# On the conditions of equality in an integral inequality

By I. DANCS and B. UHRIN (Budapest)

## 1. Introduction

For $a, b \in \mathbf{R}_+$ (non-negative real numbers), $0 \neq \alpha \in \mathbf{R}$ and $0 \leqq \lambda \leqq 1$ denote

$$M_\alpha^\lambda(a, b) \triangleq \begin{cases} 0 & \text{if } a \cdot b = 0, \\ (\lambda a^\alpha + (1-\lambda) b^\alpha)^{1/\alpha} & \text{if } a \cdot b > 0. \end{cases} \tag{1.1}$$

The mean $M_\alpha^\lambda(a, b)$ can be extended also for $\alpha = 0$, $\alpha = -\infty$ and $\alpha = +\infty$ in the following way (see, e.g. [1]):

$$M_{-\infty}^\lambda(a, b) \triangleq \lim_{\alpha \to -\infty} M_\alpha^\lambda(a, b) = \min\{a, b\}, \tag{1.2}$$

$$M_0^\lambda(a, b) \triangleq \lim_{\alpha \to 0} M_\alpha^\lambda(a, b) = a^\lambda \cdot b^{1-\lambda}, \tag{1.3}$$

$$M_{+\infty}^\lambda(a, b) \triangleq \lim_{\alpha \to +\infty} M_\alpha^\lambda(a, b) = \begin{cases} 0 & \text{if } a \cdot b = 0 \\ \max\{a, b\} & \text{if } a \cdot b > 0. \end{cases} \tag{1.4}$$

For $\lambda$, $a$ and $b$ fixed, $M_\alpha^\lambda(a, b)$ is a non-decreasing function of $\alpha$ on $-\infty \leqq \alpha \leqq +\infty$ (see [1]).

In [2] we have proved the following

**Proposition.** *Let $f, g\colon \mathbf{R} \to \mathbf{R}_+$ be Lebesgue-measurable, bounded and not identically zero functions with*

$$\gamma \triangleq \sup_{x \in \mathbf{R}} f(x), \quad \delta \triangleq \sup_{y \in \mathbf{R}} g(y) \tag{1.5}$$

*and such that the function*

$$h_\alpha^\lambda(t) \triangleq \sup_{\lambda x + (1-\lambda) y = t} M_\alpha^\lambda(f(x), g(y)), \quad t \in \mathbf{R}, \tag{1.6}$$

*is Lebesgue-measurable, $-\infty \leqq \alpha \leqq +\infty$ and $0 \leqq \lambda \leqq 1$. Then we have*

$$\int_{-\infty}^{+\infty} h_\alpha^\lambda(t)\, dt \geqq M_\alpha^\lambda(\gamma, \delta) \cdot \left(\lambda \cdot \int_{-\infty}^{+\infty} \frac{f(x)}{\gamma}\, dx + (1-\lambda) \cdot \int_{-\infty}^{+\infty} \frac{g(y)}{\delta}\, dy\right). \tag{1.7}$$

This inequality is essentially due to HENSTOCK and MACBEATH [3], see also BRASCAMP and LIEB [4] or GUPTA [6].

*Remark.* If in the definition of $h_\alpha^\lambda$ we take ess-sup instead of sup, then $h_\alpha^\lambda$ turns to be Lebesgue-measurable (see [2] and further references there). If both $f$

and $g$ are assumed to be Borel-measurable then $h_\alpha^\lambda$ as defined in (1.6) is probably $L$-measurable (although this has not been proved yet). The Proposition remains true also for $h_\alpha^\lambda$ defined by ess-sup (see [2]). I this case the inequality is slightly sharper than (1.7). Our Theorem investigates the case of equality in this weaker case (1.7), to avoid superfluous measure theoretic considerations. We think the proof of Theorem is valid with minor changes also in the sharper ess-sup case.

The aim of this paper is to prove the necessary conditions of equality in (1.7), in the case of upper semi-continuous $f$ and $g$.

First introduce some notations. Let $f, g\colon \mathbf{R}\to\mathbf{R}_+$ and denote $A \triangleq \operatorname{supp} f \triangleq \{x\in\mathbf{R}\colon f(x)\neq 0\}$, $B\triangleq \operatorname{supp} g$.

Denote by conv $(A)$ the convex hull of $A$ and consider conv $(A)$ as a topological space with the topology generated by that of $\mathbf{R}$ (analogously for conv $(B)$). Denote by $\tilde{f}$ and $\tilde{g}$ the restriction of $f$ and $g$ to conv $(A)$ and conv $(B)$, respectively. If $\gamma \triangleq \sup_{x\in A} f(x) < +\infty$ and $\delta \triangleq \sup_{y\in B} g(y) < +\infty$, then for $\xi\in(0,1)$ denote

$$A(\xi) \triangleq \{x\colon f(x) \geqq \xi\cdot\gamma\},\ B(\xi) \triangleq \{y\colon g(y) \geqq \xi\cdot\delta\},\ A(1) \triangleq \bigcap_{0<\xi<1} A(\xi),$$

$B(1) \triangleq \bigcap_{0<\xi<1} B(\xi)$. Further, denote

$$\underline{a} \triangleq \inf\{x\colon x\in\operatorname{conv}(A)\},\ \bar{a} \triangleq \sup\{x\colon x\in\operatorname{conv}(A)\}, \tag{1.8}$$

$$\underline{b} \triangleq \inf\{y\colon y\in\operatorname{conv}(B)\},\ \bar{b} \triangleq \sup\{y\colon y\in\operatorname{conv}(B)\} \tag{1.9}$$

and if $A(1)\neq\emptyset$ and $B(1)\neq\emptyset$, respectively, then

$$a \triangleq \inf\{x\colon x\in A(1)\},\ \tilde{a} \triangleq \sup\{x\colon x\in A(1)\} \tag{1.10}$$

and

$$b \triangleq \inf\{y\colon y\in B(1)\},\ \tilde{b} \triangleq \sup\{y\colon y\in B(1)\}, \tag{1.11}$$

respectively.

If $A(1)=\emptyset$ then by definition we take

$$\textit{either}\quad a=\tilde{a}=\underline{a}\quad \textit{or}\quad a=\tilde{a}=\bar{a}, \tag{1.12}$$

depending of $f$ and conv $(A)$.

Similarly, if $B(1)=\emptyset$ then we take

$$\textit{either}\quad b=\tilde{b}=\underline{b}\quad \textit{or}\quad b=\tilde{b}=\bar{b}, \tag{1.13}$$

depending on $g$ and conv $(B)$.

(The exact meaning of the last two "vague" definitions will be clear for those $f$ and $g$ which we are dealing with.)

For the illustration of these notations see Figure 1.

The function $f\colon \mathbf{R}\to\mathbf{R}_+$ is said to be $\alpha$-concave, on $\mathbf{R}$, $-\infty\leqq\alpha\leqq+\infty$, if

$$f(\lambda x+(1-\lambda)y) \geqq M_\alpha^\lambda(f(x), f(y)) \quad \text{for all}\quad x, y\in R \quad\text{and}\quad 0\leqq\lambda\leqq 1. \tag{1.14}$$

Now, the main result of this paper is the following

**Theorem.** *Let* $f, g\colon \mathbf{R}\to\mathbf{R}_+$ *be such that* $A\triangleq\operatorname{supp} f$ *and* $B\triangleq\operatorname{supp} g$ *are bounded sets containing more than one point and* $\tilde{f}$ *and* $\tilde{g}$ *are upper semi-continuous (u.s.c.) on* conv $(A)$ *and* conv $(B)$, *respectively. Let* $-\infty<\alpha<+\infty$ *and* $0<\lambda<1$.

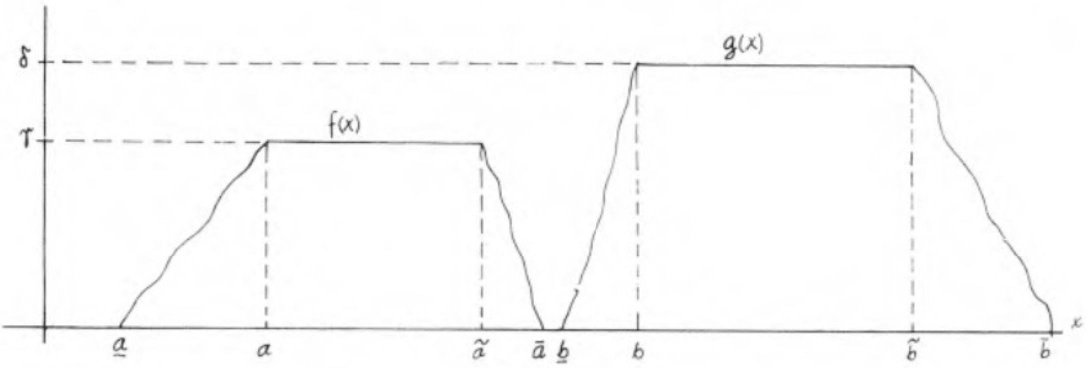


*Fig. 1.*

*If for f and g equality holds in* (1.7), *then* $A(\xi)$ *and* $B(\xi)$ *are convex for all* $\xi\in(0, 1)$ *and the following conditions hold:*

(1.15) $$\underline{a}=a \quad \text{if and only if} \quad \underline{b}=b,$$

(1.16) $$\tilde{a}=\bar{a} \quad \text{if and only if} \quad \tilde{b}=\bar{b};$$

(I): *if* $\underline{a}<a$, $\underline{b}<b$, *then*

(1.17) $$a-\underline{a}=\left(\frac{\gamma}{\delta}\right)^{\alpha}(b-\underline{b}),$$

(1.18) $$f(x)=\frac{\gamma}{\delta}\cdot g\left(\underline{b}+\left(\frac{\delta}{\gamma}\right)^{\alpha}\cdot(x-\underline{a})\right)\ \forall\, x\in(\underline{a}, a)$$

*and* $f(x)$ *is strictly increasing continuous* $\alpha$*-concave function on* $(\underline{a}, a)$;

(II): *if* $\tilde{a}<\bar{a}, \tilde{b}<\bar{b}$, *then*

(1.19) $$\bar{a}-\tilde{a}=\left(\frac{\gamma}{\delta}\right)^{\alpha}\cdot(\bar{b}-\tilde{b}),$$

(1.20) $$f(x)=\frac{\gamma}{\delta}\cdot g\left(\tilde{b}+\left(\frac{\delta}{\gamma}\right)^{\alpha}(x-\tilde{a})\right) \quad \forall\, x\in(\tilde{a}, \bar{a})$$

*and* $f(x)$ *is strictly decreasing continuous* $\alpha$*-concave function on* $(\tilde{a}, \bar{a})$.

*Remark.* It is easy to see that if the conditions (1.15) ÷ (1.20) are fulfilled then equality occurs in (1.7), hence (1.15) ÷ (1.20) *are not only necessary but also sufficient conditions of equality in (1.7)* (see also remarks at the end of the paper).

The statements (I) and (II) are independent, in the sense that $\underline{a}=a$, $\tilde{a}<\bar{a}$ or $\underline{a}<a$, $\tilde{a}=\bar{a}$ or $\underline{a}<a$, $\tilde{a}<\bar{a}$ can occur independently.

It is clear that $\underline{a}\leqq a\leqq\tilde{a}\leqq\bar{a}$ and $\underline{b}\leqq b\leqq\tilde{b}\leqq\bar{b}$.

The statement of the Theorem for $\gamma=\delta$ is illustrated on Figure 2.

For $f$ which is u.s.c. only on conv $(A)$ (and not $R$), the set $\{x\in A: f(x)=\gamma\}$ can be also empty because conv $(A)$ is not necessarily compact. Clearly, if the mentioned set is not empty, it is equal to $A(1)$ (similarly for $g$).

The statement of the theorem is especially clear if $\gamma=\delta$. In this case it simply says that: (1) moving from $\underline{a}[\underline{b}]$ to $+\infty$, the graph of $f[g]$ first strictly continuously

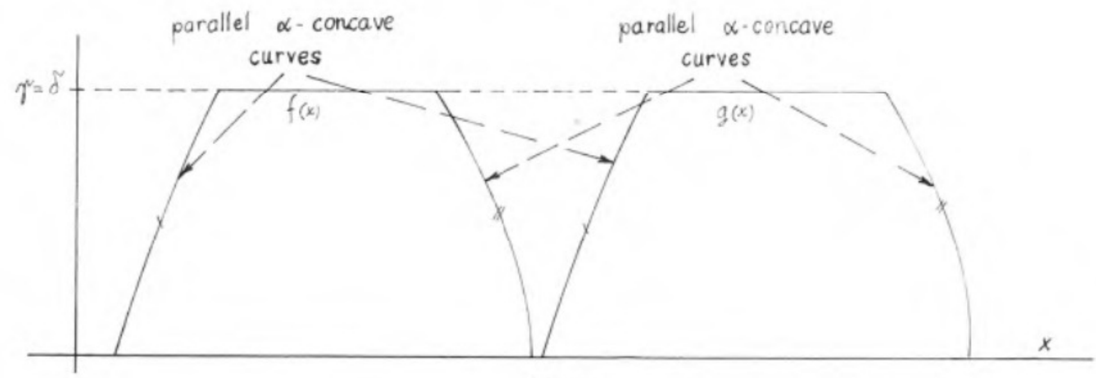


*Fig. 2.*

increases up to $\gamma[\delta=\gamma]$, after that it remains constant till $\tilde{a}[\tilde{b}]$ and strictly continuously decreases till $\bar{a}[\bar{b}]$; (2) the increasing [decreasing] "part" of the graph of $f$ is "parallel" to that of $g$; (3) both the increasing and decreasing parts of $f$ and $g$ are $\alpha$-concave.

Of course it may happen that $f[g]$ has no increasing or decreasing "part" (or neither of them).

It is easily seen, that for $f\colon \mathbf{R}\to\mathbf{R}_+$ the condition "$\{x\in R\colon f(x)\geqq\xi\}$ is convex for all $\xi\geqq 0$" is equivalent to the condition "$f$ is $-\infty$-concave" (see (1.14)). $-\infty$-concave functions are called quasi-concave. Hence, the statement of the theorem "$A(\xi)$ and $B(\xi)$ are convex for all $\xi\in(0, 1)$" can be replaced by: "*$f$ and $g$ are quasi-concave*".

In the following paragraph the proof of this theorem is given. It is by no means a simple one. It consists of two main steps. First, we prove the theorem for $\alpha=1$ and $\gamma=\delta$. In the second step the general case is transformed to the case $\alpha=1$ and $\gamma=\delta$; after "re-transforming" we get the result. To make the proof more "readable" we divided it into the number of lemmas. In the paragraph 3 some hopeful extensions of the theorem are discussed ($f$ and $g$ are u.s.c. but supp $f$ and supp $g$ are not bounded; $f$ and $g$ Lebesgue-measurable).

## 2. Proof of the theorem

In the Lemmas $1\div 4'$ it is assumed that the functions $f$ and $g$ satisfy the assumptions of the Theorem for $\alpha=1$, $0<\lambda<1$, and that $0<\gamma=\delta=1$.

**Lemma 1.** *The sets $A(\xi)$ and $B(\xi)$ are convex for all $\xi\in(0, 1)$.*

PROOF. The function $h_\alpha^\lambda(t)$ is (for fixed $t$) non-decreasing in $\alpha$, hence if equality occurs in (1.7) for $\alpha=1$ then it occurs also for $\alpha=-\infty$, i.e. denoting $h(t)\triangleq h_{-\infty}^\lambda(t)$, $t\in\mathbf{R}$, we can write

$$\int_{-\infty}^{+\infty} h(t)\,dt = \lambda\cdot \int_{-\infty}^{+\infty} f(x)\,dx+(1-\lambda)\int_{-\infty}^{+\infty} g(y)\,dy. \tag{2.1}$$

(It is clear that under the assumptions, $A$ and $B$ have positive Lebesgue-measure.) Denote $C(\xi)\triangleq\{t\in\mathbf{R}\colon h(t)\geqq\xi\}$, $\xi\in(0,1)$. Clearly, we have

$$C(\xi)\supseteq\lambda A(\xi)+(1-\lambda)B(\xi),\quad\forall\,\xi\in(0,1). \tag{2.2}$$

This implies, using 1-dimensional Brunn—Minkowski inequality (see [3]),

$$\mu\big(C(\xi)\big)\geqq\mu\big(\lambda A(\xi)+(1-\lambda)B(\xi)\big)\geqq\lambda\mu\big(A(\xi)\big)+(1-\lambda)\mu\big(B(\xi)\big),\quad\xi\in(0,1), \tag{2.3}$$

where $\mu$ means 1-dimensional Lebesgue-measure. Integrating (2.3) over $(0, 1)$ and using identities

$$\int_{-\infty}^{+\infty}f(x)\,dx=\int_0^1\mu\big(A(\xi)\big)\,d\xi,\quad\int_{-\infty}^{+\infty}g(y)\,dy=\int_0^1\mu\big(B(\xi)\big)\,d\xi, \tag{2.4}$$

$$\int_{-\infty}^{+\infty}h(t)\,dt=\int_0^1\mu\big(C(\xi)\big)\,d\xi,$$

the relation (2.1) implies

$$\int_0^1\varphi(\xi)\,d\xi=\int_0^1\psi(\xi)\,d\xi, \tag{2.5}$$

where

$$\varphi(\xi)\triangleq\mu\big(\lambda A(\xi)+(1-\lambda)B(\xi)\big), \tag{2.6}$$

$$\psi(\xi)\triangleq\lambda\mu\big(A(\xi)\big)+(1-\lambda)\mu\big(B(\xi)\big). \tag{2.7}$$

The functions $\varphi(\xi)$ and $\psi(\xi)$ are non-increasing on $(0, 1)$, hence they have at most countable many points of discontinuity (of the first kind). Denote by $F$ and $E$ the set of points of discontinuity of $\varphi$ and $\psi$, respectively. First, we prove that $F=E$.

Let $\eta\in(0,1)\setminus E$ and assume that $\eta\in F$. The point $\eta$ is a point of continuity of $\psi(\xi)$. This implies, using $\varphi(\xi)\geqq\psi(\xi)$, $\xi\in(0,1)$, that there is $\omega>0$ for which

$$\varphi(\xi)>\psi(\xi),\quad\forall\,\xi\in(\eta-\omega,\eta). \tag{2.8}$$

But this contradicts (2.5).

Similarly, if we assume that $\eta\in(0,1)\setminus F$ and $\eta\in E$, then again there is $\omega>0$ so that (2.8) hold, a contradiction. Thus, we have $(0,1)\setminus F=(0,1)\setminus E$, implying $E=F$. The relation $\varphi(\xi)\geqq\psi(\xi)$, $\xi\in(0,1)$ and (2.5) implies

$$\varphi(\xi)=\psi(\xi),\quad\forall\,\xi\in(0,1)\setminus E. \tag{2.9}$$

Really, if $\varphi(\eta)>\psi(\eta)$ for some $\eta\in(0,1)\setminus E$, then the continuity of $\varphi$ and $\psi$ implies that $\varphi(\xi)>\psi(\xi)$ in some neighbourhood of $\eta$ which contradicts (2.5). Using the conditions of equality in 1-dimensional Brunn—Minkowski inequality (see [3]), (2.9) implies

$$\begin{cases}\mu\big(\operatorname{conv}(A(\xi))\setminus A(\xi)\big)=0\\ \mu\big(\operatorname{conv}(B(\xi))\setminus B(\xi)\big)=0,\quad\forall\,\xi\in(0,1)\setminus E.\end{cases} \tag{2.10}$$

It is easy to see, that the upper semi-continuity of $f$ on $\operatorname{conv}(A)$ implies that $A(\xi)$ is closed in $\operatorname{conv}(A)$. It is also easily seen, that this implies: $\operatorname{conv}\big(A(\xi)\big)$ is closed in $\operatorname{conv}(A)$. Hence, the set $\operatorname{conv}\big(A(\xi)\big)\setminus A(\xi)$ is either empty or open in $\operatorname{conv}(A)$.

But it cannot be open, because of (2.10). The same may be said of $\operatorname{conv}(B(\xi))\setminus B(\xi)$. So, we have

$$(2.11)\qquad \begin{cases} \operatorname{conv}(A(\xi)) = A(\xi), \\ \operatorname{conv}(B(\xi)) = B(\xi), \quad \forall\, \xi\in(0,1)\setminus E. \end{cases}$$

Let $\eta\in E$. Clearly,

$$A(\eta) = \bigcap \{A(\xi)\colon\ \xi < \eta,\ \xi\in(0,1)\setminus E\}$$

and

$$B(\eta) = \bigcap \{B(\xi)\colon\ \xi < \eta,\ \xi\in(0,1)\setminus E\},$$

showing that $A(\eta)$, $B(\eta)$, are convex (as intersections of convex sets).

The lemma implies that $A(1)$ and $B(1)$ are either empty or convex. The lemma also implies, that $A$ and $B$ are convex, because

$$A = \bigcup_{0<\xi<1} A(\xi),\ B = \bigcup_{0<\xi<1} B(\xi)$$

and

$$A(\xi') \subseteq A(\xi), \quad B(\xi') \subseteq B(\xi) \quad \text{for} \quad 0 < \xi < \xi' < 1.$$

Hence, $\operatorname{conv}(A)=A$ and $\operatorname{conv}(B)=B$.

It is easily seen, that $A(1)$ can be empty only in the following two cases: either $A(\xi)=(\underline{a}, a(\xi)]$ $\forall\xi\in(0,1)$ and $\lim\limits_{\xi\to1-} a(\xi)=\underline{a}$ or $A(\xi)=[a(\xi),\bar{a})$, $\xi\in(0,1)$ and $\lim\limits_{\xi\to1-} a(\xi)=\bar{a}$. Similar criteria holds for $B(1)$. If we define $\underline{a}(\xi)\triangleq\inf\{x\colon\ x\in A(\xi)\}$ and $\bar{a}(\xi)=\sup\{x\colon\ x\in A(\xi)\}$, then we can write for the quantities defined in the previous paragraph:

$$\underline{a} = \lim_{\xi\to0+} \underline{a}(\xi),\ \bar{a} = \lim_{\xi\to0+} \bar{a}(\xi),\ a = \lim_{\xi\to1-} \underline{a}(\xi),\ \tilde{a} = \lim_{\xi\to1-} \bar{a}(\xi).$$

Similar relations hold for $\underline{b}$, $\bar{b}$, $b$, $\tilde{b}$. We see that:

$$A(1) = \emptyset \quad \textit{if and only if either} \quad \underline{a} = a = \tilde{a} \quad \textit{or} \quad a = \tilde{a} = \bar{a};$$
$$B(1) = \emptyset \quad \textit{if and only if either} \quad \underline{b} = b = \tilde{b} \quad \textit{or} \quad b = \tilde{b} = \bar{b}.$$

These clarify the definitions (1.12) and (1.13).

**Lemma 2.** *We have*

$$(2.12)\qquad \underline{a} = a \quad \textit{if and only if} \quad \underline{b} = b$$

*and*

$$(2.13)\qquad \tilde{a} = \bar{a} \quad \textit{if and only if} \quad \tilde{b} = \bar{b}.$$

PROOF. Let, say, $A=[\underline{a},\bar{a}]$, $B=[\underline{b},\bar{b}]$. We can write $A=[\underline{a},a]\cup(a,\bar{a}]$, $B=[\underline{b},b]\cup(b,\bar{b}]$ and

$$\lambda A+(1-\lambda)B = [\lambda\underline{a}+(1-\lambda)\underline{b},\ \lambda a+(1-\lambda)b]\cup(\lambda a+(1-\lambda)b,\ \lambda\bar{a}+(1-\lambda)\bar{b}].$$

Easy computations show that if for $f$ and $g$ equality occurs in (1.7) $(\alpha=1)$, then

$$(2.14)\qquad \int\limits_{\lambda\underline{a}+(1-\lambda)\underline{b}}^{\lambda a+(1-\lambda)b} \sup_{\substack{\lambda x+(1-\lambda)y=t\\ x\in[\underline{a},a]\\ y\in[\underline{b},b]}} \min\{f(x),\ g(y)\}\,dt = \lambda\cdot\int\limits_{\underline{a}}^{a} f(x)\,dx+(1-\lambda)\cdot\int\limits_{\underline{b}}^{b} g(y)\,dy.$$

This shows that neither $\underline{a}=a$, $\underline{b}<b$ nor $\underline{a}<a$, $\underline{b}=b$ can occur, hence (2.12) is true.

Similarly, in all other cases, using an appropriate partition of $A$ and $B$, we can prove (2.12). The proof of (2.13) is analogous.

**Lemma 3.** *Let* $\underline{a}<a$, $\underline{b}<b$. *Then*

$$a-\underline{a}=b-\underline{b}, \tag{2.15}$$

$$f(x)=g(\underline{b}-\underline{a}+x) \quad \forall\, x\in(\underline{a}, a), \tag{2.16}$$

*and* $f(x)$ *is strictly increasing and continuous on* $(\underline{a}, a)$.

Proof. Denote

$$\underline{A} \triangleq \begin{cases} [\underline{a}, a] & \text{if} \quad A=[\underline{a}, \bar{a}] \quad \text{or} \quad A=[\underline{a}, \bar{a}) \quad \text{and} \quad a<\bar{a}, \\ A & \text{if} \quad A=[\underline{a}, \bar{a}) \quad \text{and} \quad a=\bar{a}, \\ (\underline{a}, a] & \text{if} \quad A=(\underline{a}, \bar{a}] \quad \text{or} \quad A=(\underline{a}, \bar{a}) \quad \text{and} \quad a<\bar{a}, \\ A & \text{if} \quad A=(\underline{a}, \bar{a}) \quad \text{and} \quad a=\bar{a}. \end{cases}$$

The set $\underline{B}$ is defined analogously.

The sets $\underline{A}$ and $\underline{B}$ are closed in $A$ and $B$, respectively. Because $\tilde{f}[\tilde{g}]$ is u.s.c. on $A[B]$, if $a<\bar{a}[b<\bar{b}]$ then $f(a)=1$ $[g(b)=1]$. Hence, we have

$$\sup_{x\in\underline{A}} f(x)=\sup_{y\in\underline{B}} g(y)=1. \tag{2.17}$$

It is clear that the restriction of $f[g]$ to $\underline{A}[\underline{B}]$ is u.s.c. in $\underline{A}[\underline{B}]$.

Denote

$$\underline{k}(t)=\sup_{\substack{\lambda x+(1-\lambda)y=t\\ x\in\underline{A},\, y\in\underline{B}}} (\lambda f(x)+(1-\lambda)g(y)), \quad t\in\lambda\underline{A}+(1-\lambda)\underline{B},$$

and

$$\underline{h}(t)=\sup_{\substack{\lambda x+(1-\lambda)y=t\\ x\in\underline{A},\, y\in\underline{B}}} \min\{f(x), g(y)\}, \quad t\in\lambda\underline{A}+(1-\lambda)\underline{B}.$$

It is clear that if for $f$ and $g$ equality occurs in (1.7) (with $\alpha=1$, $0<\lambda<1$), then also

$$\int_{\lambda\underline{A}+(1-\lambda)\underline{B}} \underline{k}(t)\,dt=\int_{\lambda\underline{A}+(1-\lambda)\underline{B}} \underline{h}(t)\,dt=\lambda\int_{\underline{A}} f(x)\,dx+(1-\lambda)\int_{\underline{B}} g(y)\,dy. \tag{2.18}$$

Clearly, (2.18) holds also in the case when instead of the function $g(y)$, $y\in\underline{B}$, we take the function $g(b-a+y)$, $y\in\underline{B}-b+a$ $(g(y)$ is "translated" by $(b-a))$.

Hence, we can assume without the loss of generality, that $\underline{A}$ and $\underline{B}$ are located so that $a=b$. It is easily seen, that $f[g]$ is non-decreasing, and consequently because of its u.s.-continuity on $\underline{A}[\underline{B}]$, right semi-continuous on $\underline{A}[\underline{B}]$. This implies that $\underline{k}(t)$ is also non-decreasing and right semi-continuous on $\lambda\underline{A}+(1-\lambda)\underline{B}$.

Using the convergence relation $(\lambda u^\alpha+(1-\lambda)v^\alpha)^{1/\alpha}\xrightarrow[\alpha\to-\infty]{}\min\{u, v\}$, we can see that also $\underline{h}(t)$ is non-decreasing and right semi-continuous on $\lambda\underline{A}+(1-\lambda)\underline{B}$. It is clear that $\underline{k}(t) \geqq \underline{h}(t)$, $t \in \lambda \underline{A} + (1-\lambda)B$. This implies, using the right semi-continuity of $\underline{k}$ and $\underline{h}$, and (2.18), that

$$\underline{k}(t) = \underline{h}(t), \quad \forall\, t \in \lambda \underline{A} + (1-\lambda)\underline{B}, \tag{2.19}$$

that means

$$\lambda f(x) + (1-\lambda) g(y) \leqq \underline{h}(t), \ \forall\, t \in \lambda \underline{A} + (1-\lambda)\underline{B}, \ \forall\, \lambda x + (1-\lambda) y = t, \ x \in \underline{A}, \ y \in \underline{B} \tag{2.20}$$

Now, we prove: if (2.20) holds, then

(2.21) $\{f$ and $g$ are strictly increasing continuous functions on $\underline{A}$ and $\underline{B}$, respectively$\}$

PROOF of this statement:

Let $\xi \in (0, 1)$ and denote $s(\xi) \triangleq \{(x, y) \in \mathbf{R}^2 \colon y = \xi\}$ (a line "parallel" to the axis $x$). Denote by $\mathrm{gr}\,(f)$, $\mathrm{gr}\,(g)$ and $\mathrm{gr}\,(\underline{h})$ the graphs of $f$, $g$ and $\underline{h}$, respectively.

Denote $I \triangleq s(\xi) \cap \mathrm{gr}\,(f)$, $J \triangleq s(\xi) \cap \mathrm{gr}\,(g)$ and $L \triangleq s(\xi) \cap \mathrm{gr}\,(\underline{h})$.

Each set $I$, $J$ or $L$ is either empty or {one point} or of the form

$$\tau \triangleq \{(x, y) \in R^2 \colon y = \xi, \ c_1 \leqq x < c_2\}$$

("left closed interval").

It is easily seen, that $L = \lambda I + (1-\lambda) J$ (taking by definition $\emptyset + H = H$, $H \subset \mathbf{R}$, if necessary).

Careful analysis show, that if, say, $I = \tau$, then (2.20) cannot be true. Similarly, each of the cases $I = \emptyset$, $J = \emptyset$ and $J = \tau$ leads to a contradiction with (2.20).

We see, that both $I$ and $J$ consist of one point, only. But this proves our statement, because $f$ and $g$ are right semi-continuous on $\underline{A}$ and $\underline{B}$, respectively.

The statement (2.21) implies also that $\underline{h}(t)$ and, because of (2.19), $\underline{k}(t)$ are strictly increasing and continuous on $\lambda \underline{A} + (1-\lambda)\underline{B}$. Hence, all the functions have inverse functions which are defined on $(0, 1)$, continuous and nondecreasing there. Denote them by

$$\varphi(\xi) \triangleq f^{-1}(\xi) = \{x \in \underline{A} \colon f(x) = \xi\}, \ \psi(\xi) \triangleq g^{-1}(\xi) = \{y \in \underline{B} \colon g(y) = \xi\},$$

$$\omega_1(\xi) \triangleq \underline{k}^{-1}(\xi) = \{t \in \lambda \underline{A} + (1-\lambda)\underline{B} \colon \underline{k}(t) = \xi\}, \quad \omega_2(\xi) \triangleq \underline{h}^{-1}(\xi) =$$

$$= \{t \in \lambda \underline{A} + (1-\lambda)\underline{B} \colon \underline{h}(t) = \xi\}, \quad (\xi \in (0, 1)).$$

We can easily see that $\omega_1(\xi) = \omega_2(\xi)$, $\xi \in (0, 1)$ implies

$$\begin{aligned} &\lambda\varphi\big(\lambda\alpha + (1-\lambda)\beta\big) + (1-\lambda)\psi\big(\lambda\alpha + (1-\lambda)\beta\big) \geqq \\ &\geqq \lambda\varphi(\alpha) + (1-\lambda)\psi(\beta), \quad \forall\, 0 < \alpha, \beta < 1. \end{aligned} \tag{2.22}$$

We can write (2.22) with $\alpha$ and $\beta$ reversed. Addig up these two inequalities we get

$$\begin{aligned} &\lambda\varphi\big(\lambda\alpha + (1-\lambda)\beta\big) + (1-\lambda)\psi\big(\lambda\alpha + (1-\lambda)\beta\big) + \\ &+ \lambda\varphi\big(\lambda\beta + (1-\lambda)\alpha\big) + (1-\lambda)\psi\big(\lambda\beta + (1-\lambda)\alpha\big) \geqq \\ &\geqq \lambda\varphi(\alpha) + (1-\lambda)\psi(\alpha) + \lambda\varphi(\beta) + (1-\lambda)\psi(\beta), \quad \forall\, 0 < \alpha, \beta < 1. \end{aligned} \tag{2.23}$$

After the repeating application of (2.23) (i.e. taking $\alpha' = \lambda\alpha + (1-\lambda)\beta$, $\beta' = \lambda\beta + (1-\lambda)\alpha$ and writing (2.23) with $\alpha'$, $\beta'$, e.t.c.), taking into account continuity of $\varphi$ and $\psi$, we see that the function $\lambda\varphi+(1-\lambda)\psi$ is Jensen-concave $\left(\text{i.e. concave with } \mu=\frac{1}{2}\right)$, consequently, being continuous, concave on $(0, 1)$.

Standard arguments (see, e.g., [5]) show, that the concavity implies: there is $K$ such that

$$(2.24)\quad |\lambda\varphi(\alpha)+(1-\lambda)\psi(\alpha)-\lambda\varphi(\beta)-(1-\lambda)\psi(\beta)| \leqq K|\alpha-\beta| \quad \forall\, 0<\alpha, \beta<1,$$

i.e. the function satisfies the Lipschitz-condition. Since $\varphi$ and $\psi$ are non-decreasing functions, the condition (2.24) implies that both $\varphi$ and $\psi$, separately, satisfy the Lipschitz-condition (with the same constant $K$). This in turn implies that both $\varphi$ and $\psi$ are absolutely continuous on $(0, 1)$, consequently for all $\xi\in(0, 1)$ we have

$$(2.25)\qquad \varphi(\xi)-\varphi(0)=\int_0^\xi \varphi'(\eta)\,d\eta \quad\text{and}\quad \psi(\xi)-\psi(0)=\int_0^\xi \psi'(\eta)\,d\eta.$$

It is clear that $\varphi'(\xi)$ and $\psi'(\xi)$ exist a.e. on $(0, 1)$ (in fact, because of monotonity, there are only at most countable many points where $\varphi'(\xi)$ or $\psi'(\xi)$ does not exist). Let $\xi\in(0, 1)$ be a point where both $\varphi'(\xi)$ and $\psi'(\xi)$ exist. Using (2.22), we can write for $\Delta>0$ sufficiently small

$$(2.26)\qquad \lambda\varphi(\xi)+(1-\lambda)\psi(\xi)\geqq\lambda\varphi(\xi-\Delta)+(1-\lambda)\psi\left(\xi+\frac{1}{1-\lambda}\Delta\right).$$

This implies

$$(2.27)\qquad \frac{-\varphi(\xi)+\varphi(\xi-\Delta)}{-\Delta}\geqq\frac{1-\lambda}{\lambda}\,\frac{\psi\left(\xi+\frac{\lambda}{1-\lambda}\Delta\right)-\psi(\xi)}{\Delta},$$

and letting tend $\Delta$ to $0+$ we get

$$(2.28)\qquad \varphi'(\xi)\geqq\psi'(\xi).$$

On the other hand using (2.22) again, we have

$$(2.29)\qquad \frac{-\psi(\xi)+\psi(\xi-\Delta)}{-\Delta}\geqq\frac{\lambda}{1-\lambda}\,\frac{\varphi\left(\xi+\frac{1-\lambda}{\lambda}\Delta\right)-\varphi(\xi)}{\Delta}$$

implying

$$(2.30)\qquad \psi'(\xi)\geqq\varphi'(\xi).$$

The relations (2.28) and (2.29) together give

$$(2.31)\qquad \varphi'(\xi)=\psi'(\xi).$$

Because of (2.25) we have

$$(2.32)\qquad \varphi(\xi)-\varphi(0)=\psi(\xi)-\psi(0)\quad\forall\,\xi\in(0, 1).$$

$$\left(\text{Here}\quad \varphi(0)\triangleq\lim_{\xi\to0+}\varphi(\xi),\ \psi(0)=\lim_{\xi\to0+}\psi(\xi)\right).$$

The assumption $a=b$ and the definition of $\varphi$ and $\psi$ imply $\sup_{\xi\in(0,1)}\varphi(\xi)=\sup_{\xi\in(0,1)}\psi(\xi)$ and this yields, using (2.32) that $\varphi(0)=\psi(0)$.

This proves the lemma, because (2.32) (where $\varphi(0)=\psi(0)$) and the definition of $\varphi$ and $\psi$ imply

$$\underline{a}=\underline{b} \quad \text{and} \quad f(x)=g(x) \quad \forall\, x\in(\underline{a},a). \tag{2.33}$$

If $a\neq b$, then (2.33) holds with $f(x)$, $x\in\underline{A}$, and $g(y)\doteq g(b-a+y)$, $y\in B\doteq\underline{B}-b+a$, hence we get (2.15) and (2.16).

Using an analogous proof we can prove a similar statement for the "right end" of the functions $f$ and $g$ (now the functions involved will be strictly decreasing).

**Lemma 3′.** *Let* $\tilde{a}<\bar{a}$, $\tilde{b}<\bar{b}$. *Then*

$$\bar{a}-\tilde{a}=\bar{b}-\tilde{b}, \tag{2.34}$$

$$f(x)=g(\tilde{b}-\tilde{a}+x) \quad \forall\, x\in(\tilde{a},\bar{a}), \tag{2.35}$$

*and* $f(x)$ *is strictly decreasing and continuous on* $(\tilde{a},\bar{a})$.

**Lemma 4.** *Let* $\underline{a}<a$, $\underline{b}<b$. *Then the function* $f(x)$ *is concave on* $(\underline{a},a)$.

Proof. Using (2.15) and (2.16) we can write

$$\int\limits_{\lambda\underline{A}+(1-\lambda)\underline{B}} \underline{k}(t)\,dt = \int\limits_{\lambda\underline{a}+(1-\lambda)\underline{b}}^{\lambda a+(1-\lambda)b} \sup_{\substack{\lambda x+(1-\lambda)(b-a+x')=t\\ x,x'\in(\underline{a},a)}} \big(\lambda f(x)+(1-\lambda)f(x')\big)\,dt = \tag{2.36}$$

$$= \int\limits_{\underline{a}}^{a} \sup_{\substack{\lambda x+(1-\lambda)x'=\tau\\ x,x'\in(\underline{a},a)}} \big(\lambda f(x)+(1-\lambda)f(x')\big)\,d\tau$$

and

$$\lambda\int\limits_{\underline{A}} f(x)\,dx+(1-\lambda)\int\limits_{\underline{B}} g(y)\,dy = \int\limits_{\underline{a}}^{a} f(\tau)\,d\tau. \tag{2.37}$$

According to (2.18) we have

$$\int\limits_{\underline{a}}^{a} \sup_{\substack{\lambda x+(1-\lambda)x'=\tau\\ x,x'\in(\underline{a},a)}} \big(\lambda f(x)+(1-\lambda)f(x')\big)\,d\tau = \int\limits_{\underline{a}}^{a} f(\tau)\,d\tau, \tag{2.38}$$

that implies

$$f(\tau)=\sup_{\substack{\lambda x+(1-\lambda)x'=\tau\\ x,x'\in(\underline{a},a)}} \big(\lambda f(x)+(1-\lambda)f(x')\big), \quad \forall\, \tau\in(\underline{a},a). \tag{2.39}$$

Similar statement holds on $(\tilde{a},\bar{a})$:

**Lemma 4′.** *Let* $\tilde{a}<\bar{a}$, $\tilde{b}<\bar{b}$. *Then the function* $f(x)$ *is concave on* $(\tilde{a},\bar{a})$.

The lemmas together give the statement of the Theorem in the case $\gamma=\delta$ and $\alpha=1$. The general case can be proved using the transformations below.

**Lemma 5.** *Let* $-\infty<\alpha<+\infty$, $0<\lambda<1$, *and* $f, g\colon \mathbf{R}\to\mathbf{R}_+$ *be functions satisfying the assumptions of the Proposition. Then, equality occurs in* (1.7) *if and only if*

$$(2.40)\qquad \int_{-\infty}^{+\infty} \sup_{\omega x+(1-\omega)y=t} M_\alpha^\omega(\varphi(x), \psi(y))\,dt = M_1^\omega\Big(\int_{-\infty}^{+\infty}\varphi(x)\,dx, \int_{-\infty}^{+\infty}\psi(y)\,dy\Big),$$

*where*

$$(2.41)\qquad \varphi(x) \triangleq \frac{1}{\gamma}\cdot f\Big(\frac{\gamma^\alpha}{\lambda\gamma^\alpha+(1-\lambda)\delta^\alpha}\cdot x\Big),\ \psi(x) \triangleq \frac{1}{\delta}\, g\Big(\frac{\delta^\alpha}{\lambda\gamma^\alpha+(1-\lambda)\delta^\alpha}\cdot x\Big),\quad x\in\mathbf{R},$$

*and*

$$(2.42)\qquad \omega \triangleq \frac{\lambda\gamma^\alpha}{\lambda\gamma^\alpha+(1-\lambda)\delta^\alpha}.$$

Proof. Denote $A\triangleq \operatorname{supp} f$, $B\triangleq \operatorname{supp} g$, $\xi\triangleq\lambda+(1-\lambda)\Big(\dfrac{\delta}{\gamma}\Big)^\alpha$,

$$\eta\triangleq\lambda\Big(\frac{\gamma}{\delta}\Big)^\alpha+1-\lambda,\ C\triangleq\xi\cdot A,\quad D\triangleq\eta\cdot B.$$

Then, the following relations are easily checked:

$$(2.43)\qquad \int_A \frac{f(x)}{\gamma}\,dx = \frac{1}{\xi}\int_C \varphi(u)\,du,\ \int_B \frac{g(y)}{\delta}\,dy = \frac{1}{\eta}\int_D \psi(v)\,dv,$$

$$(2.44)\qquad \omega C+(1-\omega)D = \lambda A+(1-\lambda)B$$

and

$$\sup_{\substack{\lambda x+(1-\lambda)y=t\\ x\in A,\, y\in B}} \frac{(\lambda f^\alpha(x)+(1-\lambda)g^\alpha(y))^{1/\alpha}}{M_\alpha^\lambda(\gamma,\delta)} = \sup_{\substack{\omega u+(1-\omega)v=t\\ u\in C,\, v\in D}} (\omega\varphi^\alpha(u)+(1-\omega)\psi^\alpha(v))^{1/\alpha}$$

holds for all $t\in\omega C+(1-\omega)D$. These prove the lemma.

The lemma shows that it is enough to investigate the conditions of equality in (1.7) for the functions such that $\gamma=\delta=1$. In this case the right hand side of (1.7) does not depend on $\alpha$, hence if in (1.7) equality holds for some $\alpha$, then it holds also for all $\alpha'<\alpha$ (being $h_\alpha^\lambda(t)$ non-decreasing in $\alpha$). This shows that if $\alpha>-1$ and equality holds in (1.7), then equality holds also for $\alpha=-1$. The following transformation shows that also the cases $-\infty<\alpha<-1$ can be transformed to the case $\alpha=-1$.

**Lemma 6.** *Let* $-\infty<\alpha<-1$, $0<\lambda<1$, $f, g\colon \mathbf{R}\to\mathbf{R}_+$ *be functions satisfying the assumptions of the Proposition and* $\gamma=\delta=1$. *If equality holds in* (1.7), *then*

$$(2.45)\qquad \int_{-\infty}^{+\infty} \sup_{\lambda x+(1-\lambda)y=t} M_{-1}^\lambda(\varphi(x), \psi(y))\,dt = M_1^\lambda\Big(\int_{-\infty}^{+\infty}\varphi(x)\,dx, \int_{-\infty}^{+\infty}\psi(y)\,dy\Big),$$

*where*

$$(2.46)\qquad \varphi(x)\triangleq f^{|\alpha|}(x),\ \psi(x)\triangleq g^{|\alpha|}(x),\quad x\in\mathbf{R}.$$

PROOF. Denote $A \triangleq \operatorname{supp} f$, $B \triangleq \operatorname{supp} g$, $A(\xi) \triangleq \{x \in A: f(x) \geqq \xi\}$,

$$B(\xi) \triangleq \{y \in B: g(y) \geqq \xi\},\ C(\xi) \triangleq \{t \in \lambda A+(1-\lambda)B: h_\alpha^\lambda(t) \geqq \xi\},$$

$$E(\xi) \triangleq \{t \in \lambda A+(1-\lambda)B: k(t) \geqq \xi\},\ F(\xi) \triangleq \{x \in A: \varphi(x) \geqq \xi\},$$

$$G(\xi) \triangleq \{y \in B: \psi(y) \geqq \xi\},$$

where

$$k(t) \triangleq \sup_{\substack{\lambda x+(1-\lambda)y=t \\ x \in A,\, y \in B}} \left(\lambda\varphi^{-1}(x)+(1-\lambda)\psi^{-1}(y)\right)^{-1}, \quad t \in \lambda A+(1-\lambda)B.$$

It is easy to see that $k(t)=\left(h_\alpha^\lambda(t)\right)^{|\alpha|}$, $t \in \lambda A+(1-\lambda)B$. Using the relations (2.2), (2.3) and (2.4) (with $h(t) \triangleq h_\alpha^\lambda(t)$), equality in (1.7) implies

$$\mu(C(\xi)) = \lambda\mu(A(\xi))+(1-\lambda)\mu(B(\xi)), \quad \text{a.e.} \quad \xi \in (0, 1),$$

and from this we have

$$\mu(E(\eta)) = \lambda\mu(F(\eta))+(1-\lambda)\mu(G(\eta)), \quad \text{a.e.} \quad \eta \in (0, 1).$$

Integrating the last equality over $(0, 1)$ we get (2.45).

This lemma shows that it is enough to investigate the conditions of equality in (1.7) for the case $\gamma=\delta=1$ and $\alpha=-1$. In the last step we transform the case $\alpha=-1$ to the case $\alpha=1$.

**Lemma 7.** *Let $0<\lambda<1$ and $f, g$: $\mathbf{R} \to \mathbf{R}_+$ be functions satisfying the assumptions of the Proposition and such that $\gamma=\delta=1$. If equality holds in* (1.7) *for $\alpha=-1$, then for arbitrary (sufficiently small) $\varepsilon>0$ we have*

$$\text{(2.47)} \qquad \int\limits_{\lambda A(\varepsilon)+(1-\lambda)B(\varepsilon)} \sup_{\substack{\lambda x+(1-\lambda)y=t \\ x \in A(\varepsilon),\, y \in B(\varepsilon)}} \left(\lambda\varphi_\varepsilon(x)+(1-\lambda)\psi_\varepsilon(y)\right) dt =$$

$$= \lambda \int\limits_{A(\varepsilon)} \varphi_\varepsilon(x)\, dx+(1-\lambda) \int\limits_{B(\varepsilon)} \psi_\varepsilon(y)\, dy,$$

where $A(\varepsilon) \triangleq \{x \in \operatorname{supp} f: f(x) \geqq \varepsilon\}$, $B(\varepsilon) \triangleq \{y \in \operatorname{supp} g: g(y) \geqq \varepsilon\}$,

$$\text{(2.48)} \qquad \varphi_\varepsilon(x) \triangleq \begin{cases} \dfrac{1}{\varepsilon}-\dfrac{1}{f(x)}, & x \in A(\varepsilon) \\ 0 & x \notin A(\varepsilon), \end{cases} \quad x \in \mathbf{R},$$

and

$$\text{(2.49)} \qquad \psi_\varepsilon(y) \triangleq \begin{cases} \dfrac{1}{\varepsilon}-\dfrac{1}{g(y)}, & y \in B(\varepsilon) \\ 0 & y \notin B(\varepsilon), \end{cases} \quad x \in \mathbf{R}.$$

PROOF. Denote $A \triangleq \operatorname{supp} f$, $B \triangleq \operatorname{supp} g$, $h(t) \triangleq h_{-1}^\lambda(t)$, $t \in \lambda A+(1-\lambda)B$ (see (1.6)),

$$\text{(2.50)} \qquad h_\varepsilon(t) \triangleq \sup_{\substack{\lambda x+(1-\lambda)y=t \\ x \in A(\varepsilon),\, y \in B(\varepsilon)}} \left(\frac{\lambda}{f(x)}+\frac{1-\lambda}{g(y)}\right)^{-1},\ t \in \lambda A(\varepsilon)+(1-\lambda)B(\varepsilon),$$

and

$$(2.51)\qquad k_\varepsilon(t) \triangleq \sup_{\substack{\lambda x+(1-\lambda)y=t\\ x\in A(\varepsilon),\, y\in B(\varepsilon)}} \big(\lambda\varphi_\varepsilon(x)+(1-\lambda)\psi_\varepsilon(y)\big), \quad t\in\lambda A(\varepsilon)+(1-\lambda)B(\varepsilon).$$

Clearly, we have

$$(2.52)\qquad \{t\in\lambda A+(1-\lambda)B: h(t)\geqq\xi\}\supseteq\{t\in\lambda A(\varepsilon)+(1-\lambda)B(\varepsilon): h_\varepsilon(t)\geqq\xi\}\supseteq$$

$$\supseteq \lambda A(\xi)+(1-\lambda)B(\xi), \quad \forall\xi\in(\varepsilon, 1).$$

This shows, using (2.3), (2.4) and the fact equality holds in (1.7), that

$$(2.53)\qquad \mu\big(\{t\in\lambda A(\varepsilon)+(1-\lambda)B(\varepsilon): h_\varepsilon(t)\geqq\xi\}\big)=$$

$$=\lambda\mu\big(A(\xi)\big)+(1-\lambda)\mu\big(B(\xi)\big), \quad \text{a.e.}\quad \xi\in(\varepsilon, 1).$$

We can see easily that (2.53) implies

$$(2.54)\qquad \mu\big(\{t\in\lambda A(\varepsilon)+(1-\lambda)B(\varepsilon): k_\varepsilon(t)\geqq\eta\}\big)=\lambda\mu\big(\{x\in A(\varepsilon): \varphi_\varepsilon(x)\geqq\eta\}\big)+$$

$$+(1-\lambda)\mu\big(\{y\in B(\varepsilon): \psi_\varepsilon(y)\geqq\eta\}\big), \quad \text{a.e.}\quad \eta\in\left(0, \frac{1}{\varepsilon}-1\right).$$

Integrating (2.54) over $\left(0, \frac{1}{\varepsilon}-1\right)$ we get (2.47).

This lemma shows, that it is enough to investigate the conditions of equality in (1.7) for the case $\gamma=\delta=1$ and $\alpha=1$. But for this case the Theorem has been already proved (see Lemmas $1\div 4'$).

Thus, the proof of the Theorem for the general case $\gamma\neq\delta$ and $-\infty<\alpha<+\infty$ consists of a careful "re-transformation" of the results of Lemmas 5, 6 and 7. In this, we distinguish two cases:

$$\text{(a)}:\quad \inf_{x\in A} f(x)>0 \quad\text{and}\quad \inf_{y\in B} g(y)>0$$

and

$$\text{(b)}:\quad \min\big\{\inf_{x\in A} f(x),\ \inf_{y\in B} g(y)\big\}=0.$$

In the case (a) take:

$$\varepsilon\triangleq\min\big\{\inf_{x\in A} f(x),\ \inf_{y\in B} g(y)\big\}$$

and in the case (b):

$$0<\varepsilon\leqq\min\{\gamma, \delta\}.$$

Denote $A(\varepsilon)\triangleq\{x\in\operatorname{supp} f\colon f(x)\geqq\varepsilon\}$, $B(\varepsilon)\triangleq\{y\in\operatorname{supp} g\colon g(y)\geqq\varepsilon\}$,

$$\beta=\begin{cases}1 & \text{if}\quad \alpha\geqq-1,\\ |\alpha| & \text{if}\quad -\infty<\alpha<-1\end{cases}$$

and define on **R** the two functions as follows:

$$\varphi_\varepsilon(x)=\begin{cases}0, & \text{if} \quad x\notin\left(\lambda+(1-\lambda)\left(\frac{\delta}{\gamma}\right)^\alpha\right)\cdot A(\varepsilon),\\ \frac{1}{\varepsilon}-\frac{\gamma^\beta}{f^\beta\left(\frac{\gamma^\alpha}{\lambda\gamma^\alpha+(1-\lambda)\delta^\alpha}x\right)}, & \text{if} \quad x\in\left(\lambda+(1-\lambda)\left(\frac{\delta}{\gamma}\right)^\alpha\right)\cdot A(\varepsilon),\end{cases}$$

$$\psi_\varepsilon(y)=\begin{cases}0, & \text{if} \quad y\notin\left(\lambda\left(\frac{\gamma}{\delta}\right)^\alpha+1-\lambda\right)\cdot B(\varepsilon),\\ \frac{1}{\varepsilon}-\frac{\delta^\beta}{g^\beta\left(\frac{\delta^\alpha}{\lambda\gamma^\alpha+(1-\lambda)\delta^\alpha}y\right)}, & \text{if} \quad y\in\left(\lambda\left(\frac{\gamma}{\delta}\right)^\alpha+1-\lambda\right)\cdot B(\varepsilon).\end{cases}$$

If $f$ and $g$ were u.s.c. on $\operatorname{conv}(\operatorname{supp} f)$ and $\operatorname{conv}(\operatorname{supp} g)$, respectively, then, clearly, $\varphi_\varepsilon$ and $\psi_\varepsilon$ are u.s.c. on $\operatorname{conv}(A(\varepsilon))$ and $\operatorname{conv}(B(\varepsilon))$, respectively. Further, $\sup_{x\in A(\varepsilon)}\varphi_\varepsilon(x)=\sup_{y\in B(\varepsilon)}\psi_\varepsilon(y)=\frac{1}{\varepsilon}-1$ and according to Lemmas 5, 6 and 7 we have (see (2.47)):

$$\int\limits_{\omega A(\varepsilon)+(1-\omega)B(\varepsilon)}\sup_{\substack{\omega x+(1-\omega)y=t\\ x\in A(\varepsilon),\, y\in B(\varepsilon)}}\left(\omega\varphi_\varepsilon(x)+(1-\omega)\psi_\varepsilon(y)\right)dt= \tag{2.55}$$

$$=\omega\int\limits_{A(\varepsilon)}\varphi_\varepsilon(x)\,dx+(1-\omega)\int\limits_{B(\varepsilon)}\psi_\varepsilon(y)\,dy,$$

where

$$\omega=\frac{\lambda\gamma^\alpha}{\lambda\gamma^\alpha+(1-\lambda)\delta^\alpha}. \tag{2.56}$$

Applying the Theorem to $\varphi_\varepsilon(x)$ and $\psi_\varepsilon(y)$ we get the results (1.15) ÷ (1.20) for the functions $f_\varepsilon$ and $g_\varepsilon$ which are the restrictions of $f$ and $g$ to $A(\varepsilon)$ and $B(\varepsilon)$, respectively (clearly, $\sup_{x\in A(\varepsilon)} f_\varepsilon(x)=\gamma$ and $\sup_{y\in B(\varepsilon)} g_\varepsilon(y)=\delta$).

If the case (a) holds, then we are ready, because $A(\varepsilon)=\operatorname{supp} f$ and $B(\varepsilon)=\operatorname{supp} g$ in this case.

In the case (b) letting tend $\varepsilon\to 0+$ we can easily see that (1.15) ÷ (1.20) remain true for $f$ ad $g$. By this the Theorem is completely proved.

## 3. Remarks

1. Let $f, g\colon \mathbf{R}\to\mathbf{R}_+$ fulfil the assumptions of the Theorem, but $A$ or [and] $B$ may be unbounded. The definitions of $\underline{a}$, $a$, $\tilde{a}$ and $\bar{a}$ are meaningful also in this case and $\underline{a}\leqq a\leqq\tilde{a}\leqq\bar{a}$ also holds, but now $\underline{a}$ or $a$ and $\tilde{a}$ or $\bar{a}$ may have the value $-\infty$ and $+\infty$, respectively (the same for $\underline{b}$, $b$, $\tilde{b}$ and $\bar{b}$). A standard "restrict to bounded sets and after that let tend to infinity"-type analysis would probably show that the statement of the Theorem remain true also for this case. We did not write

down exact calculations for this analysis, for two reasons: first, the proof of the bounded case is already quite complicated and lengthy; secondly, we are not convinced of the non-existence of a proof which is shorter and such more "elegant" than ours and which would include also the non-bounded case.

For $-1\leqq\alpha\leqq+\infty$ the right hand side of the inequality (1.7) can be decreased so that the following weaker inequality is true

$$\int_{-\infty}^{+\infty} h_\alpha^\lambda(t)\,dt \geqq M^\lambda_{\frac{\alpha}{1+\alpha}}\left(\int_{-\infty}^{+\infty} f(x)\,dx,\ \int_{-\infty}^{+\infty} g(y)\,dy\right). \tag{3.1}$$

(See [2], [3]. This is a simple onsequence of (1.7) and the Hölder inequality.)

The second named author, analyzing the proof of (3.1) due to HENSTOCK and MACBEATH [3], proved: if $f$ and $g$ are Lebesgue-measurable and equality holds in (3.1) (for $-1<\alpha<+\infty$), then we have (of course, $0<\gamma<+\infty$ and $0<\delta<+\infty$ are also assumed)

$$\operatorname{supp} f \overset{\text{a.e.}}{=} [a_1, a_2],\quad \operatorname{supp} g \overset{\text{a.e.}}{=} [b_1, b_2], \tag{3.2}$$

$$\left(\frac{\gamma}{\delta}\right)^\alpha = \frac{a_2-a_1}{b_2-b_1}, \tag{3.3}$$

$$f(x) = \frac{\gamma}{\delta}\, g\left(b_1+\left(\frac{\delta}{\gamma}\right)^\alpha (x-a_1)\right) \quad \text{a.e.} \quad x\in[a_1, a_2], \tag{3.4}$$

and

$$f(\omega x'+(1-\omega)x'') \geqq M_\alpha^\omega(f(x'), f(x'')), \quad \text{a.e.} \quad x', x''\in[a_1, a_2], \tag{3.5}$$

where

$$\omega = \frac{\lambda\gamma^\alpha}{\lambda\gamma^\alpha+(1-\lambda)\delta^\alpha}. \tag{3.6}$$

The method of proof of (3.2) ÷ (3.6) is quite different from that used in this paper.

If equality holds in (3.1), then obviously the right hand sides of (3.1) and (1.7) are equal. This easily implies:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \left(\frac{\gamma}{\delta}\right)^{1+\alpha} \int_{-\infty}^{+\infty} g(y)\,dy. \tag{3.7}$$

From this it follows (using the notations of paragraph 1):

$$\tilde{a}-a = \left(\frac{\gamma}{\delta}\right)^\alpha \quad (\tilde{b}-b). \tag{3.8}$$

It is easy to see that in this case the conditions (1.17) ÷ (1.20) can be writen in the form

$$\bar{a}-\underline{a} = \left(\frac{\gamma}{\delta}\right)^\alpha \quad (\bar{b}-\underline{b}), \tag{3.9}$$

$$f(x) = \frac{\gamma}{\delta}\, g\left(\underline{b}+\left(\frac{\delta}{\gamma}\right)^\alpha (x-\underline{a})\right) \quad \forall x\in(\underline{a}, \bar{a}), \tag{3.10}$$

and

$$f(\mu x'+(1-\mu)x'') \geqq M_\alpha^\mu(f(x'), f(x'')) \tag{3.11}$$

holds for all $x', x'' \in (\underline{a}, \bar{a})$ and $0 \leqq \mu \leqq 1$.

We see that the results (1.15)—(1.20) and (3.2)—(3.6) are essentially the same. In fact, it was the result (3.2) ÷ (3.6) which suggested that a similar result might be true if equality holds only in the sharper inequality (1.7) (so that (3.7) is not satisfied). It turned out, that the method used for the proof of (3.2)—(3.6) does not work in this sharper case. On the other hand, the application of the method used in this paper to the proof of the "measurable" (i.e. not u.s.c.) case seems to be an especially difficult problem. The question, if the Theorem (or a similar result like (3.2)—(3.6)) is true for Lebesgue-measurable functions $f$ and $g$, is still open.

## References

[1] G. H. Hardy—J. E. Littlewood—G. Pólya, Inequalities, Cambridge Univ. Press, *London*, 1951.

[2] I. Dancs—B. Uhrin, On a class of Integral Inequalities and their Measure-Theoretic Consequences, *J. of Math. Anal. Appl.*, **74** (1980), 388—400

[3] R. Henstock—A. M. Macbeath, On the Measure of Sum-Sets (I) The theorems of Brunn, Minkowski and Lusternik, *Proc. Lond. Math. Soc., Ser. III.*, **3** (1953), 182—194.

[4] H. J. Brascamp—E. H. Lieb, On extensions of the Brunn—Minkowski and Prékopa—Leindler Theorems, including inequalities for log concave functions, and with an application to the diffusion equation, *J. of Functional Analysis* **22** (1976), 366—389.

[5] I. P. Natanson, Theory of Functions of Real Variable Nauka, Moscow, 1974 (in Russian).

[6] S. Das Gupta, Brunn—Minkowski Inequality and its Aftermath, *J. of Multivariate Analysis*, **10** (1980), 295—318.

DEPARTMENT OF MATHEMATICS
ELTE UNIVERSITY, BUDAPEST 1088
HUNGARY

COMPUTER AND AUTOMATION INSTITUTE
HUNGARIAN ACADEMY OF SCIENCES
1502 BUDAPEST, PF. 63, HUNGARY

*(Received January 3, 1979)*