Publ. Math. Debrecen
50 / 1-2 (1997), 177–179

On a conjecture concerning additive arithmetical functions II

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Abstract. If f is completely additive and $f(2n+A)-f(n)$ is monotonic from some number on, then $f(n)=c\log n$.

In 1946 Erdös [2] proved the following theorem:

Theorem 1 (Erdös). *If a real valued additive function f is monotonically increasing, then $f(n)=c\log n$.*

As a possible generalization of this result I proposed the following conjecture:

Conjecture. *Let f be an additive function. If $f(an+b)-f(cn+d)$ is monotonic from some number on, then $f(n)=c\log n$ for all n coprime to $ac(ad-bc)$.*

If f is bounded, then $f(an+b)-f(cn+d)$ is convergent and the conjecture is true by a theorem of Elliott [1]. In [3] we proved some special cases of the conjecture, including the following theorem:

Theorem 2. *Let f be an additive function and let a be an integer. If $f(n+a)-f(n)$ is monotonic or it is of constant sign from some number on, then $f(n)=c\log n$ for all n coprime to a. If f is completely additive, then $f(n)=c\log n$ for all n.*

Here we prove the conjecture in a further special case:

Mathematics Subject Classification: 11A25.
Key words and phrases: characterization of additive functions.
Partially supported by the Hungarian National Foundation for Scientific Research Grant No. T 017433, by CEC grant No. CIPA-CT92-4022 and by Fulbright Grant

Theorem. *Let f be a completely additive function. If*

$$f(2n+A)-f(n) \tag{1}$$

is monotonic from some number on, then $f(n)=c\log n$ for all n.

Proof of the Theorem. We may assume that A is odd in (1). (Otherwise by Theorem 2 $f(n)=c\log n$ for all n.)

We consider now that (1) is monotonically increasing.
We put $z=n\left(n+\frac{3+A}{2}\right)$. By comparing the value of (1) at the pairs of numbers (z,n) we have

$$f(2z+A)-f(n)-f\Big(n+\frac{A+3}{2}\Big)>f(2n+A)-f(n). \tag{2}$$

By connecting the pair $\left(2z+A,n+\frac{3-A}{2}\right)$ we obtain

$$f(4z+3A)-f(2z+A)>f(2n+3)-f\Big(n+\frac{3-A}{2}\Big). \tag{3}$$

We observe that $4z+3A=4n^2+2(A+3)n+3A=(2n+A)(2n+3)$, therefore adding the rows (2) and (3) we obtain

$$f\Big(n+\frac{A-3}{2}\Big)-f\Big(n+\frac{A+3}{2}\Big)>0.$$

Applying Theorem 2, we have $f(n)=c\log n$.

Background of the proof: We put $z=(n+a_1)(n+a_2)$. We apply (1) for the pair (z,N_1) (we determine N_1 later). So we have

$$f(2z+A)-f(n+a_1)-f(n+a_2)>f(2N_1+A)-f(N_1). \tag{4}$$

To cancel the $f(2z+A)$ term by addition, we apply (1) for the pair $(2z+A,N_2)$ (we determine N_2 later). So we have

$$f(4z+3A)-f(2z+A)>f(2N_2+A)-f(N_2). \tag{5}$$

If we had $4z+3A=4n^2+bn+c=(2n+A_1)(2n+A_2)$ with integers A_1 and A_2, and also $2n+A_1=2N_1+A$ and $2n+A_2=2N_2+A$, then we could cancel some terms. We can do it only if A_1 and A_2 are odd. Then adding the rows (4) and (5), we have

$$-f(n+a_1)-f(n+a_2)>-f\Big(n+\frac{A_1-A}{2}\Big)-f\Big(n+\frac{A_2-A}{2}\Big).$$

If $a_1=\frac{A_1-A}{2}$ and $a_2\neq\frac{A_2-A}{2}$, then we arrive at a special case of Theorem 2.

By solving the equation $4z+3A=4(n+a_1)(n+a_2)+3A=0$, we get $A_{1,2}=a_1+a_2\pm\sqrt{(a_1-a_2)^2-3A}$. To satisfy the condition $a_1=\frac{A_1-A}{2}$, we need the choice $a_2=\frac{A+1}{2}+1+a_1$. Therefore $A_1=2a_1+A$ and $A_2=2a_1+3$ are odd integers and we arrive at

$$f(n+a_2-A)-f(n+a_2)>0,$$

i.e. by Theorem 2 $f(n)=c\log n$.

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(Received August 2, 1996; revised September 26, 1996)