Multiplier algebras and minimal embeddability

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1. Introduction

Let K be an algebra over a commutative ring k with 1. We will say that K is *minimally embeddable* (and designate it m.e.) if whenever there is a ring A such that $K \lhd A$ then there exists some $B \lhd A$ such that either $A = K \oplus B$ or else there is some $z \in A$ such that A is generated by $K \oplus B$ and z, with $A/(K \oplus B) \cong k$. If K has this property but only when A is a k-algebra then we will say K is *algebra minimally embeddable* (designated a.m.e.). Clearly any algebra with unit is an m.e. algebra and in a recent paper [1] simple m.e. algebras without unit were constructed over any $k = Z_p$. In the present paper we will show that there exist simple a.m.e. algebras without unit over an arbitrary field k. We will give a method for constructing such algebras and show that there are simple m.e. algebras without unit, different from those of [1], including certain algebras of J. C. ROBSON [2].

Our construction begins with any k-algebra R with unit containing a regular element a. Let I be the right ideal of R generated by a, and let T the idealizer of I in R. We show that there exists a certain "algebra of multipliers" which we will call "Mult-I", and prove that the algebra Mult-I is k-isomorphic with T. It then follows that for any $K \cong I$ imbedded as an ideal of a k-algebra A, the k-monomorphism $K \to T$ can be lifted via Mult-I to a k-homomorphism $A \to T$. As a consequence we obtain a criterion for minimal embeddability: A necessary and sufficient condition for K to be an a.m.e. algebra over a field k (or an m.e. algebra if $k = Z_p$) is that either $T = I$ or $T = I + k$.

2. Multiplier algebras

Let k be a commutative ring with 1 and suppose there exists a k-algebra R with unit containing a regular element a. Write $I = aR$ and let T be the idealizer of I in R, that is $T = \{x \in R \mid xa \in I\}$. Writing k-lin I for the algebra of all k-module endomorphisms of I we will call a pair $(\alpha', \alpha'') \in (k\text{-lin } I) \oplus (k\text{-lin } I)^{\mathrm{op}}$ a *multiplier* of I if it has the property:

$$x\alpha'(y) = \alpha''(x)y \tag{1}$$

for all $x, y \in I$. Clearly the set of all multipliers of I is a sub-k-algebra of $(k\text{-lin } I) \oplus$ $\oplus (k\text{-lin } I)^{\mathrm{op}}$ and we will call it *Mult-I*. For $t \in T$ let L_t and R_t denote left and right multiplication by t, then from associativity in R it is clear that the pair $(L_t, R_t) \in \text{Mult-}I$.

Lemma 1. *The map $\varphi\colon T\to Mult\text{-}I$ given by $t\to(L_t, R_t)$ is a k-algebra isomorphism.*

Proof. Clearly φ is a k-algebra homomorphism and if $t\in\operatorname{Ker}\varphi$ then $L_t(a)=$ $=ta=0$ implies $t=0$. It thus suffices to show that φ is surjective. Suppose $(\sigma', \sigma'')\in$ Mult-I and let $\sigma'(a)=af$, $\sigma''(a)=ag$. Then from (1) we have $a\sigma'(a)=$ $=\sigma''(a)a$ so $a^2f=aga$. By the regularity of a we get $af=ga$ and thus $g\in T$. Then for any arbitrary $au\in I$ we have $\sigma'(au)=aw$ for some $w\in R$, and so

$$a^2w = a\sigma'(au) = \sigma''(a)au = agau.$$

Thus $\sigma'(au)=aw=gau=L_g(au)$. Also $\sigma''(au)=av$ for some $v\in R$ so

$$ava = \sigma''(au)a = au\sigma'(a) = auaf.$$

Thus $va=uaf=uga$ and so $v=ug$. It follows that $\sigma''(au)=av=aug=R_g(au)$, that is $(\sigma', \sigma'')=(L_g, R_g)$.

3. Minimal embeddability

Theorem 2. *Let $K\lhd A$ where K and A are k-algebras. If $K\cong I$ then there exists a k-homomorphism $A\to T$ extending the monomorphism $K\to I\subseteq T$.*

Proof. Let $\Theta\colon K\to I$ be the given k-isomorphism. For an arbitrary $z\in A$ the mappings $\sigma'=\Theta L_z\Theta^{-1}$ and $\sigma''=\Theta R_z\Theta^{-1}$ are k-linear maps of I and by associativity in A the pair $(\sigma', \sigma'')\in$ Mult-I. Clearly the map $A\to$ Mult-I given by $z\to(\sigma', \sigma'')$ is a k-homomorphism so $\alpha\colon A\to T$ where $\alpha(z)=\varphi^{-1}(\sigma', \sigma'')$ is a k-homomorphism of A into T. Now by definition, if $\sigma''(a)=at$ then $\varphi^{-1}(\sigma', \sigma'')=t$. But $\sigma''(a)=\Theta R_z\Theta^{-1}(a)=\Theta(\Theta^{-1}(a)z)$. Thus if $z\in K$ we have $\sigma''(a)=a\Theta(z)$ so $\alpha(z)=\Theta(z)$ and hence α extends $K\to I\subseteq T$.

Corollary 3. *Let $K\cong I$ where K and I are Z_p-algebras for some prime p. Then if $K\lhd A$ for A an arbitrary ring, there exists a homomorphism $A\to T$ extending the monomorphism $K\to I\subseteq T$.*

Proof. Since K has characteristic p we may regard K as an ideal of the Z_p-algebra A/pA and map $A\to A/pA\to T$.

For the case k a field we can now establish a criterion for minimal embeddability.

Theorem 4. *Let k be a field and R any k-algebra with unit containing a regular element a. Write $I=aR$ and let T be the idealizer of I in R. Then all k-algebras $K\cong I$ are algebra minimally embeddable if and only if either $T=I$ or $T=I+k$. Moreover, if $k=Z_p$ for some prime p then for "algebra minimally embeddable" may be substituted "minimally embeddable".*

Proof. If $T=I$ then any $K\cong I$ contains a unit and so is minimally embeddable. Thus suppose $T=I+k$. Let $K\lhd A$ where A is a k-algebra and let $B=\operatorname{Ker}\alpha$ where $\alpha\colon A\to T$ is the map defined in the proof of Theorem 2. Clearly $K\cap B=0$ and $\operatorname{Im}\alpha\cong A/B$ contains the image $\Theta(K)=I$. Thus $A/K\oplus B\cong\operatorname{Im}\alpha/I$. Since $\operatorname{Im}\alpha$ is a k-algebra it follows that either $A=K\oplus B$ or $A/K\oplus B\cong T/I\cong k$.

Note that when $k=Z_p$ then, via the mapping of Corollary 3, $\mathrm{Im}\,\alpha$ is a k-algebra even when A is an arbitrary ring. Thus in this case K is minimally embeddable.

Conversely, if I is algebra minimally embeddable (or minimally embeddable if $k=Z_p$) then $I\lhd T$ and $I\cap B=0$ (where $B\lhd T$) implies $B=0$. Thus either $T=I$ so I contains a unit or else $T/I\cong k$ so $T=I+k$.

Remark that while the above construction is closed under isomorphisms, it is open as to whether or not this is true of the class of all a.m.e. algebras.

Corollary 5. [1, Cor. 2.] *If K is a simple m.e. algebra and A a ring such that $K\lhd A$ and $A/K\cong K$, then $A=K\oplus K$. The same result holds when K is a simple a.m.e. k-algebra and A is a k-algebra.*

Corollary 6. [1, Theorem 1 and 1′.] *Let M be a class of simple rings such that if $k\in M$ then either K has a unit or K is an m.e. Z_p-algebra for some prime p. Let W be the set of all such p, then the upper radical defined by M is hereditary if and only if $Z_p\in M$ for all $p\in W$.*

Note that any such radical for non-empty W is a non-special supernilpotent radical.

4. Examples and applications

Example 1. The simple algebra of [1, Section 2].

Note that [1, Lemmas 1—5] can be replaced by our Theorem 2 and Corollary 3. Also [1, Lemma 6] establishes the sufficient condition of Theorem 4, and applies (as was noted in [1, Remarks 1 and 2]) to R over any field Z_p. It is clear, in fact, that a similar proof can be carried out for R over an arbitrary field. Thus the additional result: *There exists a primitive a.m.e. algebra without unit over any field k.*

Example 2. The skew polynomial algebra of [2]. This is the ring $(x-1)S$ where S is the ring of non-commutative polynomials in x and x^{-1} over a field F subject to $fx=xf^\sigma$ for all $f\in F$, where F is the field of rational functions in indeterminates $\{t_n|n\in Z\}$ over a field k in which σ is the automorphism defined by $t_n^\sigma = t_{n+1}$.

It is easy to see that $x-1$ is regular in S so that Theorem 2 or Corollary 3 applies to the k-algebra $(x-1)S$. Now to show that Theorem 4 applies let $g\in T$ so that $g(x-1)=(x-1)f$ for some $f\in S$. It is easy to check that any $g\in S$ can be written

$$g=(x-1)\mu+g'$$

for some $\mu\in S$ and $g'\in F$. But then $g'(x-1)=(x-1)r$ for some $r\in S$ where (by a degree argument) $r\in F$. Then $xg'^\sigma-g'=xr-r$ so $r=g'^\sigma=g'$. But clearly the only elements of F fixed by σ are members of k. Thus $g\in(x-1)S+k$ so $T=(x-1)S+k$. We can therefore say: *Any k-algebra $K\cong(x-1)S$ is an a.m.e. algebra for an arbitrary field k, and is an m.e. algebra when $k=Z_p$ for some prime p.*

Note that since $(x-1)S$ contains no idempotents it is distinct from the rings of Example 1. Also note (see [2]) that $(x-1)S$ is a simple algebra (Noetherian when k is a finite field) and hence could be used in Corollaries 4 and 5.

Example 3. [5, page 55.] Let k be any field of characteristic zero and let A be the ring of polynomials in x over $F=k(y)$ where for all $f \in F$ we have $xf=fx+f'$ with f' the formal derivative. Thus $xf=gx$ for $f, g \in F$ implies $f=g$ and $g'=0$, that is $g \in k$. Since A admits a division algorithm and x is regular in A, an argument similar to that of Example 2 shows that xA is an a.m.e. algebra. Since A is simple, xA is another example of a simple a.m.e. algebra without unit. It is clearly distinct from the algebra of Example 1 and is also not isomorphic to the algebra of Example 2 since there we have the relation $uv=v-u$ where $u=(x-1)x^{-1}$ and $v=x-1$. However, a degree argument shows that such a relation is impossible for non-zero elements of xA.

Example 4. [5, page 113.] To show that there exist non-simple m.e. and a.m.e. algebras over any field k let F be the same field as in Example 2 but now let S be the skew polynomials in x over F with $fx=xf^{\sigma}$ for all $f \in F$. Since S admits a division algorithm and as before $f=f^{\sigma}$ if and only if $f \in k$, the argument of Example 2 again shows that $(x-1)S$ is an a.m.e. algebra over k (or an m.e. algebra if $k=Z_p$). An easy degree argument shows that $(x-1)S$ has proper ideals, such as the ideal generated by $(x-1)x$.

As an application of our minimally embeddable algebras we will answer certain open questions.

In [3] a ring R is called *almost nilpotent* if every proper homomorphic image is nilpotent, and we will call R *almost nil* if every such image is a nil ring. Note that by this definition all simple rings are almost nilpotent.

Application 1. It is well-known that the ring of all linear transformations of an infinite dimensional vector space can be embedded as the heart of a subdirectly irreducible almost nilpotent ring. T. L. Jenkins has asked (see [4]) if this is true for every simple ring without unit. Our minimally embeddable rings (Example 1 or 2) clearly give a negative answer, and also the same answer for a similar question: can every simple ring without unit be the maximal ideal of an almost nilpotent local ring?

Application 2. Let P be an arbitrary (Kuroš—Amitsur) radical class with semisimple class SP. In [3] were considered radicals with property:

(L) Every almost nilpotent ring is either in P or in SP, and we will also consider the property:

(L′) *Every almost nil ring is either in P or in SP.*

It is easy to see that if P contains all zero rings then P has property (L), and Richard Wiegandt has asked [4] if the converse is true at least for P a hereditary radical. That the answer is again negative follows from:

Proposition 7. *Let $L\{K\}$ be the lower radical defined by $\{K\}$ where K is a non-nil simple m.e. algebra over the field k. Then $L\{K\}$ is a hereditary radical containing no zero rings and having properties* (L) *and* (L′).

Proof. Since K is simple it is well-known that $L\{K\}$ is hereditary and contains no zero rings. Thus suppose R is an almost nil ring with $R \notin SL\{K\}$. Then R has a non-zero radical and hence an accessible subring $I \cong K$. But then I is simple so $I \lhd R$. Now if $I \neq R$ then we would either have a non-zero $B \lhd R$ with $I \oplus B = R$ so $R/B \cong I$ would be a non-nil proper image, or else for some $B \lhd R$ we would have $R/(I \oplus B) \cong k$ which would also be a non-nil proper image. We conclude that if $R \notin SL\{K\}$ then $R = I \in L\{K\}$.

References

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