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Connected Hausdorff topologies weaker than the Euclidean topology of the rationals

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Abstract. We prove that for every countable regular space without isolated points there exists a weaker topology such that the new space is either first countable, connected, locally connected Hausdorff (respectively anti-Urysohn, Urysohn, or almost regular) or first countable connected Hausdorff (respectively Urysohn, or almost regular) with a dispersion point.

In [6] M. Rajagopalan and A. Wilansky conjecture that every weaker Hausdorff topology for the space of rational numbers is homeomorphic to the Euclidean topology. A negative answer to this conjecture can be given using the countable connected Urysohn almost regular space with a dispersion point, constructed by V. Kannan and M. Rajagopalan in [3]. Since there is an one-to-one continuous map from the space of rationals with the usual Euclidean topology onto the above space, it follows that there exists a connected Urysohn almost regular topology with a dispersion point on the rationals which is strictly weaker than the usual topology and not homeomorphic to the usual topology. A similar result holds for all countable regular non-compact spaces. The Rajagopalan-Wilansky's conjecture we mentioned above can be also answered by a Theorem of S.F. Čvid [1], stating that if (X,τ) is a countable completely regular non-compact space, then there exists a topology $\tau_0 \subseteq \tau$ such that (X,τ_0) is a connected Urysohn space with a dispersion point.

We extend the previous results by proving that for every countable regular space M without isolated points there exists a weaker topology on M such that the new space is first countable connected and locally connected Hausdorff (resp. anti-Urysohn, Urysohn, or almost regular).

For the construction of each one of these spaces we need first to construct an appropriate pair of countable spaces (X,τ), (X,τ^*) having the

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following properties: (1) The space (X,τ) is first countable regular without isolated points. (2) $\tau^* \subseteq \tau$. (3) The space (X,τ) is first countable Hausdorff (resp. Urysohn, almost regular) containing two points not separated by a continuous map from (X,τ^*) into (X,τ). For each such a pair of spaces, applying the method in [2], or [4] and the SIERPIŃSKI's Theorem [8] stating that all countable first countable regular spaces without isolated points are homeomorphic, we finally construct the required countable connected, locally connected spaces.

We note that by the construction of these spaces it follows that we can include the corresponding cases with a dispersion point, that is, we also prove that for every countable regular space M without isolated points there exists a weaker topology on M such that the new space is first countable connected Hausdorff (respectively Urysohn or almost regular) with a dispersion point.

A space X is called: (1) Anti-Urysohn, if it is Hausdorff and no two points of X have disjoint closed neighbourhoods, (2) Urysohn, if every pair of distinct points of X have disjoint closed neighbourhoods, (3) Almost regular, if X contains a dense subset at every point of which the space is regular.

A point x of a connected space X is called a dispersion point if $X\setminus\{x\}$ is totally disconnected.

1. The auxiliary spaces

We first construct three pairs of countable spaces which will be used in the sequel for the construction of the required countable connected spaces.

(A) The spaces (X,ϱ), (X,ϱ^*). On the set

$$X = \{(k,y):\ k=1,2,\dots, y\in Q\}\cup\{p,q\},$$

where Q is the space of rational numbers, we define a topology ϱ as follows: For every point (k,y) a basis of open neighbourhoods is the collection of sets

$$V_\epsilon((k,y)) = \Big\{(k,t) : |t-y|<\epsilon\Big\}.$$

For the point p a basis of open neighbourhoods is the collection of sets

$$V_n(p) = \{p\}\cup\bigcup_{k=n}^{\infty} V_{\frac{1}{n}}(k,-1),\quad n=1,2,\dots\ .$$

For the point q a basis of open neighbourhoods is the collection of sets

$$V_n(q) = \{q\} \cup \bigcup_{k=n}^{\infty} V_{\frac{1}{n}}(k,1), \quad n = 1, 2, \dots \ .$$

On the set X we define a second topology ϱ^* as follows: Every point (k, y) has the same basis of open neighbourhoods as in the topology ϱ. For the point p a basis of open neighbourhoods is the collection of sets

$$W_n(p) = \{p\} \cup \Big\{(k, y) : k \geq n, \ y < 0\Big\}, \quad n = 1, 2, \dots \ .$$

For the point q a basis of open neighbourhoods is the collection of sets

$$W_n(q) = \{q\} \cup \Big\{(k, y) : k \geq n, \ y > 0\Big\}, \quad n = 1, 2, \dots \ .$$

It is obvious that the countable spaces $(X, \varrho), (X, \varrho^*)$ have the following properties:

$(A1)$ (X, ϱ) is first countable regular without isolated points.

$(A2)$ $\varrho^* \subseteq \varrho$.

$(A3)$ (X, ϱ^*) is first countable totally disconnected Hausdorff but not Urysohn since the points p, q cannot be separated by disjoint closed neighbourhoods.

$(A4)$ For every continuous map f from (X, ϱ^*) into (X, ϱ), $f(p) = f(q)$.

(B) The spaces (Y, σ), (Y, σ^*). Since the space (X, ϱ^*) is not Urysohn we construct a pair of spaces (Y, σ), (Y, σ^*) having all corresponding properties of (Y, ϱ), (Y, ϱ^*) and in addition the space (Y, σ^*) is Urysohn.

Let C_k, $k = 0, \pm 1, \pm 2, \dots$ be disjoint dense subsets of the space Q of rational numbers. On the set

$$Y = \Big\{(r, k) : r \in C_k, \ k = 0, \pm 1, \pm 2, \dots\Big\} \cup \{p, q\}$$

we define a topology σ as follows: For every point (r, k) a basis of open neighbourhoods is the collection of sets

$$V_\epsilon((r, k)) = \Big\{(t, k) : |t - r| < \epsilon\Big\}.$$

For the point p a basis of open neighbourhoods is the collection of sets

$$V_n(p) = \{p\} \cup \Big\{(r, k) : r > 2n, \ k \geq 2n\Big\}, \ n = 1, 2, \dots \ .$$

For the point q a basis of open neighbourhoods is the collection of sets

$$V_n(q) = \{q\} \cup \Big\{(r,k) : r > 2n,\ k \leq -2n\Big\},\ n = 1, 2, \dots\ .$$

On the set Y we define a second topology σ^* as follows: The points p, q and every point of the form $(r, 2k)$ have the same bases of open neighbourhoods as in the topology σ. For every point of the form $(r, 2k-1)$ a basis of open neighbourhoods is the collection of sets

$$W_\epsilon((r, 2k-1)) = \Big\{(t,m) : |t-r| < \epsilon,\ m = 2k-2,\ 2k-1,\ 2k\Big\}.$$

It is obvious that the countable spaces (Y,σ), (Y,σ^*) have the following properties:

$(B1)$ (Y,σ) is first countable regular without isolated points.

$(B2)$ $\sigma^* \subseteq \sigma$.

$(B3)$ (Y,σ^*) is first countable totally disconnected Urysohn, regular at the points p, q but not almost regular.

$(B4)$ For every continuous map f from (Y,σ^*) into (Y,σ), $f(p) = f(q)$.

(C) The spaces $(Z,\tau), (Z,\tau^*)$. Since the space (Y,σ^*) is not almost regular we construct a pair of spaces $(Z,\tau), (Z,\tau^*)$ having all corresponding properties of $(Y,\sigma), (Y,\sigma^*)$ and in addition the space (Z,τ^*) is almost regular. For the construction of these spaces we will use the previous spaces (Y,σ) and (Y,σ^*).

Consider the space Q of rational numbers where we fix a point d. We set $T = Q \setminus \{d\}$ and let $T_y,\ y \in Y \setminus \{p,q\}$ be disjoint copies of the space T. To every point $y \in Y \setminus \{p,q\}$ we attach a copy T_y setting the point y in the place of the point d.

In the sequel, if $t \in T$ then by t_y we denote the copy of the point t in T_y, and if $O_\epsilon(t)$ is a basis of open neighbourhoods of t in T then by $(O_\epsilon(t))_y$ we denote the copy of $O_\epsilon(t)$ in T_y. By $(D_\epsilon(d))_y$, where $D_\epsilon(d) = O_\epsilon(d) \setminus \{d\}$ we denote the copy in T_y of the basis of deleted open neighbourhoods of the point d in T.

On the set $Z = Y \cup \bigcup\limits_{t \in Y \setminus \{p,q\}} T_t$ we define a topology τ corresponding to the topology σ of Y, as follows: For every point $t_y \in T_y$ a basis of open neighbourhoods is the collection of set $((O_\epsilon(t))_y$. For every point $y \in Y \setminus \{p,q\}$ a basis of open neighbourhoods is the collection of sets

$$O_\epsilon(y) = V_\epsilon(y) \cup (D_\epsilon(d))_y \cup \bigcup_{\substack{t \in V_\epsilon(y)\\ t \neq y}} T_y.$$

For the point p a basis of open neighbourhoods is the collection of sets

$$O_n(p) = V_n(p) \cup \bigcup_{\substack{t \in V_n(p) \\ t \neq p}} T_t, \quad n = 1, 2, \dots \ .$$

For the point q a basis of open neighbourhoods is the collection of sets

$$O_n(q) = V_n(q) \cup \bigcup_{\substack{t \in V_n(q) \\ t \neq q}} T_t, \quad n = 1, 2, \dots \ .$$

On the set Z we define a second topology τ^* corresponding to the topology σ^* of Y, as follows: The points of T_y as well as the points p, q and every point of the form $(r, 2k)$ have the same bases of open neighbourhoods as in the topology τ. For every point $y = (r, 2k-1)$ a basis of open neighbourhoods is the collection of sets

$$U_\epsilon(y) = W_\epsilon(y) \cup (D_\epsilon(d))_y \bigcup_{\substack{t \in W_\epsilon(y) \\ t \neq y}} T_t.$$

It is obvious that the countable spaces (Z, τ), (Z, τ^*) have the following properties:

$(C1)$ (Z, τ) is first countable regular without isolated points.

$(C2)$ $\tau^* \subseteq \tau$.

$(C3)$ (Z, τ^*) is first countable totally disconnected Urysohn, regular at the points p, q and almost regular.

$(C4)$ For every continuous map f from (Z, τ^*) into (Z, τ), $f(p) = f(q)$.

2. Results

Each one of the spaces (X, ϱ^*), (Y, σ^*), (Z, τ^*) can be used for the construction of a countable first countable connected locally connected Hausdorff space and of a countable first countable connected Hausdorff space with a dispersion point. This can be done by attaching to the space (X, ϱ) (resp. (Y, σ), (Z, τ)) disjoint copies of itself, attaching the points p, q on distinct pairs of points of (X, ϱ) (resp, (Y, σ), (Z, τ)), and by repeating continously this process. More precisely by applying Theorem 1 and Corollary 1 of [2], for the case of locally connected spaces and by Theorem 2 and Corollary 3 of [2] for the case of connected spaces with a dispersion point. Thus for each pair of spaces

$$(X, \varrho),\ (X, \varrho^*),\ (\text{resp. } (Y, \sigma),\ (Y, \sigma^*) \ \text{ or } \ (Z, \tau),\ (Z, \tau^*))$$

we can construct a corresponding pair of countable spaces

$$(I(X), a_\infty), (I(X), a^*_\infty),$$
$$(\text{resp. } (I(Y), b_\infty), (I(Y), b^*_\infty) \text{ or } (I(Z), c_\infty), (I(Z), c^*_\infty))$$

and a corresponding pair of countable spaces

$$(I(X), \alpha_\infty), (I(X), \alpha^*_\infty),$$
$$(\text{resp. } (I(Y), \beta_\infty), (I(Y), \beta^*_\infty) \text{ or } (I(Z), \gamma_\infty), (I(Z), \gamma^*_\infty))$$

having respectively the following properties:

(1) The topologies $a_\infty,\ b_\infty,\ c_\infty$ are regular first countable without isolated points.

(2) $a^*_\infty \subseteq a_\infty,\ b^*_\infty \subseteq b_\infty,\ c^*_\infty \subseteq c_\infty$.

(3)(i) $(I(X), a^*_\infty)$ is first countable connected locally connected Hausdorff anti-Urysohn.

(ii) $(I(Y), b^*_\infty)$ is first countable connected locally connected Urysohn but not almost regular.

(iii) $(I(Z), c^*_\infty)$ is first countable connected locally connected Urysohn almost regular.

(4) The topologies $\alpha_\infty,\ \beta_\infty,\ \gamma_\infty$ are regular first countable without isolated points.

(5) $\alpha^*_\infty \subseteq \alpha_\infty,\quad \beta^*_\infty \subseteq \beta_\infty,\ \gamma^*_\infty \subseteq \gamma_\infty$.

(6)(i) $(I(X), \alpha^*_\infty)$ is first countable connected Hausdorff with a dispersion point which cannot be separated by disjoint closed neighbourhoods with any other point of the space.

(ii) $(I(X), \beta^*_\infty)$ is first countable connected Urysohn not almost regular with a dispersion point.

(iii) $(I(Z), \gamma^*_\infty)$ is first countable connected Urysohn almost regular with a dispersion point.

Remark 1. Comparing (3)(i) and (6)(i) we note that a connected Hausdorff space X with a dispersion point x cannot be anti-Urysohn. For if A, B are disjoint open-and-closed subsets of $X \setminus \{x\}$, then every pair of points $a, b,\ a \in A,\ b \in B$ can be separated by disjoint closed neighbourhoods.

Remark 2. The above regular first countable spaces $(I(X), a_\infty)$, $(I(Y), b_\infty)$, $(I(Z), c_\infty)$, can be also constructed by the method (c-process) of V. Kannan and M. Rajagopalan [4]. In this case since the final c-space is nowhere first countable [4, Theorem 1.2.5. (c)], we need to consider a regular first countable topology, weaker than the c-topology and simultaneously finer than each one of the required connected topology.

All the above lead to the following Theorem.

Theorem. *For every countable regular space M without isolated points, there exists a weaker topology on M which is homeomorphic to the topology ϱ^* (resp. σ^*, τ^*) or to the topology*

$$a^*_\infty \text{ (resp. } b^*_\infty,\ c^*_\infty \quad \text{and} \quad \alpha^*_\infty,\ \beta^*_\infty,\ \gamma^*_\infty).$$

Proof. Let (M, μ) be a countable regular space without isolated points. By Proposition 5.1 of [5] there exists a topology $\nu \subseteq \mu$ such that (M, ν) is a first countable regular space without isolated points. Hence by Sierpiński's Theorem [8] and property $(A1)$ of the space (X, ϱ), there exists a homeomorphism h of (M, ν) onto (X, ϱ). Therefore the collection $h^{-1}(\varrho^*) = \{h^{-1}(U) : U \in \varrho^*\}$ is the required topology.

Similarly are proved all the other cases.

Remark 3. The space (Y, σ^*) of Section 2, is a modification of the space E constructed by P. Roy in [7] which is countable first countable connected Urysohn (but not almost regular) with a dispersion point. It is obvious that, if e^* denotes the topology on E, then there exists a finer topology e on E such that (E, e) is a countable first countable regular space without isolated points. Hence if (M, ν) is the space in the proof of the Theorem above then again by the Sierpiński's Theorem it follows that there exists a homeomorphism f of (M, ν) onto (E, e). But then the collection $f^{-1}(e^*) = \{f^{-1}(U) : U \in e^*\}$ is a weaker topology for ν such that $(E, f^{-1}(e^*))$ is a countable first countable connected Urysohn (not almost regular) space with a dispersion point. Thus we get the same result as in the case of the topology β^*_∞ of the Theorem above, which is similar to the Čvid's result in [1].

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