On comparison of mean values

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1. Introduction

The concepts of *deviation* and *deviation mean* have been introduced by DARÓCZY [6], [7] in 1972. The class of deviation means contains as a special case the class of quasiarithmetic means (HARDY—LITTLEWOOD—PÓLYA [9]) and the class of quasiarithmetic means with weightfunction (BAJRAKTAREVIC [2], ACZÉL—DARÓCZY [1]). Thus the investigation of deviation means enriches the theory of these two classes of means too.

In this paper we solve the comparison problem of deviation means. Up till now this problem has been solved only in case of differentiable deviations. (DARÓCZY [6], [7].) The comparison problem of quasiarithmetic means with weightfunction has been raised by BAJRAKTAREVIC [2] in 1958 and solved under differentiability conditions by DARÓCZY—LOSONCZI [8] (see also BAJRAKTAREVIC [3] where a necessary condition was given for the comparison). As an application of our general results we solve this special comparison problem without differentiability.

2. Deviation means

Let $I\subseteq\mathbf{R}$ be an open interval. The function $E\colon I^2\to\mathbf{R}$ is said to be a *deviation* on I if it has the following properties:

(E1) The function $y\to E(x, y)$ is strictly decreasing and continuous on I for all $x\in I$.

(E2) $E(x, x)=0$ for all $x\in I$.

Let us denote by $\varepsilon(I)$ the set of all deviations on I. To define the deviation mean we need the following lemma (DARÓCZY [6]):

Lemma 2.1. *If* $E\in\varepsilon(I)$, $\underline{x}=(x_1, \dots, x_n)\in I^n$ $(n\in\mathbf{N})$, *then the equation*

$$\sum_{i=1}^{n} E(x_i, y) = 0 \tag{2.1}$$

has exactly one solution $y_0\in I$ *and this solution satisfies the inequality*

$$\min(\underline{x}) \doteq \min\{x_1, \dots, x_n\} \leqq y_0 \leqq \max\{x_1, \dots, x_n\} \doteq \max(\underline{x}). \tag{2.2}$$

Definition. Let $E\in\varepsilon(I)$, $\underline{x}\in I^n$ $(n\in\mathbf{N})$. The unique solution

$$y_0 \doteq \mathfrak{M}_{n,E}(\underline{x}) \tag{2.3}$$

of (2.1) is said to be the *deviation mean* of $\underline{x}\in I^n$. Inequality (2.2) shows that (2.3) is a mean value indeed.

Denote by $\Omega(I)$ the set of real valued functions which are *continuous and strictly monoton* on I. Let further $P(I)$ be the class of *positive* real valued functions on I. If $\varphi\in\Omega(I)$, $f\in P(I)$ then the function

$$E(x, y) \doteq E_{\varphi, f}(x, y) \doteq \varepsilon_\varphi f(x)[\varphi(x)-\varphi(y)] \quad (x, y\in I) \tag{2.4}$$

is a *deviation on I* where

$$\varepsilon_\varphi \doteq \begin{cases} 1 & \text{if } \varphi \text{ is increasing} \\ -1 & \text{if } \varphi \text{ is decreasing.} \end{cases} \tag{2.5}$$

For this deviation (2.4) we find that the unique solution y_0 of (2.4) has the form

$$y_0 \doteq \mathfrak{M}_{n, E_{\varphi, f}}(\underline{x}) \doteq M_{n,\varphi}(\underline{x})_f = \varphi^{-1}\left[\sum_{i=1}^{n} f(x_i)\varphi(x_i)\Big/\sum_{i=1}^{n} f(x_i)\right] \tag{2.6}$$

where φ^{-1} is the inverse function of φ. The quantity $M_{n,\varphi}(\underline{x})_f$ defined by (2.6) will be called *quasiarithmetic mean with weightfunction* (Bajraktarevic [2], Aczél—Daróczy [1], Daróczy [5]).

If $f(x)=p=$positive constant in (2.6) we obtain the well-known *quasiarithmetic means*

$$M_{n,\varphi}(\underline{x}) \doteq \varphi^{-1}\left[\frac{1}{n}\sum_{i=1}^{n}\varphi(x_i)\right]. \tag{2.7}$$

The theory of these mean values can be found in the book of Hardy—Littlewood—Pólya [9] (see also the books Beckenbach—Bellman [4], Mitrinovic [10]).

3. Weighted deviation means

To define the weighted deviation means we need the following result.

Lemma 3.1. *Let $E\in\varepsilon(I)$, $\lambda\in[0, 1]$ then for all $x, y\in I$ the equation*

$$\lambda E(x, t)+(1-\lambda)E(y, t) = 0 \tag{3.1}$$

has a unique solution $t_0\in I$ and this solution satisfies the inequality

$$\min\{x, y\} \leqq t_0 \leqq \max\{x, y\}. \tag{3.2}$$

Proof. Let

$$e(t) = \lambda E(x, t)+(1-\lambda)E(y, t) \quad (t\in I).$$

With the notations $m=\min\{x, y\}$, $M=\max\{x, y\}$ using (E1), (E2) we have

$$e(m) = \lambda E(x, m)+(1-\lambda)E(y, m) \geqq \lambda E(x, x)+(1-\lambda)E(y, y) = 0$$

and

$$e(M) = \lambda E(x, M)+(1-\lambda)E(y, M) \leqq \lambda E(x, x)+(1-\lambda)E(y, y) = 0.$$

Since $e: I\to\mathbf{R}$ is strictly monotonic and continuous there exists a unique $t_0\in[m, M]\subset I$ satisfying $e(t_0)=0$.

Definition. The unique solution

$$t_0 \doteq \mathfrak{M}_{2,E}(x, y; \lambda, 1-\lambda)$$

of (3.1) is called the *deviation mean* of x and y *weighted* by λ and $1-\lambda$.

Lemma 3.2. *Let* $E\in\varepsilon(I)$, $T\in I$. *The inequality*

$$\mathfrak{M}_{2,E}(x, y; \lambda, 1-\lambda) \leqq T \tag{3.3}$$

is valid if and only if the inequality

$$\lambda E(x, T)+(1-\lambda)E(y, T) \leqq 0 \tag{3.4}$$

is true.

Proof. (i) From (3.3) using the property (E1) we get that

$$E(x, T) \leqq E(x, e) \quad \text{and} \quad E(y, T) \leqq E(y, e)$$

where $e \doteq \mathfrak{M}_{2,E}(x, y; \lambda, 1-\lambda)$. Hence we obtain

$$\lambda E(x, T)+(1-\lambda)E(y, T) \leqq \lambda E(x, e)+(1-\lambda)E(y, e) = 0$$

i.e. (3.4) is satisfied.

(ii) Suppose that (3.4) is valid but (3.3) is not true, that is

$$e > T.$$

(E1) implies that

$$E(x, T) > E(x, e) \quad \text{and} \quad E(y, T) > E(y, e)$$

thus

$$\lambda E(x, T)+(1-\lambda)E(y, T) > 0$$

which contradicts to (3.4).

A result analogous to Lemma 3.2 has been proved by Daróczy [6] for symmetric deviation means.

Lemma 3.3. *Let* $E\in\varepsilon(I)$, $T\in I$ *then the inequality*

$$\mathfrak{M}_{n,E}(\underline{x}) \leqq T \quad (\underline{x}\in I^n)$$

is satisfied if and only if

$$\sum_{i=1}^{n} E(x_i, T) \leqq 0$$

is true.

Lemma 3.4. *Suppose that* $E\in\varepsilon(I)$ *and* $x<y$ *(or* $x>y$*)* $x, y\in I$. *Then the function*

$$e(\lambda) \doteq \mathfrak{M}_{2,E}(x, y; \lambda, 1-\lambda) \quad (\lambda\in[0, 1])$$

is strictly decreasing (increasing) and continuous.

Proof. We prove in the case when $x<y$ (the other case is similar). Suppose that e is not strictly decreasing. Then there exist values λ, μ such that $0\leqq\lambda<\mu\leqq 1$ but $e(\lambda)\leqq e(\mu)$. By Lemma 3.2

$$\lambda E(x, e(\mu))+(1-\lambda)E(y, e(\mu)) \leqq 0 = \mu E(x, e(\mu))+(1-\mu)E(y, e(\mu)),$$

therefore

$$E(x, e(\mu)) \geqq E(y, e(\mu)).$$

Since $E(x, e(\mu)) \leqq 0$, $E(y, e(\mu)) \geqq 0$ we get that $E(x, e(\mu))=E(y, e(\mu))=0$ i.e. $x=y$. This is a contradiction which proves that e is strictly decreasing.

To prove the continuity of e we show that the function e: $[0, 1] \to I$ assumes every intermediate value $s \in]e(1)=x, e(0)=y[$. From the equation $e(\lambda)=s$ i.e. from

$$\lambda E(x, s)+(1-\lambda) E(y, s) = 0$$

we obtain

$$\lambda = \frac{E(y, s)}{E(y, s)-E(x, s)}.$$

Since $E(y, s)>0$, $E(x, s)<0$ we have that $\lambda \in [0, 1]$ i.e. $e(\lambda)=s$.

4. Comparison of deviation means

The comparison problem of deviation means is the following. Find necessary and sufficient conditions for the inequalities

$$\mathfrak{M}_{n,F}(\underline{x}) \leqq \mathfrak{M}_{n,E}(\underline{x}) \quad (\underline{x} \in I^n,\ n \in \mathbf{N}) \tag{4.1}$$

to hold where $F, E \in \varepsilon(I)$. This problem has been solved by Daróczy [6] in case of differentiable deviations.

Lemma 4.1. *Suppose that* $F, E \in \varepsilon(I)$ *and* (4.1) *is satisfied for all* $\underline{x} \in I^n$, $n \in \mathbf{N}$. *Then the inequality*

$$\mathfrak{M}_{2,F}(x, y; \lambda, 1-\lambda) \leqq \mathfrak{M}_{2,E}(x, y; \lambda, 1-\lambda) \tag{4.2}$$

holds for all $x, y \in I$ *and* $\lambda \in [0, 1]$.

Proof. (a) Let first $\lambda = \frac{m}{n} \in [0, 1]$ be a rational number. Since for $\lambda=0$ or $\lambda=1$ (4.2) is valid we may assume that $0<\frac{m}{n}<1$. Let x, y be arbitrary elements of I and

$$\underline{x} = (\underbrace{x, \ldots, x}_{m}, \underbrace{y, \ldots, y}_{n-m}) \in I^n.$$

With the notations

$$f \doteq \mathfrak{M}_{n,F}(\underline{x}), \quad e \doteq \mathfrak{M}_{n,E}(\underline{x})$$

we get

$$mF(x, f)+(n-m) F(y, f) = 0$$

and

$$mE(x, e)+(n-m) E(y, e) = 0.$$

This shows that

$$f = \mathfrak{M}_{2,F}(x, y; \lambda, 1-\lambda), \quad e = \mathfrak{M}_{2,E}(x, y; \lambda, 1-\lambda)$$

and from (4.1) $f \leqq e$ i.e. (4.2) holds for every *rational* $\lambda \in [0, 1]$.

(b) Suppose now that $\lambda\in[0, 1]$ is *irrational.* Let $\lambda_n\in[0, 1]$ be a sequence of rational numbers which tends to λ and is decreasing or increasing. We have then

$$\mathfrak{M}_{2,F}(x, y; \lambda_n, 1-\lambda_n) \leqq \mathfrak{M}_{2,E}(x, y; \lambda_n, 1-\lambda_n)$$

thus by $n\to\infty$ using Lemma 3.4 we obtain the inequality (4.2).

Lemma 4.2. *Suppose that $F, E\in\varepsilon(I)$. If the inequality (4.2) holds for all $x, y\in I$, $\lambda\in[0, 1]$ then*

$$F(x, y)E(z, y) \leqq F(z, y)E(x, y) \tag{4.3}$$

is true for all $x, y, z\in I$ satisfying the condition $x\leqq y\leqq z$.

PROOF. Let x, y, z be elements of I such that $x\leqq y\leqq z$. We may suppose that $x<z$ since for $x=z$ (4.3) is obviously true. Then $E(z, y)\geqq 0$, $E(x, y)\leqq 0$ and $E(z, y)-E(x, y)>0$ hence

$$\lambda \doteq \frac{E(z, y)}{E(z, y)-E(x, y)}\in[0, 1] \tag{4.4}$$

and

$$\lambda E(x, y)+(1-\lambda)E(z, y) = 0.$$

By (4.2) we get

$$y = \mathfrak{M}_{2,E}(x, z; \lambda, 1-\lambda) \geqq \mathfrak{M}_{2,F}(x, z; \lambda, 1-\lambda)$$

thus by Lemma 3.2

$$\lambda F(x, y)+(1-\lambda)F(z, y) \leqq 0.$$

This shows that

$$\frac{F(z, y)}{F(z, y)-F(x, y)} \leqq \lambda = \frac{E(z, y)}{E(z, y)-E(x, y)}$$

which implies (4.3).

The main result of our paper is

Theorem 1. *Assume that $F, E\in\varepsilon(I)$. The inequality*

$$\mathfrak{M}_{n,F}(\underline{x}) \leqq \mathfrak{M}_{n,E}(\underline{x}) \tag{4.1}$$

holds for all $I^n\in I$, $n\in\mathbf{N}$ if and only if the inequality

$$F(x, y)E(z, y) \leqq F(z, y)E(x, y) \tag{4.3}$$

is satisfied for all $x\leqq y\leqq z$ $(x, y, z\in I)$.

PROOF. (i) If (4.1) holds then by lemmas 4.1 and 4.2, (4.3) is valid.

(ii) To prove the converse let $\underline{x}=(x_1, \dots, x_n)\in I^n$ and $e\doteq\mathfrak{M}_{n,E}(\underline{x})$. We may suppose without loss of generality that $x_1\leqq x_2\leqq\dots\leqq x_n$. Let $j\in\{1, 2, \dots, n-1\}$ the subscript for which

$$x_j \leqq e \leqq x_{j+1}.$$

If $l\in\{1, 2, \dots, j\}$ and $k\in\{j+1, \dots, n\}$ then $x_l\leqq e\leqq x_k$ and from (4.3)

$$F(x_l, e)E(x_k, e) \leqq F(x_k, e)E(x_l, e).$$

Adding these inequalities we get

$$\sum_{l=1}^{j}\sum_{k=j+1}^{n} F(x_l, e)E(x_k, e) \leqq \sum_{l=1}^{j}\sum_{k=j+1}^{n} F(x_k, e)E(x_l, e)$$

hence

$$A\left[\sum_{s=1}^{n} F(x_s, e)\right] \leqq 0$$

where

$$A \doteq \sum_{k=j+1}^{n} E(x_k, e) = -\sum_{l=1}^{j} E(x_l, e) \geqq 0.$$

If $A=0$ then $x_1=x_2=\ldots=x_n=e$ and (4.1) is obviously true. If $A>0$ then dividing by A we get

$$\sum_{s=1}^{n} F(x_s, e) \leqq 0$$

therefore by Lemma 3.3

$$\mathfrak{M}_{n,F}(\underset{\sim}{x}) \leqq e = \mathfrak{M}_{n,E}(\underset{\sim}{x})$$

i.e. (4.1) holds.

Remarks. (a) The importance of theorem 1 is that it reduces the system of inequalities (4.1) to the much simpler inequality (4.3). We emphasize that *no regularity assumptions* were made concerning the deviations F, E. For differentiable deviations DARÓCZY [6], [7] proved the following result. Let $\varepsilon^*(I)$ be the class of all deviations for which the partial derivative

$$E_2(t, s) \doteq \frac{\partial E(t, s)}{\partial s} < 0$$

exists and is negative for all $t, s \in I$. If $E \in \varepsilon^*(I)$ we set

$$E^*(t, s) \doteq -\frac{E(t, s)}{E_2(s, s)} \quad (t, s \in I).$$

Let $F, E \in \varepsilon^*(I)$. *The inequality* (4.1) *holds for all* $\underset{\sim}{x} \in I^n$, $n \in \mathbf{N}$ *if and only if*

$$F^*(t, s) \leqq E^*(t, s) \tag{4.4}$$

is valid for all $t, s \in I$.

Using theorem 1 we give a new proof for this result.

Theorem 2. *Suppose that* $F, E \in \varepsilon^*(I)$. *The inequality* (4.3) *is true for all* $x \leqq y \leqq z$ $(x, y, z \in I)$ *if and only if* (4.4) *holds for all* $t, s \in I$.

PROOF. (i) Assume that (4.3) is satisfied and $t, s \in I$. We may suppose that $s<t$ (if $s=t$ (4.4) is obvious, if $s>t$ the proof is similar).
By (4.3) for $y \in]s, t[$ we get

$$F(s, y)E(t, y) \leqq F(t, y)E(s, y)$$

or

$$-F(s,y)\frac{E(t,y)-E(t,t)}{y-t} \leqq -E(s,y)\frac{F(t,y)-F(t,t)}{y-t}$$

provided that $-(y-t)>0$. Taking the limit $y\to t\ (y<t)$ we have

$$F(s,t)[-E_2(t,t)] \leqq E(s,t)[-F_2(t,t)]$$

which means that (4.4) holds.

(ii) Suppose now that (4.4) holds and $x\leqq y\leqq z$ $(x,y,z\in I)$. (4.4) gives that

$$F^*(x,y)\leqq E^*(x,y) \quad\text{and}\quad F^*(z,y)\leqq E^*(z,y).$$

Taking into consideration the inequalities $F^*(x,y)\leqq 0$, $E^*(x,y)\leqq 0$ and $F^*(z,y)\geqq 0$, $E^*(z,y)\geqq 0$ we have

$$F^*(z,y)E^*(x,y) \geqq F^*(x,y)E^*(z,y).$$

Multiplying by $[-F_2(y,y)][-E_2(y,y)]>0$ we get (4.3).

(b) Theorem 1 solves the *equality problem* of deviation means. Suppose that $F, E\in\varepsilon(I)$. The equality

$$\mathfrak{M}_{n,F}(\underline{x}) = \mathfrak{M}_{n,E}(\underline{x})$$

is valid for all $\underline{x}\in I^n$, $n\in\mathbf{N}$ if and only if

$$\frac{F(x,y)}{F(z,y)} = \frac{E(x,y)}{E(z,y)}$$

holds for all $x<y<z$ $(x,y,z\in I)$.

5. Comparison of quasiarithmetic means with weightfunction

As an application of theorem 1 we solve here the comparison problem of quasiarithmetic means with weightfunction.

Theorem 3. *Suppose that* $\varphi,\psi\in\Omega(I)$ *and* $f,g\in P(I)$. *The inequality*

$$M_{n,\varphi}(\underline{x})_f \leqq M_{n,\psi}(\underline{x})_g \tag{5.1}$$

holds for all $\underline{x}\in I^n$, $n\in\mathbf{N}$ *if and only if the functions* $F\doteq\psi\circ\varphi^{-1}$ *and* $h\doteq g\circ\varphi^{-1}/f\circ\varphi^{-1}$ *satisfy the inequality*

$$\varepsilon_\psi\frac{F(t_2)-F(t_1)}{t_2-t_1}h(t_1) \leqq \varepsilon_\psi\frac{F(t_3)-F(t_2)}{t_3-t_2}h(t_3) \tag{5.2}$$

for all $t_1<t_2<t_3$ $\big(t_i\in\varphi(I),\ i=1,2,3\big)$.

Proof. Applying theorem 1 for the deviations

$$F(x,y) \doteq \varepsilon_\varphi f(x)[\varphi(x)-\varphi(y)]$$

and

$$E(x,y) \doteq \varepsilon_\psi g(x)[\psi(x)-\psi(y)]$$

we get that (5.1) holds if and only if the inequality

$$\varepsilon_\varphi f(x)[\varphi(x)-\varphi(y)]\varepsilon_\psi g(z)[\psi(z)-\psi(y)] \leqq \tag{5.3}$$
$$\leqq \varepsilon_\varphi f(z)[\varphi(z)-\varphi(y)]\varepsilon_\psi g(x)[\psi(x)-\psi(y)]$$

is true for all $x<y<z$ $(x, y, z\in I)$. Let

$$t_1 \doteq \varphi(x),\ t_2 \doteq \varphi(y),\ t_3 \doteq \varphi(z) \tag{5.4}$$

then from (5.3) we have

$$\varepsilon_\varphi \varepsilon_\psi h(t_3)(t_2-t_1)[F(t_3)-F(t_2)] \geqq \tag{5.5}$$
$$\geqq \varepsilon_\varphi \varepsilon_\psi h(t_1)(t_3-t_2)[F(t_2)-F(t_1)].$$

If $\varepsilon_\varphi=1$ then (5.4) shows that $t_1<t_2<t_3$ and (5.5) implies (5.2). If $\varepsilon_\varphi=-1$ then from (5.4) $t_1>t_2>t_3$ and interchanging t_1 and t_3 we get (5.2) from (5.5).

Conversely, it is easy to see that (5.2) implies (5.5) and (5.3) which was necessary and sufficient for (5.1).

With $f(t)=p$, $g(t)=q$ (p, q are positive constants) we get the well-known result for quasiarithmetic means (see Hardy—Littlewood—Pólya [9], theorems 92 and 122) since now (5.2) means the convexity (if $\varepsilon_\psi=1$) or concavity (if $\varepsilon_\psi=-1$) of F.

The equality problem of quasiarithmetic means with weightfunction has been solved by Aczél—Daróczy [1]. Our investigations give a possibility to find a new proof of their theorem.

Theorem 4. *Let* $\varphi, \psi\in\Omega(I)$ *and* $f, g\in P(I)$. *The equality*

$$M_{n,\varphi}(\underline{x})_f = M_{n,\psi}(\underline{x})_g \tag{5.6}$$

holds for all $\underline{x}\in I^n$, $n\in\mathbf{N}$ *if and only if there are constants* a, b, c, d, k *with* $k(ad-bc)\neq 0$ *such that*

$$\psi(x) = \frac{a\varphi(x)+b}{c\varphi(x)+d} \tag{5.7}$$

and

$$g(x) = kf(x)[c\varphi(x)+d] \tag{5.8}$$

for all $x\in I$.

Proof. By theorem 1 the necessary and sufficient condition for (5.6) is

$$f(x)[\varphi(x)-\varphi(y)]g(z)[\psi(z)-\psi(y)] = \tag{5.9}$$
$$= f(z)[\varphi(z)-\varphi(y)]g(x)[\psi(x)-\psi(y)].$$

We know from [1] that it is enough to prove the theorem for every closed subinterval $[A, B]$ of I. Let $A<t<B$. Substituting $x=A$, $y=t$ and $z=B$ in (5.9) we get

$$f(A)[\varphi(A)-\varphi(t)]g(B)[\psi(B)-\psi(t)] =$$
$$= f(B)[\varphi(B)-\varphi(t)]g(A)[\psi(A)-\psi(t)]$$

thus

$$\psi(t) = \frac{a\varphi(t)+b}{c\varphi(t)+d} \tag{5.7}$$

where $ad-bc\neq 0$ since $\psi\in\Omega(I)$. By the continuity of φ and ψ (5.7) is true for $t=A$ and $t=B$ too.

Let now $x, z\in I$, $x\neq z$ and C be a fixed value between x and z. Substituting $y=C$ in (5.9) we obtain that

$$T(x) \doteq \frac{g(x)[\psi(x)-\psi(c)]}{f(x)[\varphi(x)-\varphi(c)]} = T(z) = K$$

where K is a constant. Taking into consideration (5.7) we get (after some calculations)

$$g(x) = Kf(x)\frac{\varphi(x)-\varphi(c)}{\psi(x)-\psi(c)} = kf(x)[c\varphi(x)+d] \tag{5.8}$$

where $k\neq 0$ is a constant.

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