|\frac{\sum_{k=1}^{n} \varepsilon_k b_{kn}}{\tau_n}\right| \geqq \varepsilon \middle| C\right) \leqq \frac{E\left(\left(\sum_{k=1}^{n} \varepsilon_k b_{kn}\right)^2 \middle| C\right)}{\varepsilon^2 E(\tau_n^2|C)} \leqq \frac{\left(\sum_{k=1}^{n} p_k b_{kn}\right)^2}{\varepsilon^2 \left(\sum_{k=1}^{n} p_k\right)^2} \to 0 \quad (n \to \infty).$$

This proves the theorem.

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(Received September 4, 1978.)