On the existence of non-archimedean valued invariant means

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ABSTRACT

Let $A_c(G)$ denote the space of all continuous almost periodic functions on a topological group G taking values in a complete rank-one non-archimedean non-trivially valued field Q. In an earlier paper it was shown that under a condition of Index Boundedness (IB-) on G an invariant mean existed. It is now shown that this condition is also necessary for the existence of such a mean. This condition is further linked up with the condition of p-freeness or p-finiteness used by MONNA—SPRINGER, SCHIKHOF and VAN ROOIJ to prove the existence of Haar measure or invariant mean. Finally it is shown that the Index Boundedness property is passed on to closed subgroups of finite index and quotients by closed normal subgroups. This phenomenon makes the Approximation Theorem of the earlier paper available for any IB-group.

1. Introduction and Terminology

This paper has as its chief aim the linking up of the results relating to the existence of invariant mean or Haar measure in MONNA—SPRINGER [4], SCHIKHOF [9, 10] and VAN ROOIJ [7] with those in [5]. Theorem 3.3, [5] gives the index boundedness condition on a topological group G to be a sufficient condition for the existence of an invariant mean on the space $A_c(G)$ of continuous almost periodic functions on G taking values in a complete (rank 1) non-archimedean non-trivially valued field Q. A topological group satisfying this condition is called an IB-group. This condition means that the valuations of the indices of the closed normal subgroups of finite index (viewed as integers in Q) have a positive lower bound. In the light of Proposition 2.6, [5] if G is a zero-dimensional compact group, any continuous function $f\colon G \to Q$ is in $A_c(G)$; it is thus suggested by Theorem 1.4, [10], Corollary 7.2, [7] and Theorem 2.2.3, [9] that the condition that G is an IB-group (with respect to Q) should be related to what Monna—Springer [4], van Rooij [7] and Schikhof [9, 10] call the condition of p-freeness or p-finiteness of G (with respect to the family of open compact or open subgroups if G is a locally-compact zero-dimensional group).

The notion of p-finiteness from above or p-freeness introduced by earlier authors ([4], [7], [9], [10]) was with reference to the family of open or open compact subgroups in the context of a locally compact zero-dimensional group G. But in the context of the function space $A_c(G)$ related to an arbitrary topological group G, the family Γ_G of closed subgroups of finite index seems to be more relevant. It is also relevant to point out here that it is immaterial whether we have in our considerations the class Γ_G or the smaller class Γ'_G of closed *normal* subgroups of finite index in G (which will be used when necessary in the sequel without explicit mention) in view of the fact that any closed subgroup H of finite index contains a closed *normal* subgroup of finite index (cf. 3, Section 2, [2]). In this paper it is shown (Theorem 2.1) that the condition that G is an IB-group is equivalent to its being *p-finite from above* (in the sense of Monna—Springer—Schikhof) with respect to the family Γ_G where p is the characteristic of the residue class field of Q, i.e. *when there exists a subgroup $H \in \Gamma_G$ such that for every $S \subset H$, $S \in \Gamma_G$, the index $[H:S]$ (the number of right or left cosets of S in H) is not divisible by p. If for every pair of subgroups $H, S \in \Gamma_G$, $S \subset H$, $[H:S]$ is not divisible by p, then we call such a group p-free (with respect to the family Γ_G).* It is also shown (Theorem 3.2) that the condition that G is an IB-group is necessary for the existence of an invariant mean on $A_c(G)$. Thus the earlier result (Theorem 3.3, [5]) about the existence of an invariant mean on $A_c(G)$ is placed now in the setting of the other results of this kind due to van Rooij and Schikhof in terms of Monna—Springer conditions. Moreover the condition of p-freeness in preference to p-finiteness by VAN ROOIJ [7] and SCHIKHOF [10] is shown (Theorem 3.3) to stem from the additional requirement of unit norm for the mean. Variants for the formulation of these two conditions are also indicated (Theorem 2.2).

Finally it is shown that the property of being an IB-group is passed on to open and closed subgroups of finite index and quotients by closed normal subgroups, thereby making an earlier approximation theorem (Theorem 7.4, [5]) for functions in $A_c(G)$ available to all IB-groups. A redundant condition relating to an earlier result (Theorem 4.4(vi), [6]) is also removed (Theorem 3.8) thereby showing that the invariant mean when it exists is nothing but the integral in the sense of Thomas (see BRUHAT [1]). The notation and terminology is in accordance with the brief details outlined in the preceding paragraphs.

2. Index-Boundedness (IB-condition)

It may be observed that Theorem 2.1.7, [9] remains true even when Γ_G is any collection of subgroups of G with the property $H_1, H_2 \in \Gamma_G \Rightarrow H_1 \cap H_2 \in \Gamma_G$ and is of finite index both in H_1 and H_2, and in particular, the family Γ_G of this paper (see 4.21(d), [3]). More explicitly, for such a Γ_G the following conditions are equivalent:

(i) G is p-finite from above with respect to Γ_G (p- a prime number);

(ii) For every $H \in \Gamma_G$, there exists a positive integer $n = n(H)$ such that for all $S \subset H$, $S \in \Gamma_G$, $[H:S] < \infty$, $[H:S]$ is not divisible by p^n;

(iii) For every sequence $H_1 \supset H_2 \supset \ldots$, $H_i \in \Gamma_G$, there exists a positive integer m such that for all $k > m$, $[H_k : H_{k+1}]$ is not divisible by p.

With this observation, the following theorem which gives the link mentioned at the outset can now be proved.

Theorem 2.1. *A topological group is an IB-group with respect to the complete (rank 1) non-archimedean non-trivially valued field Q if and only if it is p-finite from above with respect to the family Γ_G of closed subgroups of finite index in G, where p is the characteristic of the residue class field of Q.*

Proof. If $p=0$, every integer n in Q has valuation 1, so that any topological group G is an IB-group. By convention 0 is not a divisor of any integer and so any topological group is p-finite from above with respect to Γ_G.

Let now $p>0$, G an IB-group and $H_1 \supset H_2 \supset \dots$ be a descending chain of closed normal subgroups H_i $(i=1, 2, \dots)$ of finite index in G. Suppose G is not p-finite from above with respect to Γ_G. Then since $[G:H_i]=[G:H_1][H_1:H_i]$, by (iii) of the observation made just above, the powers of p occuring in $[G:H_i]$ tend to ∞ as $i\to\infty$, i.e. $|[G:H_i]|\to 0$ as $i\to\infty$ ($|\ |$ denotes the valuation on Q), which contradicts our assumption that G is an IB-group. Conversely, if G is p-finite from above, by (ii) of the same observation taking the subgroup H to be G itself we see that there exists a positive integer n such that if S is any closed subgroup of finite index, then $[G:S]$ is not divisible by p^n. Hence $[G:S]=p^k$ for some $k<n$. Since $|p|<1$, $|[G:S]|>|p|^n$ (i.e.) G is an IB-group.

The following result gives alternative formulations of the condition that G is an IB-group.

Theorem 2.2. *The following conditions on a topological group are equivalent:*

(i) *G is an IB-group with respect to a complete (rank 1) non-archimedean non-trivially valued field Q;*

(ii) *There exists an integer n such that if H is any closed subgroup of finite index in G, then $[G:H]$ is not divisible by p^n;*

(iii) *For some positive integer n there exists no continuous homomorphisms of G onto any finite group whose order is divisible by p^n.*

Proof. After Theorem 2.1 and the equivalence of conditions on a topological group observed in the beginning of this section, it suffices to prove that (ii) and (iii) are equivalent.

(ii)$\Rightarrow$(iii). Let F be any finite group and $\Theta\colon G\to F$ be any continuous homomorphism onto F. Then $\operatorname{Ker}\Theta(=H)$ is a closed normal subgroup of finite index in G. Hence by (ii) there exists an integer n such that $p^n\nmid[G:H]$ and clearly $[G:H]$ is the order of F.

(iii)$\Rightarrow$(ii). If H is any closed normal subgroup of finite index in G, then G/H is a finite group and hence $[G:H]=$order of G/H and hence there exists an integer n such that $p^n\nmid[G:H]$. If S is any closed subgroup of finite index in G, since S contains a closed normal subgroup H of finite index in G, $[G:H]=[G:S][S:H]$ and so $p^n\nmid[G:S]$.

Theorem 2.3. *The following conditions on a topological group G are equivalent:*

(i) *G is p-free with respect to the family Γ_G of all closed subgroups of finite index in G;*

(ii) *For every H in Γ_G, $p\nmid[G:H]$;*

(iii) *For every H in Γ_G, $|[G:H]|=1$.*

3. Almost periodic functions and invariant mean

Recall that $f\colon G\to Q$ is said to be *almost periodic* if there exists for each $\varepsilon>0$, a finite collection of subsets $A_1, A_2, \ldots, A_n$ with $G=\bigcup_{i=1}^{n} A_i$ and

$$|f(cxd)-f(cyd)|<\varepsilon \text{ for all } c, d\in G,\ x, y\in A_i,\ i=1, 2, \ldots, n \quad (1)$$

and that if f is almost periodic on G, $f|H$ is so on H, where H is any subgroup of G.

Theorem 3.1. *Let H be a subgroup of finite index in a group G and f an almost periodic function on H. Then f extended to G by defining it to be zero outside H, is almost periodic on G.*

PROOF. Suppose f is almost periodic on H, then corresponding to $\varepsilon>0$ there exists a normal subgroup N *(with respect to H)* of finite index in H whose cosets serve as the A_i's for the fulfilment of (1) above relative to H. Clearly N is a subgroup of finite index in G and so contains a closed *normal (in G)* subgroup S of finite index in G. Now the cosets of S in G serve as the A_i for the condition (1) above to be satisfied relative to G and this completes the proof of the theorem.

Remarks. If in Theorem 3.1, H is a closed (and hence also open) subgroup of finite index in a topological group G and f is continuous then the function extended as above belongs to $A_c(G)$. In particular, the characteristic function χ_H of such an H is a continuous almost periodic function on G.

A *left invariant mean* on $A_c(G)$ is, as is familiar, a continuous linear functional M on $A_c(G)$ such that

(i) $M(1)=1$

(ii) $M({}_af)=M(f)$ for all $a\in G$ and $f\in A_c(G)$ where ${}_af(x)=f(ax)$ for $x\in G$.

A right invariant mean is defined analogously. When a mean is both left and right invariant, it is called an *invariant mean*.

Remark. Some authors (see [7], [10]) define a left invariant mean M to be one which satisfies, in addition to the above conditions (i) and (ii), the condition

(iii) $|M(f)|\leqq\|f\|$.

One can easily see that (iii) is equivalent to the condition

(iii)′ $\|M\|=1$.

In what follows unless otherwise stated we write G to be p-finite from above or p-free without explicitly stating that these are with respect to Γ_G, the family of all closed subgroups of finite index. Theorem 3.3, [5] can now be put in its best form as follows:

Theorem 3.2. *A left invariant mean exists on $A_c(G)$ if and only if G is an IB-group or equivalently G is p-finite from above.*

PROOF. After Theorem 3.3, [5] and the equivalence (Theorem 2.1) of p-finiteness from above of G with its being an IB-group, it remains to prove that the condition that G is an IB-group is necessary for a left invariant mean to exist on $A_c(G)$. To do this, let H be a closed subgroup of finite index n in G. Then χ_H is a continuous almost periodic function on G (see remarks following Theorem 3.1). If M is a left invariant mean on $A_c(G)$, then $M(\chi_H)=M(\chi_{xH})$ for any $x\in G$, so that $1=M(\sum \chi_{xH})=n\cdot M(\chi_H)$, where the sum is taken over all the distinct left cosets of H in G, i.e. $M(\chi_H)=1/n$. Since the field is non-trivially valued, the continuity of M implies its boundedness. Thus $1/|n|=|M(\chi_H)|\leqq\|M\|\,\|\chi_H\|=\|M\|$, i.e. G is an *IB*-group.

If the mean M were to have unit norm (see the remark in the beginning of this section) the corresponding result is

Theorem 3.3. *A left invariant mean M with $\|M\|=1$ exists on $A_c(G)$ if and only if G is p-free.*

Proof. When G is a p-free, it follows that $|n|=1$ (Theorem 2.3), where n is the index of any closed subgroup of finite index in G. G is then obviously an *IB*-group with $IB(G)=1$ (see remark following Definition 3.2, [5]) and now Theorem 3.2 implies that there exists an invariant mean M on $A_c(G)$. It is evident from Proposition 4.3, [5] that $\|M\|=1$.

Conversely, if there exists a left invariant mean M on $A_c(G)$, as in the proof of Theorem 3.2, if H is a closed subgroup of finite index n in G, $|n|\geqq 1/\|M\|=1$, since $\|M\|=1$; but the valuation is non-archimedean implies that $|n|\leqq 1$ i.e. $|n|=1$. In view of Theorem 2.3 the proof is now complete.

Remark. One can easily see from the Uniqueness Theorem 4.1, [5] that as in the classical case, when a left (or right) invariant mean exists, then it is the invariant mean on $A_c(G)$.

Corollary 3.4. (Corollary 7.2, [7]; Theorem 1.4, [10].) *If G is a compact group then* (i) *there exists an invariant mean M on $BC(G)$, the space of (bounded) continuous functions on G, if and only if G is p-finite from above,* (ii) *there exists an invariant mean on $BC(G)$ with $\|M\|=1$ (G is Q-amenable in the sense of* [10]*) if and only if G is p-free.*

We need the following lemma to show that the fulfilment of the *IB*-condition or the p-finiteness condition is carried over to open and closed subgroups of finite index and quotients by closed normal subgroups.

Lemma 3.5. *If the invariant mean exists on $A_c(G)$, then it exists on $A_c(H)$ for any open and closed subgroup H of finite index in G as well as on $A_c(G/N)$ where N is a closed normal subgroup of G.*

Proof. Let $f\in A_c(H)$. Then f belongs to $A_c(G)$ if it is defined by $\bar{f}=f$ on H and $\bar{f}=0$ outside H, (see remarks following Theorem 3.1). Define $M_H(f)=(1/M(\chi_H))\cdot M(\bar{f})$, where M is the mean on $A_c(G)$. M_H is an invariant mean on $A_c(H)$.

Suppose N is a closed normal subgroup of G and $\varphi\colon G\to G/N$ is the natural homomorphism. If $g\in A_c(G/N)$, then the function defined by $f(x)=g(\varphi(x))$ for $x\in G$ is in $A_c(G)$. If f_0 is the function corresponding to the function 1 in $A_c(G/N)$, pefine $M_N(g)=(1/M(f_0))\cdot M(f)$; then M_N is the invariant mean on $A_c(G/N)$.

After Theorem 3.2, Lemma 3.5 yields the following:

Theorem 3.6. *If G is an IB-group or, equivalently, is p-finite from above, then so is every open and closed subgroup H of finite index in G and every quotient group G/N by a closed normal subgroup N.*

Remark 1. It is not difficult to see, (i) $IB(G)$ (the index bound of the group G) which is defined to be the $\inf |n|$, n varying over indeces of closed normal subgroups of finite index, is also equal to $\inf |n|$, as n varies over the indices (=the number of left or right cosets) of closed subgroups of finite indices in G; and (ii) for any closed subgroup H of finite index in G, $IB(G) \leqq IB(H)$. In view of this and Theorem 3.6, the concept of a heredirary IB-group introduced in section 7, [5], is unnecessary and so the Peter—Weyl—von Neumann Approximation Theorem 7.4, [5] is available for any IB-group.

Remark 2. If G is finitely generated IB-group, G contains only a finite number of subgroups of a given index n (see section 2, [2]). Hence such a G has only a countable number of subgroups of finite index and so there are atmost a countable number of cosets of subgroups of finite index. In view of Remark 1, for any IB-group which is finitely generated, $A_c(G)$ is a Banach space of countable type (see p. 50, [8]).

To complete the considerations in Theorem 3.6 we give the following result relating to products of groups.

Theorem 3.7. *$G_1 \times G_2$ is p-finite from above or, equivalently, is an IB-group if and only if G_1 and G_2 are so.*

PROOF. Let $f(x, y)$ be an almost periodic function on $G_1 \times G_2$. Suppose G_1 and G_2 are p-finite from above. Then there exist invariant means M_1, M_2 on $A_c(G_1)$ and $A_c(G_2)$ respectively. We define $M(f) = M_2(M_1(f^y))$ where $f^y(x) = f(x, y)$ (see section 6, [5]). Then M is an invariant mean on $A_c(G_1 \times G_2)$. Hence, by Theorem 3.2, $G_1 \times G_2$ is p-finite from above. The proof is now complete in conjunction with Theorem 6.1, [5].

Theorem 4.4(vi), [6] has the following form after removal of the undesirable redundant hypothesis.

Theorem 3.8. *Let G be an IB-group. Then the invariant mean on $A_c(G)$ is precisely the integral in the sense of Thomas (see* BRUHAT [1]*) defined on $\hat{G}$, the non-archimedean Bohr compactification of G.*

PROOF. After Theorem 4.4(iv), [6] it suffices to show that $\hat{G}$ is an IB-group if G is one. In fact, it can be shown that G is an IB-group if and only if $\hat{G}$ is one. By Theorem 3.2, if G is an IB-group, there exists a mean M on $A_c(G)$. Define $\overline{M}(\hat{f}) = M(f)$ where $\hat{f}$ is the extension of f on G to $\hat{G}$. Then $\overline{M}$ is the mean on $A_c(\hat{G})$. Therefore $\hat{G}$ is an IB-group. Conversely if $\hat{G}$ is an IB-group, by defining again $M(f) = \overline{M}(\hat{f})$ where $f(x) = \hat{f}(\varrho(x))$, $x \in G$ where ϱ is the representation associated with $\hat{G}$ we see that M is the invariant mean on $A_c(G)$, thereby proving that G is an IB-group.

Remark. It may be noted that the condition used by Thomas, to prove the coincidence of the families of integrable and continuous functions on a compact zero dimensional group (Corollary following Proposition 3, [1]) is the same as the IB-condition on G.

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