On the conditional laws of large numbers II

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1. This paper, which is a continuation of our previous publication [3], is concerned with the strong laws of large numbers for a conditional probability field of RÉNYI. We shall deal with the generalizations of some classical strong laws. The proofs are based on a generalization of Kolmogorov's inequality, and a generalization of the inequality of HAJEK and RÉNYI [2]. Naturally, in the special case of a probability space (in the sense of Kolmogorov) we arrive at some well-known laws of large numbers.

We shall use the same symbols as in [3]. A conditional probability field will be denoted by $[H, T_1, T_2, p]$. Throughout this article we employ some constant notations:

Let $\xi_1, \xi_2, \ldots$ be random variables on the conditional probability field $[H, T_1, T_2, p]$ and Q be a Borel set of the real line. Let $C \in T_2$ and let us assume that

$$\xi_n^{-1}(Q) \in T_2, \quad \xi_n^{-1}(Q) \subseteqq C \quad (n = 1, 2, \ldots).$$

Let $B_n = \xi_n^{-1}(Q)$, $p_n = p(B_n|C)$, $E_n = E(\xi_n|B_n)$, $D_n = D(\xi_n|B_n)$ provided that $E(\xi_n|B_n)$, $D(\xi_n|B_n)$ $(n=1, 2, \ldots)$ exist. Finally let

$$\varepsilon_i = \begin{cases} 1, & \xi_i \in Q \\ 0, & \xi_i \notin Q \end{cases}$$

$$\tau_n = \sum_{i=1}^{n} \varepsilon_i, \quad \text{and} \quad \nu_n = \sum_{i=1}^{n} \varepsilon_i(\xi_i - E_i).$$

If the sequence $\xi_1, \xi_2, \ldots$ converges to a random variable ξ with probability 1 (with respect to C), it will be denoted by $\xi_n \xrightarrow[C]{} \xi$.

A more detailed list of notions and notations can be found in [3].

2. The following result is a simple generalization of Kolmogorov's inequality.

Lemma 2.1. *Let $\xi_1, \xi_2, \ldots$ be independent with respect to C and assume that the conditional expectations and variances $E_i = E(\xi_i|B_i)$, $D_i = D(\xi_i|B_i)$ $(i=1, 2, \ldots)$ exist. If $\omega_1, \omega_2, \ldots$ are random variables such that $0 < \omega_1 \leqq \omega_2 \leqq \ldots$ and there exist $E(\omega_i^2|C) > 0$, then for all $\varepsilon > 0$*

$$p\left(\max_{1 \leqq k \leqq n} \left|\frac{\nu_k}{\omega_k}\right| \geqq \varepsilon | C\right) \leqq \frac{1}{\varepsilon^2} \sum_{i=1}^{n} \frac{p_i D_i^2}{E(\omega_i^2|C)}. \tag{2.1}$$

PROOF. Let n be fixed and $\xi_i^* = \varepsilon_i(\xi_i - E_i)$. Then $\nu_k = \sum_{i=1}^{k} \xi_i^*$. Let

$$A_k = \{|\nu_1| < \omega_1 \varepsilon\} \cap \{|\nu_2| < \omega_2 \varepsilon\} \cap \ldots \cap \{|\nu_{k-1}| < \omega_{k-1} \varepsilon\} \cap \{|\nu_k| \geqq \omega_k \varepsilon\}$$
$$(k = 1, 2, 3, \ldots, n)$$

and

$$A_0 = \bigcap_{i=1}^{n} \{|\nu_i| < \omega_i \varepsilon\}.$$

We need the following computations:

If $k \leqq i \leqq n$, then $\nu_i = \nu_k + \sum_{j=k+1}^{i} \xi_j^*$ and

$$\nu_i^2 = \nu_k^2 + \sum_{j=k+1}^{i} \xi_j^{*2} + 2 \sum_{j=k+1}^{i} \nu_k \xi_j^* + 2 \sum_{k<j_1<j_2\leqq i} \xi_{j_1}^* \xi_{j_2}^*. \tag{2.2}$$

Let now $h \in H$ and

$$\alpha_k(h) = \begin{cases} 1, & h \in A_k \\ 0, & h \notin A_k. \end{cases}$$

Then, if $k < i < j \leqq n$ and $p(A_k|C) \neq 0$

$$E(\xi_i^* \xi_j^* | A_k C) = \int_H \xi_i^* \xi_j^* \, dp(A|A_k C) = \int_H \xi_i^* \xi_j^* \frac{dp(AA_k|C)}{p(A_k|C)} =$$

$$= \frac{1}{p(A_k|C)} \int_H \xi_i^* \xi_j^* \alpha_k \, dp(A|C) = 0$$

because ξ_i^*, ξ_j^*, α_k are independent with respect to C and $E(\xi_i^*|C) = 0$ $(i = 1, 2, \ldots, n)$.

In a similar way we get

$$E(\nu_k \xi_j^* | A_k C) = 0 \quad (k < j \leqq n) \quad \text{if} \quad p(A_k|C) \neq 0.$$

Thus by (2.2)

$$E(\nu_i^2 | A_k C) \geqq E(\nu_i | A_k C) + K_{k,i} \quad (k \leqq i < n), \tag{2.3}$$

where

$$K_{k,i} = E\left(2\left[\sum_{j=k+1}^{i} \nu_k \xi_j^* + \sum_{k<j_1<j_2\leqq i} \xi_{j_1}^* \xi_{j_2}^*\right] \middle| A_k C\right),$$

and

$$K_{k,i} = 0 \quad \text{if} \quad p(A_k|C) \neq 0.$$

Let

$$\eta_n = \sum_{k=1}^{n-1} \nu_k^2 \left(\frac{1}{E(\omega_k^2|C)} - \frac{1}{E(\omega_{k+1}^2|C)}\right) + \frac{\nu_n^2}{E(\omega_n^2|C)}.$$

Since

$$E(\nu_i^2|C) = \sum_{j=1}^{i} p_j D_j^2$$

$$E(\eta_n|C) = \sum_{i=1}^{n} \frac{p_i D_i^2}{E(\omega_i^2|C)}. \tag{2.4}$$

By (2.3) it follows that

$$E(\eta_n|A_kC) = \sum_{i=1}^{n-1} E(v_i^2|A_kC)\left(\frac{1}{E(\omega_i^2|C)} - \frac{1}{E(\omega_{i+1}^2|C)}\right) + \frac{E(v_n^2|A_kC)}{E(\omega_n^2|C)} \geqq$$

$$\geqq \sum_{i=k}^{n-1} E(v_i^2|A_kC)\left(\frac{1}{E(\omega_i^2|C)} - \frac{1}{E(\omega_{i+1}^2|C)}\right) + \frac{E(v_n^2|A_kC)}{E(\omega_n^2|C)} \geqq$$

$$\geqq \sum_{i=k}^{n-1} E(v_n^2|A_kC)\left(\frac{1}{E(\omega_i^2|C)} - \frac{1}{E(\omega_{i+1}^2|C)}\right) + \frac{E(v_k^2|A_kC)}{E(\omega_k^2|C)} +$$

$$+ \sum_{i=k}^{n-1} K_{k,i}\left(\frac{1}{E(\omega_i^2|C)} - \frac{1}{E(\omega_{i+1}^2|C)}\right) + \frac{K_{k,n}}{E(\omega_n^2|C)} =$$

$$= \frac{E(v_k^2|A_kC)}{E(\omega_k^2|C)} + L_k \geqq \varepsilon^2 \frac{E(\omega_k^2|C)}{E(\omega_k^2|C)} + L_k = \varepsilon^2 + L_k \quad (k = 1, 2, \ldots, n),$$

where $L_k = 0$ if $p(A_k|C) \neq 0$.

Hence

$$E(\eta_n|A_kC) \geqq \varepsilon^2 + L_k \quad (k = 1, 2, \ldots, n), \tag{2.5}$$

where $L_k = 0$ if $p(A_k|C) \neq 0$.

Since $\{A_0, A_1, \ldots, A_n\}$ is a complete group of events

$$E(\eta_n|C) \geqq \sum_{k=1}^{n} E(\eta_n|A_kC)\, p(A_k|C)$$

which implies

$$E(\eta_n|C) \geqq \sum_{k=1}^{n} \varepsilon^2 p(A_k|C) + \sum_{k=1}^{n} L_k\, p(A_k|C) =$$

$$= \varepsilon^2 \sum_{k=1}^{n} p(A_k|C) = \varepsilon^2 p\left(\bigcup_{i=1}^{n} A_i|C\right).$$

Hence by (2.4)

$$\varepsilon^2 p\left(\bigcup_{i=1}^{n} A_i|C\right) \leqq \sum_{i=1}^{n} \frac{p_i D_i^2}{E(\omega_i^2|C)},$$

which proves (2.1).

Corollary. *If* $\omega_k = \sum_{i=1}^{k} \varepsilon_i$, *then Lemma* 2.1 *means that*

$$p\left(\max_{1 \leqq k \leqq n} \left|\frac{\sum_{i=1}^{k} \varepsilon_i(\xi_i - E_i)}{\sum_{i=1}^{k} \varepsilon_i}\right| \geqq \varepsilon|C\right) \leqq \frac{1}{\varepsilon^2} \sum_{i=1}^{n} \frac{p_i D_i^2}{\left(\sum_{j=1}^{i} p_j\right)^2} \tag{2.6}$$

because of $E\left(\left(\sum_{j=1}^{i} \varepsilon_j\right)^2 \Big| C\right) \geqq \left(\sum_{j=1}^{i} p_j\right)^2$ *by Lemma* 3.1 *of* [3].

Our following two theorems (Theorem 2.2 and Theorem 2.3) are analogous to a classical result of Kolmogorov [6, Theorem 2.7.5a]. A similar result was obtained by Rényi who proved the following (see [4, pp. 401]):

If $\xi_1, \xi_2, \dots$ are independent with respect to C with finite expectation and variance $E_k = E(\xi_k|B_k)$, $D_k = D(\xi_k|B_k) (k=1, 2, \dots)$, and

$$\text{a)} \quad \sum_{k=1}^{\infty} p_k = \infty,$$

$$\text{b)} \quad \sum_{k=1}^{\infty} \frac{p_k D_k^2}{\left(\sum_{j=1}^{k} p_j\right)^2} < \infty,$$

then

$$\frac{\sum_{i=1}^{k} \varepsilon_i \xi_i}{\sum_{i=1}^{k} \varepsilon_i} \xrightarrow[C]{} E \quad (k \to \infty).$$

Our theorems will be proved under somewhat different conditions and by a totally different method.

Theorem 2.2. *Let the random variables $\xi_1, \xi_2, \dots$ be independent with respect to C and suppose that E_i, D_i exist $(i=1, 2, \dots)$. Moreover let us assume that*

$$1) \quad \sum_{k=1}^{\infty} \frac{D_k^2}{k^2} < \infty, \qquad 2) \quad 0 < d \leq p_i \quad (i = 1, 2, \dots).$$

Then

$$\frac{\nu_k}{\tau_k} = \frac{\sum_{i=1}^{k} \varepsilon_i (\xi_i - E_i)}{\sum_{i=1}^{k} \varepsilon_i} \xrightarrow[C]{} 0 \quad (k \to \infty).$$

Proof. Let

$$\delta_N = \sup_{n \geq N} \left|\frac{\nu_n}{\tau_n}\right|, \; \delta_{a,b} = \max_{a \leq n \leq b} \left|\frac{\nu_n}{\tau_n}\right|.$$

If $\delta_N > \varepsilon$ and $2^s \leq N < 2^{s+1}$ for an integer s, then there exists $l \geq s$ such that $\delta_{2^l 2^{l+1}} \geq \frac{\varepsilon}{2}$, and so

$$p(\delta_N \geq \varepsilon | C) \leq \sum_{l=s}^{\infty} p\left(\delta_{2^l, 2^{l+1}} \geq \frac{\varepsilon}{2} \middle| C\right) \quad \text{if} \quad 2^s \leq N < 2^{s+1}.$$

Furthermore

$$p(\delta_{2^l,2^{l+1}} \geqq \varepsilon|C) = p\left(\max_{2^l \leqq k < 2^{l+1}} \left|\frac{\nu_k}{\tau_k}\right| \geqq \frac{\varepsilon}{2}\middle|C\right) \leqq p\left(\max_{2^l \leqq k < 2^{l+1}} \sum_{i=1}^{k} p_i \left|\frac{\nu_k}{\tau_k}\right| \geqq \right.$$

$$\left. \geqq \frac{\varepsilon}{2} \sum_{i=1}^{2^l} p_i \middle| C\right) = p\left(\max_{2^l \leqq k < 2^{l+1}} \left|\frac{\nu_k}{\tau_k'}\right| \geqq \frac{\varepsilon}{2} \sum_{i=1}^{2^l} p_i \middle| C\right) \leqq$$

$$\leqq p\left(\max_{1 \leqq k < 2^{l+1}} \left|\frac{\nu_k}{\tau_k'}\right| \geqq \frac{\varepsilon}{2} \sum_{i=1}^{2^l} p_i \middle| C\right) \leqq \frac{4}{\varepsilon^2}\left(\sum_{k=1}^{2^l} p_k\right)^{-2} \sum_{k=1}^{2^{l+1}-1} \frac{p_k D_k^2}{E(\tau_k'^2|C)} \leqq$$

$$\leqq \frac{4}{\varepsilon^2}\left(\sum_{k=1}^{2^l} p_k\right)^{-2} \sum_{k=1}^{2^{l+1}-1} p_k D_k^2,$$

where $\tau_k' = \dfrac{\tau_k}{\sum_{i=1}^{k} p_i}$. Hence

$$p(\delta_N \geqq \varepsilon|C) \leqq \frac{4}{\varepsilon^2} \sum_{l=s}^{\infty} \frac{\sum_{k=1}^{2^{l+1}-1} p_k D_k^2}{\left(\sum_{k=1}^{2^l} p_k\right)^2} \leqq \frac{4}{\varepsilon^2 d^2} \sum_{l=s}^{\infty} \frac{1}{2^{2l}} \sum_{k=1}^{2^{l+1}-1} D_k^2 \leqq$$

$$\leqq \frac{4}{\varepsilon^2 d^2}\left(\frac{1}{4^{s-1}} \sum_{k=1}^{2^{s+1}-1} D_k^2 + 16 \sum_{k=2^s+1}^{\infty} \frac{D_k^2}{k^2}\right).$$

Now by the Kronecker lemma $\dfrac{\sum_{i=1}^{n} D_i^2}{n^2} \to 0 \ (n \to \infty)$, consequently the last expression converges to zero which implies our theorem.

The classical form of the following lemma is due to HAJEK and RÉNYI [2].

Lemma 2.3. *Let us assume that the conditional expectations and variances of* $\xi_1, \xi_2, \ldots$ *exist, i.e.* $E_i = E(\xi_i|B_i) < \infty$, $D_i = D(\xi_i|B_i) < \infty$, *and*

$$\sum_{i=1}^{\infty} p_i = \infty, \quad \sum_{i=1}^{\infty} \frac{p_i D_i^2}{\left(\sum_{j=1}^{i} p_j\right)^2} < \infty.$$

Then for every $\varepsilon > 0$ *and* $\sum_{i=1}^{n} p_i > 0$

$$p\left(\sup_{k \geqq n} \left|\frac{\nu_k}{\tau_k}\right| \geqq \varepsilon|C\right) \leqq \frac{1}{\varepsilon}\left(\frac{\sum_{i=1}^{n} p_i D_i^2}{E(\tau_n^2|C)} + \sum_{k=n+1}^{\infty} \frac{p_k D_k^2}{E(\tau_k^2|C)}\right). \tag{2.7}$$

PROOF. First of all we remark that the right hand side of equation (2.7) is defined because by Lemma 3.1 of [3] we have

$$E(\tau_n^2|C) \geqq \left(\sum_{i=1}^{n} p_i\right)^2 \quad (n = 1, 2, \ldots). \tag{2.8}$$

We need some calculations. Let $\xi_i^* = \varepsilon_i(\xi_i - E_i)$, then

$$E(\nu_k^2|C) = E\left(\left(\sum_{i=1}^{k} \xi_i^*\right)^2 \middle| C\right) = \sum_{i=1}^{k} \int_{B_i} (\xi_i - E_i)^2 dP(A|C) +$$

$$+2 \sum_{1 \leqq i < j \leqq k} E(\xi_i^*, \xi_j^*|C) = \sum_{i=1}^{k} \int_{B_i} (\xi_i - E_i)^2 p_i dP(A|B_i) = \sum_{i=1}^{k} p_i D_i^2$$

because $E(\xi_i^*|C) = 0$ $(i = 1, 2, \ldots)$ and ξ_i^*, ξ_j^* are independent with respect to C. Hence

$$E(\nu_k^2|C) = \sum_{i=1}^{k} p_i D_i^2. \tag{2.9}$$

Let now

$$\eta_n = \sum_{k=n}^{\infty} \nu_k^2 \left(\frac{1}{E(\tau_k^2|C)} - \frac{1}{E(\tau_{k+1}^2|C)}\right).$$

Then

$$E(\eta_n|C) = \frac{E(\nu_n^2|C)}{E(\tau_n^2|C)} - \frac{E(\nu_n^2|C)}{E(\tau_{n+1}^2|C)} + \frac{E(\nu_{n+1}^2|C)}{E(\tau_{n+1}^2|C)} - \frac{E(\nu_{n+1}^2|C)}{E(\tau_{n+2}^2|C)} + (-) \ldots =$$

$$= \frac{\sum_{i=1}^{n} p_i D_i^2}{E(\tau_n^2|C)} - \frac{\sum_{i=1}^{n} p_i D_i^2}{E(\tau_{n+1}^2|C)} + \frac{\sum_{i=1}^{n+1} p_i D_i^2}{E(\tau_{n+1}^2|C)} - \frac{\sum_{i=1}^{n+1} p_i D_i^2}{E(\tau_{n+2}^2|C)} + (-) \ldots =$$

$$= \frac{\sum_{i=1}^{n} p_i D_i^2}{E(\tau_n^2|C)} + \sum_{k=n+1}^{\infty} \frac{p_k D_k^2}{E(\tau_k^2|C)}$$

and these series are convergent by the Kronecker lemma and our conditions. Thus

$$E(\eta_n|C) = \frac{\sum_{i=1}^{n} p_i D_i^2}{E(\tau_n^2|C)} + \sum_{k=n+1}^{\infty} \frac{p_k D_k^2}{E(\tau_k^2|C)}. \tag{2.10}$$

Let us denote the events A_m $(m = n, n+1, \ldots)$ by

$$A_m = \left\{\left|\frac{\nu_n}{\tau_n}\right| < \varepsilon, \left|\frac{\nu_{n+1}}{\tau_{n+1}}\right| < \varepsilon, \ldots, \left|\frac{\nu_{m-1}}{\tau_{m-1}}\right| < \varepsilon, \left|\frac{\nu_m}{\tau_m}\right| \geqq \varepsilon\right\}.$$

Since $E(\nu_m^2|A_m C) \geqq E(\tau_m^2 \varepsilon^2|A_m C) \geqq E(\tau_m^2 \varepsilon^2|C) = \varepsilon^2 E(\tau_m^2|C)$ we can state

$$E(\nu_m^2|A_m C) \geqq \varepsilon^2 E(\tau_m^2|C) \quad (m = n, n+1, \ldots). \tag{2.11}$$

On the other hand, if $k \geqq m$ then

$$E(v_k^2|A_m C) = E\big(v_m^2 + 2v_m(v_k - v_m) + (v_k - v_m)^2|A_m C\big) \geqq$$

$$\geqq E(v_m^2|A_m C) + E\big(2v_m(v_k - v_m)|A_m C\big),$$

where

$$v_m = \sum_{i=1}^{m} \varepsilon_i(\xi_i - E_i) \quad \text{and} \quad v_k - v_m = \sum_{i=m+1}^{k} \varepsilon_i(\xi_i - E_i)$$

are independent with respect to C.

Let

$$\alpha_m(a) = \begin{cases} 1, & a \in A_m \\ 0, & a \notin A_m. \end{cases}$$

Then $v_k - v_m$ and α_m are also independent with respect to C, so if $P(A_m|C) \neq 0$ we get

$$E\big(v_m(v_k - v_m)|A_m C\big) = \int_H v_m(v_k - v_m)\, dP(A|A_m C) =$$

$$= \int_H \frac{v_m(v_k - v_m)}{P(A_m|C)}\, dP(AA_m|C) = \int_H \frac{v_m(v_k - v_m)}{P(A_m|C)}\, \alpha_m\, dP(A|C) =$$

$$= \frac{1}{P(A_m|C)}\, E(v_m \alpha_m|C)\, E(v_k - v_m|C) = 0.$$

Denoting $K_{m,k}$ $\big(m \leqq k$ by $K_{m,k} = E(2v_m(v_k - v_m)|A_m C)\big)$ we have

$$(2.12) \quad K_{m,k} = \begin{cases} 2 \sum\limits_{1 \leqq i < j \leqq k} E(\xi_i^* \xi_j^*|A_m C) & \text{if} \quad m < k \quad \text{and} \quad P(A_m|C) = 0 \\ 0 & \text{if} \quad m = k \quad \text{or} \quad P(A_m|C) \neq 0. \end{cases}$$

Hence

$$E(v_k^2|A_m C) \geqq E(v_m^2|A_m C) + K_{m,k} \quad (k \geqq m)$$

so with the help of (2.11)

$$E(\eta_n|A_m C) = \sum_{k=n}^{\infty} E(v_k^2|A_m C)\left(\frac{1}{E(\tau_k^2|C)} - \frac{1}{E(\tau_{k+1}^2|C)}\right) \geqq$$

$$\geqq \sum_{k=m}^{\infty} E(v_m^2|A_m C)\left(\frac{1}{E(\tau_k^2|C)} - \frac{1}{E(\tau_{k+1}^2|C)}\right) + \sum_{k=m}^{\infty} K_{m,k}\left(\frac{1}{E(\tau_k^2|C)} - \frac{1}{E(\tau_{k+1}^2|C)}\right) \geqq$$

$$\geqq \sum_{k=m}^{\infty} \varepsilon^2 E(\tau_m^2|C)\left(\frac{1}{E(\tau_k^2|C)} - \frac{1}{E(\tau_{k+1}^2|C)}\right) + \sum_{k=m}^{\infty} K_{m,k}\left(\frac{1}{E(\tau_k^2|C)} - \frac{1}{E(\tau_{k+1}^2|C)}\right) =$$

$$= \varepsilon^2 + \sum_{k=m}^{\infty} K_{m,k}\left(\frac{1}{E(\tau_k^2|C)} - \frac{1}{E(\tau_{k+1}^2|C)}\right).$$

Let

$$L_m = \sum_{k=m}^{\infty} K_{m,k}\left(\frac{1}{E(\tau_k^2|C)} - \frac{1}{E(\tau_{k+1}^2|C)}\right),$$

then

(2.13) $$E(\eta_n|A_mC) \geqq \varepsilon^2 + L_m \quad (m = n, n+1, \dots).$$

Now it is evident by (2.12) that

(2.14) $$L_m = 0 \quad \text{if} \quad P(A_m|C) \neq 0.$$

Let $A_0 = C \setminus \bigcup_{i=n}^{\infty} A_i$. Then $E_0, E_n, E_{n+1}, \dots$ is a complete group of events, so there follows

$$E(\eta_n|C) = \sum_{m=n}^{\infty} E(\eta_n|A_mC)P(A_m|C) + E(\eta_n|A_0C)P(A_0|C) \geqq$$

$$\geqq \sum_{m=n}^{\infty} E(\eta_n|A_mC)P(A_m|C) \geqq \sum_{m=n}^{\infty} (\varepsilon^2 + L_m)P(A_m|C) =$$

$$= \varepsilon^2 \sum_{m=n}^{\infty} P(A_m|C) + \sum_{m=n}^{\infty} L_m P(A_m|C) =$$

$$= \varepsilon^2 P\left(\bigcup_{m=n}^{\infty} A_m|C\right) = \varepsilon^2 P\left(\sup_{k\geqq n}\left|\frac{\nu_k}{\tau_k}\right| \geqq \varepsilon|C\right).$$

Hence

(2.15) $$E(\eta_n|C) \geqq \varepsilon^2 P\left(\sup_{k\geqq n}\left|\frac{\nu_k}{\tau_k}\right| \geqq \varepsilon|C\right).$$

Finally, by (2.10) and (2.15) our statement follows.

As a straightforward consequence of the previous lemma, we can now state the following result.

Theorem 2.4. *Let $\xi_1, \xi_2, \dots$ be independent with respect to C and assume that $E_k = E(\xi_k|B_k)$, $D_k = D(\xi_k|B_k)$ exist. We also assume that*

$$\text{a)} \quad \sum_{k=1}^{\infty} p_k = \infty$$

and

$$\text{b)} \quad \sum_{k=1}^{\infty} \frac{p_k D_k^2}{\left(\sum_{j=1}^{k} p_j\right)^2} < \infty.$$

Then

$$\frac{\nu_n}{\tau_n} = \frac{\sum_{i=1}^{n} \varepsilon_i(\xi_i - E_i)}{\sum_{i=1}^{n} \varepsilon_i} \xrightarrow[C]{} 0 \quad (n \to \infty).$$

PROOF. It is enough to prove that

$$\frac{\sum_{i=1}^{n} p_i D_i^2}{E(\tau_n^2|C)} + \sum_{k=n+1}^{\infty} \frac{p_k D_k^2}{E(\tau_k^2|C)} \to 0 \quad (n \to \infty).$$

But from (2.8)

$$\frac{\sum_{i=1}^{n} p_i D_i^2}{E(\tau_n^2|C)} + \sum_{k=n+1}^{\infty} \frac{p_k D_k^2}{E(\tau_k^2|C)} \leqq \frac{\sum_{i=1}^{n} p_i D_i^2}{\left(\sum_{i=1}^{n} p_i\right)^2} + \sum_{k=n+1}^{\infty} \frac{p_k D_k^2}{\left(\sum_{j=1}^{k} p_j\right)^2}$$

and the right side of this inequality tends to zero by the Kronecker lemma.

Remark. Beside the hypotheses of Theorem 2.4, assume that

$$\sum_{n=1}^{\infty} \frac{\sum_{i=1}^{n} p_i |E_i - E|}{\sum_{i=1}^{n} p_i} < +\infty.$$

Then the reader can readily verify that

$$\frac{\sum_{i=1}^{n} \varepsilon_i \xi_i}{\tau_n} \xrightarrow[C]{} E \quad (n \to \infty).$$

In the following three theorems we shall prove that the strong law of large numbers holds not only for mutually independent random variables. If $\xi_1, \xi_2, \dots$ are not assumed to be mutually independent then, of course, we need some stronger hypotheses.

Theorem 2.5. *Let $\xi_1, \xi_2, \dots$ be identically distributed random variables and let any four of them be independent with respect to C. Suppose that*

a) $E(\xi_k^4|B_k) = E^{(4)} < \infty,\ E(\xi_k|B_k) = 0 \quad (k = 1, 2, \dots),$

b) $$\sum_{i=1}^{\infty} \frac{1}{\left(\sum_{j=1}^{i} p_j\right)^2} < \infty.$$

Then $\dfrac{\sum_{i=1}^{n} \varepsilon_i \xi_i}{\tau_n} \xrightarrow[C]{} 0 \quad (n \to \infty).$

PROOF. Apply Lemma 3.1 and Lemma 3.2 of [3]. Then

$$P\left(\left|\frac{\sum_{i=1}^{n}\varepsilon_i\xi_i}{\tau_n}\right|\geqq\varepsilon|C\right)\leqq\frac{1}{\varepsilon^4}\frac{E(\nu_n^4|C)}{E(\tau_n^4|C)}\leqq\frac{1}{\varepsilon^4}\frac{E(\nu_n^4|C)}{\left(\sum_{j=1}^{n}p_j\right)^4}=$$

$$=\frac{\sum_{i=1}^{n}p_iE^{(4)}+6D^4\sum_{1\leqq i<j\leqq n}p_ip_j}{\varepsilon^4\left(\sum_{i=1}^{n}p^i\right)^4}\leqq\frac{K\left(\sum_{i=1}^{n}p_i+2\sum_{1\leqq i<j\leqq n}p_ip_j\right)}{\left(\sum_{i=1}^{n}p_i\right)^4}=$$

$$=\frac{K\left(\sum_{i=1}^{n}p_i\right)+\left(\sum_{i=1}^{n}p_i\right)^2}{\left(\sum_{i=1}^{n}p_i\right)^4}\leqq\frac{2K'\left(\sum_{i=1}^{n}p_i\right)^2}{\left(\sum_{i=1}^{n}p_i\right)^4}=\frac{2K'}{\left(\sum_{i=1}^{n}p_i\right)^2}$$

where the constants K, K' do not depend on n. Hence

$$\sum_{n=1}^{\infty}P\left(\left|\frac{\nu_n}{\tau_n}\right|\geqq\varepsilon|C\right)\leqq 2K'\sum_{i=1}^{\infty}\frac{1}{\left(\sum_{j=1}^{i}p_j\right)^2}<\infty$$

which implies the theorem.

Theorem 2.6. *Let* $\xi_1,\xi_2,\dots$ *be identically distributed random variables with finite conditional expectations and variances*

$$E=E(\xi_i|B_i),\quad D=D(\xi_i|B_i)\quad(i=1,2,\dots).$$

Furthermore, let us suppose that a) $\xi_1,\xi_2,\dots$ *are pairwise uncorrelated random variables, i.e.,* $E\big(\varepsilon_i(\xi_i-E(\xi_i|B_i))\varepsilon_j(\xi_j-E(\xi_j|B_j))|C\big)=0$ $(i\neq j)$, *b)* $0<d\leqq p_i$ $(i=1,2,\dots)$. *Then*

$$\frac{\sum_{i=1}^{n}\varepsilon_i\xi_i}{\tau_n}\xrightarrow[C]{}E\quad(n\to\infty).$$

PROOF. Let $\xi_i^*=\varepsilon_i(\xi_i-E)$ $(i=1,2,\dots)$. For every positive integer N there exists n such that $n^2\leqq N<(n+1)^2$. Thus

$$\frac{\left|\sum_{i=1}^{N}\xi_i^*\right|}{\tau_N}\leqq\frac{|\xi_1^*+\xi_2^*+\dots+\xi_{n^2}^*|}{\tau_{n^2}}+\frac{|\xi_{n^2+1}^*+\dots+\xi_N^*|}{\tau_{n^2}}.$$

This inequality shows that it is enough to prove that

$$\frac{\nu_{n^2}}{\tau_{n^2}} = \frac{\sum_{i=1}^{n^2} \xi_i^*}{\tau_{n^2}} \xrightarrow[C]{} 0 \quad (n \to \infty) \tag{2.16}$$

and

$$\frac{\nu_N - \nu_{n^2}}{\tau_{n^2}} \xrightarrow[C]{} 0 \quad (n \to \infty). \tag{2.17}$$

By Lemma 3.1 and Lemma 3.2 of [3]

$$P\left(\left|\frac{\nu_{n^2}}{\tau_{n^2}}\right| \geqq \varepsilon | C\right) \leqq \frac{1}{\varepsilon^2} \frac{E(\nu_{n^2}^2|C)}{E(\tau_{n^2}|C)} \leqq \frac{D^2 \sum_{i=1}^{n^2} p_i}{\left(\sum_{i=1}^{n^2} p_i\right)^2} = \frac{D^2}{\sum_{i=1}^{n^2} p_i}.$$

Hence

$$\sum_{n=1}^{\infty} P\left(\left|\frac{\nu_{n^2}}{\tau_{n^2}}\right| \geqq \varepsilon | C\right) \leqq D^2 \sum_{n=1}^{\infty} \frac{1}{\sum_{i=1}^{n^2} p_i} \leqq \frac{D^2}{d} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

which implies that (2.16) holds.

On the other hand

$$P\left(\max_{n^2 \leqq N < (n+1)^2} \frac{|\nu_N - \nu_{n^2}|}{\tau_{n^2}} < \varepsilon | C\right) \leqq P\left(\frac{|\nu_N - \nu_{n^2}|}{\tau_{n^2}} < \varepsilon | C\right)$$

thus it is sufficient to show for (2.17) that

$$P\left(\max_{n^2 \leqq N < (n+1)^2} \frac{|\nu_N - \nu_{n^2}|}{\tau_{n^2}} < \varepsilon | C\right) \to 1 \quad (n \to \infty).$$

But this last condition is fulfilled if

$$\sum_{n=1}^{\infty} P\left(\max_{n^2 \leqq N < (n+1)^2} \frac{|\nu_N - \nu_{n^2}|}{\tau_{n^2}} \geqq \varepsilon | C\right) < \infty. \tag{2.18}$$

By Lemma 3.2 of [3]

$$P\left(\max_{n^2 \leqq N < (n+1)^2} \frac{\left|\sum_{i=n^2+1}^{N} \xi_i^*\right|}{\tau_{n^2}} \geqq \varepsilon | C\right) \leqq \sum_{N=n^2}^{(n+1)^2-1} P\left(\frac{\left|\sum_{i=n^2+1}^{N} \xi_i^*\right|}{\tau_{n^2}} \geqq \varepsilon | C\right) \leqq$$

$$\leqq \sum_{N=n^2}^{(n+1)^2-1} \frac{1}{\varepsilon^2} D^2 \frac{\sum_{i=n^2+1}^{N} p_i}{\left(\sum_{i=1}^{n^2} p_i\right)^2} \leqq \frac{D^2}{\varepsilon^2} \sum_{N=n^2}^{(n+1)^2-1} \frac{N-n^2}{d^2 n^4} \leqq \frac{D^2}{\varepsilon^2 d^2} \sum_{N=n^2+1}^{(n+1)^2-1} \frac{2n}{n^4} = \frac{4D^2}{\varepsilon^2 d^2} \frac{1}{n^2}.$$

Hence

$$\sum_{n=1}^{\infty} P\left(\max_{n^2 \leqq N < (n+1)^2} \frac{|\nu_N - \nu_{n^2}|}{\tau_{n^2}} \geqq \varepsilon | C\right) \leqq \frac{4D^2}{\varepsilon^2 d^2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

i.e., (2.18) is fulfilled which was to be proved.

Theorem 2.7. *Suppose that the following conditions are fulfilled for the random variables* $\xi_1, \xi_2, \ldots$: *a) There exist* $E_i = E(\xi_i | B_i)$, $D_i = D(\xi_i | B_i)$ $(i = 1, 2, \ldots)$, *b)* $\xi_1, \xi_2,$ *are pairwise uncorrelated,*

$$c)\ 0 < d \leqq p_i \quad (i = 1, 2, \ldots), \qquad d)\ \sum_{i=1}^{\infty} p_i D_i^2 \Big/ \left(\sum_{j=1}^{i} p_j\right)^{\frac{3}{2}} < \infty.$$

Then

$$\frac{\nu_n}{\tau_n} = \frac{\sum_{c=1}^{n} \varepsilon_i (\xi_i - E_i)}{\tau_n} \xrightarrow[C]{} 0 \quad (n \to \infty).$$

PROOF. It is sufficient to show that (2.16) and (2.17) hold. Using Lemma 3.2 of [3]

$$\sum_{n=1}^{\infty} P\left(\frac{|\nu_{n^2}|}{\tau_{n^2}} \geqq \varepsilon | C\right) \leqq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{\sum_{i=1}^{n^2} p_i D_i^2}{\left(\sum_{i=1}^{n^2} p_i\right)^2} = \frac{p_1 D_1^2}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{\left(\sum_{i=1}^{n^2} p_i\right)^2} +$$

$$+ \frac{1}{\varepsilon^2} \sum_{m=2}^{\infty} p_m D_m^2 \sum_{n=m}^{\infty} \left(\sum_{i=1}^{n^2} p_i\right)^{-2}$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^2} < 1$ and $0 < d \leqq p_i$ $(i = 1, 2, \ldots)$ we have

$$\sum_{n=m}^{\infty} \left(\sum_{i=1}^{n^2} p_i\right)^{-2} \leqq \frac{1}{d^2} \left(\frac{1}{m^4} + \frac{1}{(m+1)^4} + \ldots\right) \leqq \frac{1}{d^2} \frac{1}{m^2} \leqq \frac{1}{d^2} \frac{1}{m^{3/2}} \leqq \frac{1}{d^2} \frac{1}{(p_1 + \ldots + p_m)^{3/2}}.$$

Hence

$$\sum_{n=1}^{\infty} P\left(\frac{|\nu_{n^2}|}{\tau_{n^2}} \geqq \varepsilon | C\right) \leqq K + \frac{1}{\varepsilon^2 d^2} \sum_{m=2}^{\infty} \frac{p_m D_m^2}{\left(\sum_{i=1}^{m} p_i\right)^{\frac{3}{2}}} < \infty,$$

i.e. (2.16) holds.

For (2.17) it is enough to prove that

$$\sum_{n=1}^{\infty} P\left(\max_{n^2 < N < (n+1)^2} \frac{|\nu_N - \nu_{n^2}|}{\tau_{n^2}} \geqq \varepsilon | C\right) < \infty.$$

But

$$\sum_{n=1}^{\infty} P\left(\max_{n^2<N<(n+1)^2} \frac{|\nu_N-\nu_{n^2}|}{\tau_{n^2}} \geqq \varepsilon|C\right) \leqq \sum_{n=1}^{\infty} \sum_{N=n^2+1}^{(n+1)^2-1} P\left(\frac{\left|\sum_{i=n^2+1}^{N} \xi_i^*\right|}{\tau_{n^2}} \geqq \varepsilon|C \leqq\right)$$

$$\leqq \sum_{n=1}^{\infty} \sum_{N=n^2+1}^{(n+1)^2-1} \frac{1}{\varepsilon^2} \frac{\sum_{i=n^2+1}^{N} p_i D_i^2}{\left(\sum_{i=1}^{n^2} p_i\right)^2} \leqq \sum_{m=1}^{\infty} \frac{2\sqrt{m}\, p_m D_m^2}{(p_1+\ldots+p_m)^2} \leqq$$

$$\leqq 2 \sum_{m=1}^{\infty} \frac{\sqrt{p_1+\ldots+p_m}\; p_m D_m^2}{\sqrt{d}\,(p_1+\ldots+p_m)^2} = \frac{2}{\sqrt{d}} \sum_{m=1}^{\infty} \frac{p_m D_m^2}{\left(\sum_{i=1}^{m} p_i\right)^{\frac{3}{2}}} < \infty,$$

which implies (2.17). Thus the theorem is proved.

References

[1] V. I. GNEDENKO, The theory of probability, *New York*, 1962.

[2] J. HAJEK, A. RÉNYI, Generalization of an inequality of Kolmogorov, *Acta Math. Acad. Sci. Hung.* **6** (1955), 281—283.

[3] I. G. KALMÁR, On the conditional laws of large numbers I, *Publ. Math. (Debrecen)* **28** (1981), 299—315.

[4] A. RÉNYI, A valószínűségszámítás új axiomatikus felépítése, *MTA III. Mat. és Fiz. Oszt. Közl.* **4** (1954), 369—428.

[5] A. RÉNYI, On a new axiomatic theory of probability, *Acta Math. Acad. Sci. Hung.* **6** (1955), 285—335.

[6] P. RÉVÉSZ, The laws of large numbers, *Budapest*, 1967.

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