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# Remarks on convergence of types theorems on finite dimensional vector spaces

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**Abstract.** The Convergence-of-Types theorem for finite dimensional vector spaces turned out to be a most powerful tool for investigations of limit theorems in probability theory. A convenient formulation is the following: Let $\mathcal{B}$ be a group of affine or linear transformations ("admissible normalizations") acting on the vector space $E$ and let $\mathcal{F} \subseteq \mathcal{M}^1(E)$ be a suitable set of probabilities ("full or non-degenerate w.r.t. $\mathcal{B}$"). Let further $(\mu_n), \mu, \nu \subseteq \mathcal{F}$ and $(\alpha_n) \subseteq \mathcal{B}$. Then $\mu_n \to \mu$ and $\alpha_n \mu_n \to \nu$ imply that $(\alpha_n)$ is relatively compact. And then $\nu = \alpha\mu$ for any accumulation point $\alpha$ of $(\alpha_n)$.

In the language of transformation groups this means that the group of normalizations $\mathcal{B}$ acts properly or perfectly on the set of full measures $\mathcal{F}$.

Unifying previous investigations we present a method to construct suitable classes of full measures $\mathcal{F}$ for a given group $\mathcal{B}$. And in the sequel we apply this method to concrete examples of groups. As a by-product a new proof of the convergence-of-types theorem for nilpotent simly connected Lie-groups is obtained, based on linear algebra and avoiding any deeper knowledge of Lie-group theory.

The history of convergence-of-types theorems started with KHINTCHINE's investigations [K] on the behaviour of the distributions of normalized real random variables, see also e.g. [L] or [GK]. Since then the ideas and techniques developed there found various applications and were generalized in different directions. See e.g. [LS], [Ba], [JM] for recent surveys on the literature.

A new branch of investigations started with [F], independently [Bi], [Sh]: Now the limit behaviour of distributions of vector valued random variables normalized by *linear operators* resp. *affine transformations* is considered with applications to operator-stability (see e.g. [Sh], [HMV], [M]), operator-semi-stability (e.g. [Ja], [Lu], [S]), self-decomposability (e.g. [U], [J]), operator- self-similar processes ([HM], see also [W], [W1]). For generalizations to infinite dimensional spaces and applications see e.g. [LS], [JM], [Si] and the literature mentioned there, for applications to the theory of extreme values see e.g. [P1], [P2]. For similar concepts in the framework of probabilities on groups see e.g. [B], [HN], [N], [Ha1-3], for nilpotent groups and compact extensions, and see [D] for general connected Lie groups. For applications to stability and semi-stability on locally compact groups see e.g. [B], [DG], [Ha1], [Ha4-6], [N], [Sch1-2], for totally disconnected groups, e.g. for $p$-adic-vector spaces see [Sha] or [T].

Convergence of types theorems are usually formulated as follows: Let $\boldsymbol{E}$ be a vector space (or a group), let $\mathcal{B}$ be a group of linear or affine transformations (or automorphisms) acting on $\boldsymbol{E}$. Then $\mathcal{B}$ acts canonically on the probabilities $\mathcal{M}^1(\boldsymbol{E})$. (Call $\mathcal{B}$ the group of admissible normalizations). Let $\mathcal{F} \subseteq \mathcal{M}^1(\boldsymbol{E})$ be a subset of measures (in the sequel called *full with respect to* $\mathcal{B}$, in short: $\mathcal{B}$-*full*). Then $\mathcal{F}$ and $\mathcal{B}$ fulfil the *convergence of types condition* (C-T)

(C-T) if $(\mu_n), \mu, \nu \in \mathcal{F}, (\alpha_n) \subseteq \mathcal{B}, \mu_n \to \mu$ and $\alpha_n \mu_n \to \nu$ imply that $(\alpha_n)$ is relatively compact in $\mathcal{B}$.

And then $\alpha\mu = \nu$ for any accumulation point $\alpha$ of $(\alpha_n)$.

In most investigations mentioned above $\mathcal{B}$ is the whole group of linear operators $\mathbf{Gl}(\boldsymbol{E})$, resp. the corresponding group of affine transformations $\mathbf{AG}(\boldsymbol{E})$. Then the corresponding class of full measures $\mathcal{F} = \mathcal{F}_{\mathcal{B}}$ are those which are not concentrated on a proper subspace resp. on a proper hyperplane ([Sh]). And if the group $\mathcal{B}$ is the group of homothetical transformations resp. the corresponding affine group then (C-T) holds with $\mathcal{F} = \mathcal{M}^1(\boldsymbol{E})\backslash\{\varepsilon_0\}$ resp. $\mathcal{M}^1(\boldsymbol{E})\backslash\{\varepsilon_x,\ x \in \boldsymbol{E}\}$. But already the investigations for nilpotent groups ([N], [HN]) – the first examples of non-abelian groups – lead in a natural way to the more general concept: Given a group $\mathcal{B}$ acting on $\mathbf{E}$, try to construct a suitable large class of "full" measures $\mathcal{F}$ such that $(\mathcal{F}, \mathcal{B})$ fulfil condition (C-T). And it is natural to expect that the size of $\mathcal{F}$ depends on the size of $\mathcal{B} : \mathcal{F}$ increases if $\mathcal{B}$ decreases.

In investigations of limit distributions on finite dimensional vector spaces usually the limit measure is supposed to fulfil the strong fullness condition of Sharpe [Sh], whereas the proofs often only depend on the validity of the convergence of types theorem, see e.g. [HM], [W], [W1], [JM]. Hence it is possible to obtain analogous results in more general situations without changing the proofs. This remark is essential for investigations of the behaviour of group-valued random variables: Here it turns out that problems on groups often can be solved by studying equivalent problems on vector spaces (and vice versa). Only the group of admissible normalizations and the corresponding fullness concept have to be changed in a proper way. So, the following may be considered as an attempt to unify previous investigations and may serve as a tool box for further applications.

The investigations in [D] and [Ha3] lead into different directions: For a general connected Lie group $G$ we fix $\mathcal{F}$ to be the set of probabilities not concentrated on a proper connected subgroup. Then it is shown that under restrictions on $\mathcal{B}$, e.g. $\mathcal{B}$ being contained in the connected component of the group of automorphisms $\mathbf{Aut}(G)$, $\mathcal{F}$ and $\mathcal{B}$ fulfil the convergence of types condition (C-T).

We start (§1) with a general formulation of (C-T) conditions within the framework of topological transformation groups. It is pointed out that (C-T) means exactly that the transformation group acts properly (in the sense of Bourbaki [Bou]) resp. perfectly ([E]) on a suitable set of "full measures" $\mathcal{F} \subseteq \mathcal{M}^1(\boldsymbol{E})$. Hence various properties of full measures follow directly from well known results in the theory of transformation groups. We collect some of these results in Theorem 1.4.

In the following (§2. ff) we restrict our considerations to groups of linear or affine transformations acting on a finite dimensional (hence locally compact) vector space. It turns out that – from the point of view of transformation groups – it is not necessary to distinguish between linear and affine transformations (§2). We show then that (C-T) may be replaced by weaker boundedness conditions (W-C-T) (§2) resp. (W-C-T$^*$) (§3), and we present a method to construct full measures $\mathcal{F} = \mathcal{F}_{\mathcal{B}}$ fulfilling these boundedness conditions for a given group $\mathcal{B}$ (§3).

Then, in §4, we show for concrete examples that this method solves the problem of finding a suitable class of full measures $\mathcal{F}_{\mathcal{B}}$ for a given group of normalizations $\mathcal{B}$, unifying previous investigations. Moreover, we show that in most examples the so constructed class $\mathcal{F}_{\mathcal{B}}$ of full measures is maximal. Indeed, in these cases we obtain the following characterization : $\mu$ is full iff the invariance group $\mathcal{J}_{\mathcal{B}}(\mu)$ is compact.

As a special application of t the methods we tain finally – in Theorem 4.8. – a new proof for the convergence of types theorem for simply connected nilpotent Lie groups ([HN], [N]); a proof which applies the above-mentioned vector space methods to the tangent space of the group, hence a finite-dimensional vector space, thus avoiding any deeper methods from Lie group theory.

## §1 Types and transformation groups

We start with some general considerations concerning types of probabilities and convergence of types within the context of topological transformation groups. The idea to consider convergence of types and type-spaces of probabilities from a topological point of view appears already in [Doe]. See also [Ba] for a survey on the history. To avoid difficulties with non-Hausdorff topologies on quotient spaces we restrict the considerations to suitable subsets $\mathcal{F} = \mathcal{F}_{\mathcal{B}}$ of probabilities called "full w.r.t. $\mathcal{B}$" or in short "$\mathcal{B}$-full".

Let $\boldsymbol{E}$ be a topological space, let $\mathcal{B}$ be a topological group acting continuously on $\mathbf{E}$, i.e. $\theta : \mathcal{B} \times \boldsymbol{E} \to \boldsymbol{E} \times \boldsymbol{E}, (\alpha, x) \mapsto (x, \alpha x)$ is continuous. In order to simplify notations and to avoid more or less trivial problems we always assume $\boldsymbol{E}$ and $\mathcal{B}$ to be Polish. (In fact, in the sequel (§2 − §4) we shall restrict our considerations to matrix groups acting on finite dimensional vector spaces.) Let $\mathcal{M}^1(\boldsymbol{E})$ be the set of probabilities on $\boldsymbol{E}$, endowed with the topology of weak convergence. So, $\mathcal{B}$ acts continuously on $\mathcal{M}^1(\boldsymbol{E})$ in a canonical way: For $f \in C^b(\boldsymbol{E}), \mu \in \mathcal{M}^1(\boldsymbol{E}), \alpha \in \mathcal{B}$, we put $\langle \alpha\mu, f\rangle := \int f d\alpha\mu := \langle \mu, f \circ \alpha\rangle$. Define $\theta : \mathcal{B} \times \mathcal{M}^1(\boldsymbol{E}) \to \mathcal{M}^1(\boldsymbol{E}) \times \mathcal{M}^1(\boldsymbol{E})$ by $(\alpha, \mu) \mapsto (\mu, \alpha\mu)$, then $\theta$ is continuous. I.e. $\mathcal{B}$ acts as transformation group on $\mathcal{M}^1(\boldsymbol{E})$.

*Definition 1.1.* The action of $\mathcal{B}$ on $\mathcal{M}^1(\boldsymbol{E})$ defines an equivalence relation $\simeq_{\mathcal{B}}$ (in short: $\simeq$) on $\mathcal{M}^1(\boldsymbol{E}) : \mu \simeq \nu$ iff $\mathcal{B}\mu = \mathcal{B}\nu$, i.e. iff $\mu = \alpha\nu$ for some $\alpha \in \mathcal{B}$. Following the usual notations in probability theory we define *$\mathcal{B}$-type* of $\mu$ (in short: type) to be the orbit $\mathcal{B}\mu$. Furthermore, the quotient space $\mathcal{T} := \mathcal{M}^1(\boldsymbol{E})/\simeq$, endowed with the quotient topology is called *type space*. $\pi : \mathcal{M}^1(\boldsymbol{E}) \to \mathcal{T}$ denotes the canonical projection $\mu \mapsto \mathcal{B}\mu$.

Let $\mathcal{F} \subseteq \mathcal{M}^1(\boldsymbol{E})$ be $\mathcal{B}$-invariant, considered as topological space w.r.t. the induced topology. Then $\simeq$ restricted to $\mathcal{F}$ defines an equivalence relation on $\mathcal{F}$. Let $\mathcal{T}(\mathcal{F}) := \mathcal{F}/\simeq$ be the quotient space, endowed with the quotient topology, the *type space of* $\mathcal{F}$, let $\pi_{\mathcal{F}} : \mathcal{F} \to \mathcal{T}(\mathcal{F})$ be the canonical projection and let finally $\boldsymbol{i} : \mathcal{F} \to \mathcal{M}^1(\boldsymbol{E})$ resp. $\boldsymbol{j} : \mathcal{T}(\mathcal{F}) \to \mathcal{T}$ be the inclusion maps.

For $\mu \in \mathcal{M}^1(\boldsymbol{E})$ put $\mathcal{J}(\mu) := \mathcal{J}_{\mathcal{B}}(\mu) = \{\alpha \in \mathcal{B} : \alpha\mu = \mu\}$. $\mathcal{J}(\mu)$ is usually called *invariance group* resp. *symmetry group* (linear resp. affine normalizations) in probabilistic language, *isotropy group* or *stabilisator* in the language of transformation groups.

**Proposition 1.2.** *a) The map* $\theta : (\alpha, \mu) \mapsto (\mu, \alpha\mu), \mathcal{B} \times \mathcal{M}^1(\boldsymbol{E}) \to \mathcal{M}^1(\boldsymbol{E}) \times \mathcal{M}^1(\boldsymbol{E})$ *and the inclusions* $\boldsymbol{i} : \mathcal{F} \to \mathcal{M}^1(\boldsymbol{E})$ *and* $\boldsymbol{j} : \mathcal{T}(\mathcal{F}) \to \mathcal{T}$ *are continuous.*

*b) The quotient maps* $\pi$ *and* $\pi_{\mathcal{F}} : \mathcal{F} \to \mathcal{T}(\mathcal{F})$ *are continuous and open.*

*c) If* $\mathcal{F}$ *is closed or open in* $\mathcal{M}^1(\boldsymbol{E})$ *then on* $\mathcal{T}(\mathcal{F})$ *the quotient topology of* $\mathcal{F}/\simeq$ *and the induced topology of* $\mathcal{T}$ *coincide, i.e.* $\boldsymbol{j}$ *is a homeomorphism* $\boldsymbol{j} : \mathcal{T}(\mathcal{F}) \longleftrightarrow \boldsymbol{j}(\mathcal{T}(\mathcal{F})) \subseteq \mathcal{T}$. *Hence in this situation we may use the abbreviation* $\pi$ *instead of* $\pi_{\mathcal{F}}$.

*d) The quotient topology on* $\mathcal{T}(\mathcal{F})$ *is T1 iff the orbits* $\mathcal{B}\mu$ *are closed in* $\mathcal{F}$. $\mathcal{T}(\mathcal{F})$ *is a Hausdorff space iff the relation* $\simeq$ *has a closed graph* $\Delta = \{(\mu,\nu) : \mathcal{B}\mu = \mathcal{B}\nu\}$ *in* $\mathcal{F} \times \mathcal{F}$.

Proof. *a*) See e.g. [Bou] I §3 N°6. *b*), *c*) see [Bou] I §5 N°2, Propositon 3, 4; resp. III §2 N°4, Lemma 2, or [RD] Ch. 4, Lemma 4.4, 4.5, resp. [E] Ch. 2, 2.4. *d*) See [Bou] III §8 N°3, Propositon 8, resp. [E] Ch. 2, 2.4. □

Recall that a continuous map $f : X \to Y$ between topological spaces is called *proper* if for any topological space $Z$ the map $f \otimes \mathrm{id}_Z : X \times Z \to Y \times Z$ is closed $\llbracket$cf. [Bou] §10 N°1$\rrbracket$. A transformation group $G$ acts properly on a topological space $X$ iff the map $\theta : (g,x) \mapsto (x, gx), G \times X \to X \times X$, is proper $\llbracket$cf. [Bou] III §4 N°1$\rrbracket$.

*Remark.* Continuous proper maps resp. actions on a Hausdorff-space are "*perfect*" in the sense of [E] Ch. 3, 3.7. See also [RD]. Especially, proper maps $\varphi : X \to Y$ are closed and $\varphi^{-1}(K)$ is compact for any compact $K \subseteq$Y. $\llbracket$Cf. [Bou] I §10 N°2, Th. 1 or [E] Ch. 3, Th. 3.7.2.$\rrbracket$

For our purposes we prefer equivalent formulations ($G := \mathcal{B}$, $X := \mathcal{F} \subset \mathcal{M}^1(\boldsymbol{E})$):

**Proposition 1.3.** $\mathcal{B}$ *acts properly (or perfectly) on* $\mathcal{F}$ *iff*

*a)* $\theta : (\alpha,\mu) \mapsto (\mu, \alpha\mu)$ *is a closed map and if the invariance group* $\mathcal{J}(\mu)$ *(cf. Definition 1.1) is compact for* $\mu \in \mathcal{F}$, *equivalently, iff*

*b) for any sequence* $(\alpha_n, \mu_n)$ *in* $\mathcal{B} \times \mathcal{F}$ *such that* $(\mu_n, \alpha_n\mu_n) \to (\mu, \nu) \in \mathcal{F} \times \mathcal{F}$ *the sequence* $(\alpha_n)$ *is relatively compact in* $\mathcal{B}$.

*And then for any accumulation point* $\alpha$ *of* $(\alpha_n)$ *we have* $\alpha\mu = \nu$.

Proof. *a*) See e.g. [Bou] I §10 N°2, Théorème 1: Note that $\mathcal{B}$ and $\mathcal{F}$ are supposed to be Hausdorff spaces, hence quasi-compact sets are compact, and furthermore that $\theta^{-1}(\mu,\nu) = \{(\beta,\mu) : \beta\mu = \nu\} = \beta_0\mathcal{J}(\mu) \times \{\mu\}$ for some $\beta_0 \in \mathcal{B}$ with $\beta_0\mu = \nu$.
*b*) See [Bou] III §4 N°1, N°2, Proposition 4. See also [E] Ch. 3, Theorem 3.7.13 □

For further use we collect some facts concerning properly acting transformation groups:

**Theorem 1.4.** *Assume $\mathcal{B}$ to act properly on $\mathcal{F}$. Then*

a) *$\mathcal{T}(\mathcal{F})$ is Hausdorff, especially the orbits $\mathcal{B}\mu$ are closed in $\mathcal{F}$ for any $\mu \in \mathcal{F}$*
b) *for any $\mu \in \mathcal{F}$ the map $\alpha \mapsto \alpha\mu : \mathcal{B} \to \mathcal{F}$ is proper*
c) *moreover we have $\mathcal{B}/\mathcal{J}_{\mathcal{B}}(\mu) \cong \mathcal{B}\mu$, where*
d) *$\mathcal{J}_{\mathcal{B}}(\mu)$ is a compact group*
e) *for any compact set $K \subseteq \mathcal{F}$ and any $\mu \in \mathcal{F}$ the set $\{\alpha \in \mathcal{B} : \alpha\mu \in K\}$ is compact in $\mathcal{B}$.*
f) *for any compact $K, L \subseteq \mathcal{F}$ the set $\{(\alpha, \mu) \in \mathcal{B} \times \mathcal{F} : \mu \in L$ and $\alpha\mu \in K\}$ is compact.*
g) *for any compact $K \subseteq \mathcal{F}$ and closed $F \subseteq \mathcal{B}$ the set $F.K$ is closed in $\mathcal{F}$.*
h) *If $\mathcal{B}$ is locally compact we have: $\mathcal{B}$ acts properly on $\mathcal{F}$ iff for any $\mu, \nu \in \mathcal{F}$ there exist neighbourhoods $V(\mu), W(\nu)$ such that $\{\alpha \in \mathcal{B} : \alpha V(\mu) \cap W(\nu) \neq \emptyset\}$ is relatively compact.*

Proof. *a*)–d) See e.g. [Bou] III §4 N°2, Proposition 3,4. *e*) is a consequence of *b*) and the remark below 1.2.

f) By assumption $\theta : (\alpha, \mu) \mapsto (\mu, \alpha\mu)$ is proper hence $\theta^{-1}(L \times K)$ is compact.

g) See [Bou] III §4 N°5, Corrollaire. *h*) See [Bou] III §4 N°4, Prop. 4. □

For further properties of proper (perfect) maps see also [E] Ch. 3, 3.7.

Following again the usual notations in probability theory we are led to

*Definition 1.5.* $(\mathcal{B}, \mathcal{F})$ fulfil the *convergence of types condition* if for any sequence $(\mu_n) \subseteq \mathcal{F}, (\alpha_n) \subseteq \mathcal{B}$ such that $\mu_n \to \mu$ and $\alpha_n\mu_n \to \nu$

(C-T) with $\mu, \nu \in \mathcal{F}$ the sequence $(\alpha_n)$ is relatively compact in $\mathcal{B}$. Then, for any accumulation point $\alpha$ of $(\alpha_n)$ we have $\alpha\mu = \nu$

*Remark 1.6.* In most applications $\mathcal{F}$ will be an open subset. Then (C-T) is equivalently formulated as follows:

(C-T*) Let $(\mu_n) \subseteq \mathcal{M}^1(\boldsymbol{E}), (\alpha_n) \subseteq \mathcal{B}, \mu_n \to \mu, \nu_n := \alpha_n\mu_n \to \nu$. If $\mu$ and $\nu \in \mathcal{F}$ then $(\alpha_n)$ is relatively compact in $\mathcal{B}$.

In view of Proposition 1.3.b) we obtain the following translation from the language of transformation groups into probabilistic language:

*Remark 1.7.* $\mathcal{B}$ acts properly respectively perfectly on $\mathcal{F}$ iff $(\mathcal{B}, \mathcal{F})$ fulfil the condition (C-T). Hence e.g. the assertions of Theorem 1.4 hold if (C-T) is fulfilled.

Let $(\mu_n), \mu, \nu \subseteq \mathcal{F}, (\alpha_n) \subseteq \mathcal{B}$ and $\alpha_n\mu_n \to \nu$. Then, $\pi_{\mathcal{F}}$ being continuous, the types $\pi_{\mathcal{F}}(\mu_n)$ converge to $\pi_{\mathcal{F}}(\mu)$ in the topology of the type space $\mathcal{T}(\mathcal{F})$, and conversely. Indeed, the following observation will help to understand better the notion *"convergence of types":*

**Theorem 1.8.** *Let $\mathcal{F} \subseteq \mathcal{M}^1(\boldsymbol{E})$. Assume $\pi_{\mathcal{F}}(\mu_n) \to \pi_{\mathcal{F}}(\mu)$ in the type space $\mathcal{T}(\mathcal{F})$.*

- *a) Then there exist $(\alpha_n) \subseteq \mathcal{B}$ such that $\alpha_n\mu_n \to \mu$.*
- *b) If $\mathcal{T}(\mathcal{F})$ is a Hausdorff space then for any sequence $(\beta_n) \subseteq \mathcal{B}$ such that $\beta_n\mu_n \to \nu \in \mathcal{F}$ we obtain $\nu = \beta\mu$ for some $\beta \in \mathcal{B}$, i.e. $\nu$ and $\mu$ belong to the same type.*
- *c) If moreover $(\mathcal{B}, \mathcal{F})$ fulfil (C-T) then $\alpha_n\mu_n \to \mu \in \mathcal{F}$ and $\beta_n\mu_n \to \nu \in \mathcal{F}$ imply that $\pi_{\mathcal{F}}(\mu) = \pi_{\mathcal{F}}(\nu)$ and that $(\beta_n\alpha_n^{-1})$ is relatively compact, the accumulation points belonging to $\{\gamma \in \mathcal{B} : \gamma\mu = \nu\} = \gamma_0\mathcal{J}(\mu)$, for some $\gamma_0 \in \mathcal{B}$.*

$\big[$*Note that obviously we have $\pi_{\mathcal{F}}(\alpha_n\mu_n) \to \pi_{\mathcal{F}}(\mu)$ and $\pi_{\mathcal{F}}(\beta_n\mu_n) \to \pi_{\mathcal{F}}(\nu)$ in* a), b) *and* c).$\big]$

Proof. *a)* $\pi_{\mathcal{F}}$ is continuous and open (Proposition 1.2. b)). Therefore we can choose neighbourhood bases $\mathcal{W}$ of $\mu$ in $\mathcal{F}$ and $\mathcal{V} := \{\pi_{\mathcal{F}}W : W \in \mathcal{W}\}$ of $\pi(\mu)$ in $\mathcal{T}(\mathcal{F})$. We write in short $\pi := \pi_{\mathcal{F}}$. Given $W \in \mathcal{W}$ there exists $n = n_W \in \boldsymbol{N}$, such that for $n \geq n_W$ we have $\pi\mu_n \in V = \pi W$. Hence $\mathcal{B}\mu_n \subseteq \mathcal{B}W$, therefore we have $\alpha_n\mu_n \in W$ for some $\alpha_n \in \mathcal{B}$ and for sufficiently large $n > n_W$. Considering a sequence of neighbourhoods $W_m \searrow \{\mu\}$, hence $\pi W_m \searrow \{\pi\mu\}$, we can find a suitable sequence $(\alpha_n)$ in $\mathcal{B}$ such that $\alpha_n\mu_n \to \mu$.

*b)* Assume moreover $\beta_n\mu_n \to \nu$, then $\pi\mu_n \to \pi\nu$. If $\mathcal{T}(\mathcal{F})$ is Hausdorff then $\pi\mu = \pi\nu$, i.e. $\mathcal{B}\mu = \mathcal{B}\nu$. Hence $\nu = \beta\mu$ for some $\beta \in \mathcal{B}$.

*c)* We have $\alpha_n\mu_n \to \mu$ and $(\beta_n\alpha_n^{-1})\alpha_n\mu_n \to \nu$, whence by (C-T) relative compactness of $(\beta_n\alpha_n^{-1})$ follows. The rest assertions of c) follow immediately. □

Finally, we note the following consequence of the (C-T) condition resp. of Theorem 1.4. The proof is left to the reader.

**Corollary 1.9.** *Assume $(\mathcal{B}, \mathcal{F})$ to fulfil* (C-T). *Then, for a compact set $K \subseteq \mathcal{F}$ the set $\bigcup_{\mu \in K} \mathcal{J}_{\mathcal{B}}(\mu)$ is compact in $\mathcal{B}$.* $\big[$*Cf. Theorem 1.4.*f)$\big]$. *Therefore, if $\mu_n \to \mu$ in $\mathcal{F}$, then $\mathcal{J}_{\mathcal{B}}(\mu_n) \to \mathcal{J}_{\mathcal{B}}(\mu)$. (Convergence in $\mathcal{C}(\mathcal{B})$, the set of compact subsets of $\mathcal{B}$).*

## §2 Affine and linear normalizations

Let $\boldsymbol{E} \cong \boldsymbol{R}^d$ be a finite dimensional real vector space, let $\mathbf{End}(\boldsymbol{E})$ respectively $\mathbf{Aff}(\boldsymbol{E})$ denote the semigroups of linear respectively affine transformations on $\boldsymbol{E}$, furthermore let $\mathbf{Gl}(\boldsymbol{E})$ respectively $\mathbf{AG}(\boldsymbol{E})$ be the linear group of $\boldsymbol{E}$ respectively the group of invertible affine transformations endowed with the natural topology. Let $\mathcal{B}$ be a closed subgroup of $\mathbf{Gl}(\boldsymbol{E})$ $\big[$respectively of $\mathbf{AG}(\boldsymbol{E})\big]$, hence acting in canonical way as transformation group on $\boldsymbol{E}$.

In probability theory it is usual to distinguish between linear and affine normalizations. (See e.g. the definitions of stability and strict stability in [Sh], [JM], [HMV], [Lu], [S], [Ja].) We show that from the point of view of convergence-of-types theorems this distinction is not essential:

*Definition 2.1.* Let $\boldsymbol{E} \cong \boldsymbol{R}^d, \boldsymbol{A} := \boldsymbol{E} \oplus \boldsymbol{R} \cong \boldsymbol{R}^{d+1}$. Let $\Psi$ be the affine embedding $\boldsymbol{E} \to \boldsymbol{A}, x \mapsto (x, 1)$, defining a canonical affine embedding $\Phi : \mathbf{Aff}(\boldsymbol{E}) \to \mathbf{End}(\boldsymbol{A})$ $[\mathbf{AG}(\boldsymbol{E}) \to \mathbf{Gl}(\boldsymbol{A})]$ via $\Phi(\gamma)(\Psi x) = \Psi(\gamma x)$, $\gamma \in \mathbf{Aff}(\boldsymbol{E})$, $x \in \boldsymbol{E}$ $\big[$respectively $\gamma \in \mathbf{AG}(\boldsymbol{E})$, $x \in \boldsymbol{E}$.$\big]$

The embedding $\Psi$ of $\boldsymbol{E}$ in into $\boldsymbol{A}$ induces a topological isomorphism between $\mathcal{M}^1(\boldsymbol{E})$ and $\Psi(\mathcal{M}^1(\boldsymbol{E})) = \{\rho \in \mathcal{M}^1(\boldsymbol{A}) : \operatorname{supp}(\rho) \subseteq \Psi(\boldsymbol{E}) = \boldsymbol{E} \oplus \{1\}\}$ endowed with the induced topology of $\mathcal{M}^1(\boldsymbol{A})$.

*Definition 2.2.* Let $\mathcal{F} \subseteq \mathcal{M}^1(\boldsymbol{E})$, let $\mathcal{C}$ be a subset $\subseteq \mathbf{End}(\boldsymbol{E})$ [resp. $\mathcal{C} \subseteq \mathbf{Aff}(\boldsymbol{E})$].

$(\mathcal{C}, \mathcal{F})$ fulfil the *weak convergence of types condition* (W-C-T) if for $\mu_n, \mu, y\nu \in \mathcal{M}^1(\boldsymbol{E})$ the conditions

(W-C-T) $(\alpha_n) \subseteq \mathcal{C}, \mu_n \to \mu$, $\alpha_n \mu_n \to \nu$ and $\mu \in \mathcal{F}$ imply that $(\alpha_n)$ is relatively compact (i.e. bounded) in $\mathbf{End}(\boldsymbol{E})$ $\big[$resp. in $\mathbf{Aff}(\boldsymbol{E})\big]$.

If $\mathcal{C} = \mathcal{B}$ is a group we have

**Lemma 2.3.** *Let $\mathcal{B}$ be a closed subgroup of $\mathbf{Gl}(\boldsymbol{E})$ respectively $\mathbf{AG}(\boldsymbol{E})$, let $\mathcal{F}$ be $\mathcal{B}$ – invariant in $\mathcal{M}^1(\boldsymbol{E})$. Then $(\mathcal{B}, \mathcal{F})$ fulfil* (C-T) *iff* (W-C-T) *is fulfilled.*

Proof. Assume (W-C-T) to hold. Let $\mu_n \to \mu$, $\alpha_n\mu_n \to \nu$, $\mu_n, \mu, \nu \in \mathcal{F}$. Apply (W-C-T) to "$\mu_n \to \mu$, $\alpha_n\mu_n \to \nu$, $\mu \in \mathcal{F}$" then boundedness of $(\alpha_n)$ follows. On the other hand apply (W-C-T) to "$\nu_n := \alpha_n\mu_n \to \nu$, $\mu_n := \alpha_n^{-1}(\nu_n) \to \mu$, $\nu \in \mathcal{F}$" then boundedness of $(\alpha_n^{-1})$ follows.

Conversely, (C-T) implies (W-C-T), as easily seen. □

The simple Lemma 2.3 turns out to be extremly useful. E.g. with the notations above (2.1–2.3) we obtain immediately:

**Proposition 2.4.** *Let $\mathcal{B} \subseteq \mathbf{AG}(\boldsymbol{E})$ be a closed subgroup, let $\mathcal{F} \subseteq \mathcal{M}^1(\boldsymbol{E})$ be $\mathcal{B}$-invariant. Put $\mathcal{B}^* := \Phi(\mathcal{B}) \subseteq \mathbf{Gl}(\boldsymbol{A})$ and $\mathcal{F}^* := \Psi(\mathcal{F}) \subseteq \mathcal{M}^1(\boldsymbol{A})$, where $\boldsymbol{A}, \Phi, \Psi$ are defined as in* 2.1. *Then $(\mathcal{B}, \mathcal{F})$ fulfil* (C-T) *iff $(\mathcal{B}^*, \mathcal{F}^*)$ do.*

Proof. By Lemma 2.3 it suffices to check only the weak condition (W-C-T) for $(\mathcal{B}, \mathcal{F})$ resp. for $(\mathcal{B}^*, \mathcal{F}^*)$. Whence the assertion immediately follows. □

Therefore, in the following we will restrict our considerations to closed subgroups $\mathcal{B}$ of the *linear group* $\mathbf{Gl}(\boldsymbol{E})$ acting on $\boldsymbol{E} \cong \boldsymbol{R}^d$

**Proposition 2.5.** *Let $\mathcal{B}$ be a non-compact closed subgroup of $\mathbf{Gl}(\boldsymbol{E})$ [resp. $\mathbf{AG}(\boldsymbol{E})$]. Then there exists a unique maximal open subset $\mathcal{F}_{\mathcal{B}} = \mathcal{F} \subseteq \mathcal{M}^1(\boldsymbol{E})$, such that $(\mathcal{B}, \mathcal{F})$ fulfil the convergence of types condition* (C-T).

Proof. Let $\mathcal{V} := \{\mathcal{F} \subseteq \mathcal{M}^1(\boldsymbol{E}) : \mathcal{F}$ is open and $(\mathcal{B}, \mathcal{F})$ fulfil (W-C-T)$\}$. Then obviously, $\mathcal{V}$ is stable w.r.t. finite unions and for any directed chain $(\mathcal{F}_\alpha) \subseteq \mathcal{V}$ we have $\bigcup \mathcal{F}_\alpha \in \mathcal{V}$. Hence, by Zorn's lemma, there exists a unique maximal element $\mathcal{F}^o$ in $\mathcal{V}$. Now Lemma 2.3 yields that $(\mathcal{B}, \mathcal{F}^o)$ fulfil (C-T). □

Indeed, the assertion of Proposition 2.5. is valid in more general situations considered in §1. But here we are mainly interested to obtain *explicite* constructions and descriptions of classes of $\mathcal{B}$-full measures $\mathcal{F}_{\mathcal{B}}$. This will be done in the following §3 and 4. In most examples the constructed $\mathcal{F}_{\mathcal{B}}$ will turn out to be maximal for the given group $\mathcal{B}$.

### §3 A method to construct full measures, given $\mathcal{B}$

Let $\boldsymbol{E} \cong \boldsymbol{R}^d, \boldsymbol{F} \cong \boldsymbol{R}^s$ be finite dimensional vector spaces. Put $\mathcal{F}_0(\boldsymbol{E}) := \mathcal{M}^1(\boldsymbol{E})\backslash\{\varepsilon_0\}$ and $\mathcal{F}_1(\boldsymbol{E}) := \{\mu : \mu$ not concentrated on a proper linear subspace$\}$, furthermore let $\mathcal{F}_0^s := \{\mu : \mu * \tilde{\mu} \in \mathcal{F}_0\}$ and define $\mathcal{F}_1^s$ analogously.

Note that Sharpe's "full measures" (cf. [Sh]) – called non-degenerate in [B] – are defined to be not concentrated on proper hyperplanes, hence are just the measures belonging to $\mathcal{F}_1^s(\boldsymbol{E})$.

Let $\mathcal{H}(\boldsymbol{E}) := \{x \mapsto tx : t > 0\}$ be the group of homothetical transformations on $\boldsymbol{E}$. It is well known that $(\mathbf{Gl}(\boldsymbol{E}), \mathcal{F}_1(\boldsymbol{E}))$ resp. $(\mathbf{AG}(\boldsymbol{E}), \mathcal{F}_1^s(\boldsymbol{E}))$ as well as $(\mathcal{H}(\boldsymbol{E}), \mathcal{F}_0(\boldsymbol{E}))$ fulfil (C-T). $\llbracket$See e.g. [Sh], resp. [K], [L], [GK]. Note that a proof for homothetical transformations on $\boldsymbol{R}^d$ is identical with a proof for probabilities on the real line. See also 4.4.a) below.$\rrbracket$

Remark that obviously

$$\mathcal{F}_1(\boldsymbol{E}) = \bigcap_{\varphi \in \boldsymbol{E}'\backslash\{0\}} \varphi^{-1}(\mathcal{F}_0(\boldsymbol{R})) = \bigcap_{\varphi \in \boldsymbol{E}'\backslash\{0\}} \{\mu \in \mathcal{M}^1(\boldsymbol{E}) : \varphi(\mu) \neq \varepsilon_0\}.$$

Therefore we define in an analogous way for an auxiliary space $\boldsymbol{F} \cong \boldsymbol{R}^s$:

*Definition 3.1.* Let $\boldsymbol{H} \subseteq \mathbf{Lin}(\boldsymbol{E}, \boldsymbol{F})\backslash(0), \boldsymbol{H} \neq \emptyset$. Then we define: $\mathcal{F}_{\boldsymbol{H}} := \mathcal{F}_{\boldsymbol{H}}(\boldsymbol{E}) := \{\mu \in \mathcal{M}^1(\boldsymbol{E}) : \varphi(\mu) \neq \varepsilon_0, \varphi \in \boldsymbol{H}\} = \bigcap_{\varphi \in \boldsymbol{H}} \varphi^{-1}(\mathcal{F}_0(\boldsymbol{F}))$. For latter use we define further:

$$\mathcal{F}_{\boldsymbol{H}}^s := \mathcal{F}_{\boldsymbol{H}}^s(\boldsymbol{E}) := \{\mu \in \mathcal{M}^1(\boldsymbol{E}) : \mu * \tilde{\mu} \in \mathcal{F}_{\boldsymbol{H}}\}.$$

For a subset $A \subseteq \mathbf{Lin}(\boldsymbol{E}, \boldsymbol{F})\backslash\{0\}$ put $\Gamma_A := \left\{\frac{1}{\|\varphi\|}\varphi : \varphi \in A\right\}$ and $K_A := \{t\varphi : t > 0, \varphi \in A\}$. Obviously we have $\mathcal{F}_{\boldsymbol{H}} = \mathcal{F}_{\Gamma_{\boldsymbol{H}}} = \mathcal{F}_{K_{\boldsymbol{H}}}$.

The subsequent conditions (3.1) and (3.2) are always supposed to be fulfilled:

(3.1) $\Gamma_{\boldsymbol{H}}$ is assumed to be closed, hence compact

(3.2) and $K_{\boldsymbol{H}}$ to be $\mathcal{B}$-invariant $K_{\boldsymbol{H}} = K_{\boldsymbol{H}\mathcal{B}}$ (hence $\Gamma_{\boldsymbol{H}} = \Gamma_{\boldsymbol{H}\mathcal{B}}$)

(where $\mathcal{B} \subseteq \mathbf{End}(\boldsymbol{E})$ acts in canonical way on $\mathbf{Lin}(\boldsymbol{E}, \boldsymbol{F})$ from the right).

According to the notations introduced in previous investigations, esp. in [Sh], we call probabilities belonging to $\mathcal{F}_{\boldsymbol{H}}$ [resp. $\mathcal{F}_{\boldsymbol{H}}^s$] $\mathcal{B}$-*full* [resp. $\mathcal{B}$-*shift-full*, in short $\mathcal{B}$-**S**-*full*]. Note that in the definition of full measures above $\mathcal{F}_{\boldsymbol{H}}(\boldsymbol{E})$ and $\mathcal{B} \subseteq \mathbf{End}(\boldsymbol{E})$ are connected by $\boldsymbol{H}$ and $\boldsymbol{F}$ via the conditions (3.1) and (3.2). Hence we frequently use the notation $\mathcal{F}_{\boldsymbol{H}} =: \mathcal{F}_{\mathcal{B}}$ and call the measures in short $\mathcal{B}$-full, or full w.r.t. $\mathcal{B}$.

*Remark.* Obviously, $\mathcal{F}_0(\boldsymbol{E}) = \mathcal{F}_{\boldsymbol{H}}(\boldsymbol{E})$, where $\boldsymbol{F} := \boldsymbol{E}$, $\boldsymbol{H} := \{\mathrm{id}\}$, and $\mathcal{F}_1(\boldsymbol{E}) = \mathcal{F}_{\boldsymbol{H}}(\boldsymbol{E})$, where $\boldsymbol{F} := \boldsymbol{R}$, $\boldsymbol{H} := \boldsymbol{E}'\backslash\{0\}$. Furthermore, as easily seen $\mathcal{F}_0^s(\boldsymbol{E}) = \mathcal{M}^1(\boldsymbol{E})\backslash\{\varepsilon_x : x \in \boldsymbol{E}\}$ and $\mathcal{F}_1^s(\boldsymbol{E}) = \{\mu : \mu$ not concentrated on a hyperplane$\}$. Therefore we obtain

**Proposition 3.2.** *a) $\mathcal{F}_{\boldsymbol{H}}^s(\boldsymbol{E})$ is an open ideal in the convolution semigroup $\mathcal{M}^1(\boldsymbol{E})$ and $\mathcal{F}_{\boldsymbol{H}}(\boldsymbol{E})$ is an open subset containing $\mathcal{F}_{\boldsymbol{H}}^s(\boldsymbol{E})$.*

*b) Furthermore, the inclusion $\mathcal{F}_{\boldsymbol{H}}^s \supseteq \{\mu \in \mathcal{F}_{\boldsymbol{H}} : 0 \in \operatorname{supp}(\mu)\}$ holds true.*

*c) Let $\mu \in \mathcal{F}_{\boldsymbol{H}}$ $\left[\text{resp. } \mathcal{F}_{\boldsymbol{H}}^s\right]$. Then $\{\nu \in \mathcal{M}^1(\boldsymbol{E}) : \operatorname{supp}\nu \supseteq \operatorname{supp}\mu\} \subseteq \mathcal{F}_{\boldsymbol{H}}$ $\left[\text{resp. } \subseteq \mathcal{F}_{\boldsymbol{H}}^s\right]$.*

*d) $\mathcal{F}_{\boldsymbol{H}}^s(\boldsymbol{E}) := \{\mu : \varphi(\mu) \neq \varepsilon_x$ for all $x \in \boldsymbol{F}, \varphi \in \boldsymbol{H}\} = \bigcap_{\varphi\in\boldsymbol{H}} \varphi^{-1}(\mathcal{F}_0^s(\boldsymbol{F})) = \{\mu : \psi(\mu) \neq \varepsilon_0$ for all affine $\psi : x \mapsto \varphi x + a, \varphi \in \boldsymbol{H}, a \in \boldsymbol{E}\}$.*

Proof. *a)* As easily seen, $\mathcal{F}_0^s(\boldsymbol{F}) = \mathcal{M}^1(\boldsymbol{E})\backslash\{\varepsilon_x : x \in \boldsymbol{E}\}$ is an open ideal in $M^1(\boldsymbol{F})$ and $\mathcal{F}_0(\boldsymbol{F})$ is open, $\mathcal{F}_0(\boldsymbol{F}) \supseteq \mathcal{F}_0^s(\boldsymbol{F})$. Hence $\varphi^{-1}(\mathcal{F}_0(\boldsymbol{F}))$ and $\varphi^{-1}(\mathcal{F}_0^s(\boldsymbol{F}))$ are open in $\mathcal{M}^1(\boldsymbol{E})$ for any $\varphi \in \boldsymbol{H}$, furthermore $\varphi^{-1}(\mathcal{F}_0^s(\boldsymbol{F}))$ is an ideal. Hence $\bigcap_{\varphi\in\boldsymbol{H}} \varphi^{-1}(\mathcal{F}_0^s(\boldsymbol{F}))$ is an ideal.

Further, the representations $\mathcal{F}_{\boldsymbol{H}}(\boldsymbol{E}) = \bigcap_{\varphi\in\boldsymbol{H}} \varphi^{-1}(\mathcal{F}_0(\boldsymbol{F}))$ and $\mathcal{F}_0^s(\boldsymbol{F}) = \{\mu : \mu * \tilde{\mu} \in \mathcal{F}_0(\boldsymbol{F})\}$ show that $\mathcal{F}_{\boldsymbol{H}}^s(\boldsymbol{E}) = \bigcap_{\varphi\in\boldsymbol{H}} \varphi^{-1}(\mathcal{F}_0^s(\boldsymbol{F}))$.

Hence we have to show that $\mathcal{F}_{\boldsymbol{H}}(\boldsymbol{E})$ and $\mathcal{F}_{\boldsymbol{H}}^s(\boldsymbol{E})$ are open. Since we may assume $\boldsymbol{H} = \Gamma_{\boldsymbol{H}}$ to be compact, assertion a) is a consequence of the following simple

**Lemma.** *Let $K, F, G$ be Hausdorff spaces, let $K$ be compact and $\phi : K \times F \to G$ continuous. Then for any open $W \subseteq G$ the set $\bigcap_{k\in K}\{f \in F : \phi(k, f) \in W\}$ is open in $F$.*

*b)* Obviously, $\mathcal{F}_0^s(\boldsymbol{F}) \supseteq \{\mu \in \mathcal{F}_0(\boldsymbol{F}) : 0 \in \operatorname{supp}(\mu)\}$. Let $\mu \in \mathcal{F}_{\boldsymbol{H}}(\boldsymbol{E})$ and assume $0 \in \operatorname{supp}(\mu)$. Then by definition, for $\varphi \in \boldsymbol{H}$ we have $\varphi(\mu) \in \mathcal{F}_0(\boldsymbol{F})$ and $0 = \varphi(0) \in \operatorname{supp}(\varphi(\mu))$. Hence $\varphi(\mu) \in \mathcal{F}_0^s(\boldsymbol{F})$ and therefore $\mu \in \bigcap_{\varphi\in\boldsymbol{H}} \varphi^{-1}(\mathcal{F}_0^s(\boldsymbol{F})) = \mathcal{F}_{\boldsymbol{H}}^s$.

The assertions c) and d) are proved in a similar way. □

The next – nearly obvious – result is, together with Proposition 2.4 and 4.4, the key to our construction of suitable classes of full measures fulfilling the convergence-of-types condition (C-T) for a given group $\mathcal{B}$:

**Proposition 3.3.** *Let $\mathcal{B} \subseteq \mathbf{End}(\boldsymbol{E}), \boldsymbol{E} \cong \boldsymbol{R}^d$, let $\boldsymbol{F} \cong \boldsymbol{R}^k$ be an auxiliary space. Furthermore, let $\boldsymbol{H} \subseteq \mathbf{Lin}(\boldsymbol{E}, \boldsymbol{F})\backslash\{0\}$ such that $\mathcal{F}_{\boldsymbol{H}}$ and $\mathcal{B}$ fulfil* (3.1) *and* (3.2).

Then $\mu_n \to \mu, \alpha_n\mu_n \to \nu, \mu \in \mathcal{F}_{\boldsymbol{H}}$ imply the boundedness property

$$\text{(W-C-T}^*\text{)} \qquad \sup_n \sup_{\varphi\in\Gamma_{\boldsymbol{H}}} \left\|\varphi\alpha_n\right\| < \infty.$$

(And hence, if for given $\mathcal{B}$ the set $\boldsymbol{H}$ is suitably chosen (W-C-T$^*$) implies (W-C-T). Cf. the examples in §4.)

Proof. Let $\mu_n, \mu, \nu \in \mathcal{M}^1(\boldsymbol{E}), \mu_n \to \mu, \alpha_n\mu_n \to \nu$, where $(\alpha_n) \subseteq \mathcal{B}$. Assume $\mu \in \mathcal{F}_{\boldsymbol{H}}$, and suppose w.l.o.g. $\boldsymbol{H} = K_{\boldsymbol{H}}$ and hence $\Gamma_{\boldsymbol{H}} = \Gamma_{\boldsymbol{H}\mathcal{B}}$ according to (3.1) and (3.2). $\Gamma_{\boldsymbol{H}}$ being compact, there exist $\varphi_n \in \Gamma_{\boldsymbol{H}}$ such that $\|\varphi\alpha_n\|$ attains the maximum at $\varphi_n$. Assume $\left\|\varphi_n\alpha_n\right\| \to \infty$, and assume w.l.o.g. $\varphi_n \to \varphi \in \Gamma_{\boldsymbol{H}}$. Then $\left\{\psi_n := \frac{1}{\left\|\varphi_n\alpha_n\right\|}\varphi_n\alpha_n\right\}_{n\geq 1}$ has an accumulation point $\psi^- \in \Gamma_{\boldsymbol{H}}$. But $\varphi_n\alpha_n\mu_n \to \varphi\nu$, and $1/\parallel \varphi_n\alpha_n \parallel\searrow 0$, therefore $\psi^-(\mu) = \varepsilon_0$ in contradiction to the assumption $\mu \in \mathcal{F}_{\boldsymbol{H}}$. We have proved that $\{\left\|\varphi\alpha_n\right\|\}$ is bounded for $\varphi \in \boldsymbol{H}$. The assertion follows. □

Note that in case $\boldsymbol{F} = \boldsymbol{R}, \boldsymbol{H} = \boldsymbol{E}'\backslash\{0\}$ this is just the usual proof of the convergence of types theorem.

The role of the classes $\mathcal{F}^s_{\boldsymbol{H}}$ of $S$-full measures is illustrated by the following

**Proposition 3.4.** *Let $\boldsymbol{E}, \mathcal{B}, \boldsymbol{H}$ as above. Define $\mathcal{B}_{\text{aff}} := \{\beta : x \mapsto \alpha x + c : \alpha \in \mathcal{B}, c \in \boldsymbol{E}\}$ the semidirect product of $\mathcal{B}$ and $\boldsymbol{E}$. Let $\mu_n, \mu, \nu \in \mathcal{M}^1(\boldsymbol{E})$, let further $\mu, \nu \in \mathcal{F}^s_{\boldsymbol{H}}$, and let $(\beta_n : x \mapsto \alpha_n x + c_n) \subseteq \mathcal{B}_{\text{aff}}$.*

Assume $\mu_n \to \mu, \beta_n\mu_n \to \nu$. Then
(a) $\sup_n \sup_{\varphi\in\Gamma_{\boldsymbol{H}}} \left\|\varphi\alpha_n\right\|$ and (b) $\sup_n \sup_{\varphi\in\Gamma_{\boldsymbol{H}}} \left\|\varphi(c_n)\right\|$ (S-W-C-T$^*$)
are finite.

Proof. (a) Apply 3.3 to the sequence $\mu_n * \tilde{\mu}_n$. Then we have $\mu_n * \tilde{\mu}_n \to \mu * \tilde{\mu}$, and $(\beta_n\mu_n) * (\beta_n\tilde{\mu}_n) = \alpha_n(\mu_n * \tilde{\mu}_n) \to \nu * \tilde{\nu}$ . Hence assertion (a) follows.

(b) To prove (b), observe according to (a) that$\{\varphi\alpha_n\mu_n : n \in \boldsymbol{N}, \varphi \in \Gamma_{\boldsymbol{H}}\}$ is relatively compact. On the other hand, $\Gamma_{\boldsymbol{H}}$ being compact and $\beta_n\mu_n = \alpha_n\mu_n * \varepsilon_{c_n} \to \nu$, hence $\{\varphi\alpha_n\mu_n * \varepsilon_{\varphi(c_n)} : n \in \boldsymbol{N}, \varphi \in \Gamma_{\boldsymbol{H}}\}$ is relatively compact. Therefore, boundedness of the norms $\{\left\|\varphi(c_n)\right\| : n \in \boldsymbol{N}, \varphi \in \Gamma_{\boldsymbol{H}}\}$ follows. □

*Remark 3.5.* With the notations introduced in §2 we obtain : Let $\Psi$ be the affine embedding of $\boldsymbol{E}$ into $\boldsymbol{A}$ and let $\mathcal{B}_{\text{aff}} := \{x \mapsto \alpha x + c : \alpha \in \mathcal{B}, c \in \boldsymbol{E}\}$ be the corresponding affine group. Then $\Phi(\mathcal{B}_{\text{aff}}) =: \mathcal{C} \subseteq \mathbf{Gl}(\boldsymbol{A})$. Let $\boldsymbol{H} \subseteq \mathbf{Lin}(\boldsymbol{E}, \boldsymbol{F})$, put $\boldsymbol{K} := \{(x,t) \mapsto \varphi(x) : \varphi \in \boldsymbol{H}, t \in \boldsymbol{R}\} \subseteq \mathbf{Lin}(\boldsymbol{A}, \boldsymbol{F})$. Then $\Psi(\mathcal{F}^s_{\boldsymbol{H}}(\boldsymbol{E})) = \mathcal{F}_{\boldsymbol{K}}(\boldsymbol{A}) \cap \Psi(\mathcal{M}^1(\boldsymbol{E}))$.

This again justifies our restriction to linear groups.

## §4 Examples

In the following we show for concrete examples of groups of admissible norming automorphisms $\mathcal{B}$ that the methods above enable us to find suitable classes of full measures such that the "*convergence of types theorem*" (in short: C-T-T) holds. The examples appear to be quite natural in connection with investigations of operator stability and semistability and related problems. Again $\boldsymbol{E} \cong \boldsymbol{R}^d$ is a fixed finite dimensional vector space and $\mathcal{B} \subseteq \mathbf{End}(\boldsymbol{E})$.

**4.1. C-T-T for endomorphisms.**
Let $\mathcal{B} \subseteq \mathbf{End}(E)$ be closed, e.g. $\mathcal{B} = \mathbf{End}\ (\boldsymbol{E})$. $\mathcal{B}$ need not to be a group here.) Put $\boldsymbol{F} := \boldsymbol{R}$, $\boldsymbol{H} := \boldsymbol{E}' \backslash \{0\}$, hence $\mathcal{F}_{\boldsymbol{H}}(\boldsymbol{E}) = \mathcal{F}_1(\boldsymbol{E}) := \{\mu :$ $\mu$ not concentrated on a proper linear subspace$\}$. Then, according to Proposition 3.2, $\mathcal{F}_{\boldsymbol{H}}$ is open in $\mathcal{M}^1(\boldsymbol{E})$ and (Prop. 3.3) the weak condition (W-C-T$^*$) – and hence by the choice of $\boldsymbol{H}$ also (W-C-T) – is fulfilled. And $\mathcal{F}_1^s(\boldsymbol{E})$ fulfils the (W-C-T) condition for the corresponding set of affine transformations $\mathcal{B}_{\text{aff}} := \{x \mapsto \beta x + c : \beta \in \mathcal{B},\ c \in \boldsymbol{E}\}$ (Prop. 3.4).
⟦Cf. e.g. [U], [JM] Ch. 2 for the case of affine normalizations $\mathcal{B} = \mathbf{Aff}(\boldsymbol{E})$.⟧

**4.2. C-T-T for the linear group.**
Let $\mathcal{B} = \mathbf{Gl}(\boldsymbol{E})$. Put again $\boldsymbol{F} := \boldsymbol{R}$, $\boldsymbol{H} := \boldsymbol{E}' \backslash (0)$, hence $\mathcal{F}_{\boldsymbol{H}} = \mathcal{F}_1(\boldsymbol{E})$. According to 4.1 (W-C-T) holds and, $\mathcal{B}$ being a group, (C-T) follows (Lemma 2.3). Therefore the open set $\mathcal{F}_1(\boldsymbol{E})$ [resp. $\mathcal{F}_1^s(\boldsymbol{E})$] is a suitable class of measures fulfilling the convergence of types condition for $\mathcal{B} = \mathbf{Gl}(\boldsymbol{E})$ [ resp. for the affine group $\mathcal{B} = \mathbf{AG}(\boldsymbol{E})$]. ⟦Cf. [B], [F], [Sh],[JM]⟧.

**4.3. Homothetical transformations.**
Put $\mathcal{B} := \{x \mapsto tx : t > 0\}$. $\mathcal{B}$ is a closed subgroup of $\mathbf{Gl}(\boldsymbol{E})$. Put $\boldsymbol{F} := \boldsymbol{E}, \boldsymbol{H} := \{\mathrm{id}_{\boldsymbol{E}}\}$. Hence in this case $\mathcal{F}_{\boldsymbol{H}}(\boldsymbol{E}) = \mathcal{M}^1(\boldsymbol{E}) \backslash \{\varepsilon_0\} = \mathcal{F}_0(\boldsymbol{E})$. $\mathcal{F}_0$ is an open subset in $\mathcal{M}^1(\boldsymbol{E})$ fulfilling (C-T), and $\mathcal{F}_0^s = \mathcal{M}^1(\boldsymbol{E}) \backslash \{\varepsilon_x : x \in \boldsymbol{E}\}$ fulfils (C-T) for the corresponding affine transformations $\mathcal{B}_{\text{aff}} := \{x \mapsto tx + c : t > 0, c \in \boldsymbol{E}\}$. ⟦See e.g. [K], [GK], [L]. The proof for homothetical transformations is identical to the proof for the one-dimensional case. A simple proof is given below in 4.4.a).⟧

**4.4. Quasi-contracting groups.**
Let $\mathcal{B}$ be a closed subgroup of $\mathbf{Gl}(\boldsymbol{E})$. We call $\mathcal{B}$ *quasi- contracting* if for any sequence $(\beta_n) \subseteq \mathcal{B}$ we have: $\|\beta_n\| \to \infty$ iff $\|\beta_n^{-1}\| \to 0$. (Hence $\mathcal{B} \cup \{0\}$ and $\mathcal{B} \cup \{\infty\}$ are topological semigroups and $\mathcal{B}^* = \mathcal{B} \cup \{0\} \cup \{\infty\}$ defines a 2-point compactification of $\mathcal{B}$.)

**Proposition.** *Put again* $\boldsymbol{H} := \{\mathrm{id}\}$ *and* $\mathcal{F}_{\mathcal{B}} := \mathcal{F}_{\boldsymbol{H}}(\boldsymbol{E}) := \mathcal{F}_0(\boldsymbol{E}) = \mathcal{M}^1(\boldsymbol{E})\backslash\{\varepsilon_0\}$. *Then* $(\mathcal{B}, \mathcal{F}_0(\boldsymbol{E}))$ *fulfil the* (C-T) *condition.*

Proof. Let $\mu_n \to \mu$, $\alpha_n\mu_n \to \nu$, $\mu, \nu \in \mathcal{F}_0$. It suffices to show that $\{\|\alpha_n\|\}$ and $\{\|\alpha_n^{-1}\|\}$ are bounded. For then the accumulation points of $\{\alpha_n\}$ in $\mathbf{End}(\boldsymbol{E})$ belong to $\mathcal{B}$. Assume $\{\|\alpha_n\|\}$ to be unbounded. Suppose $\|\alpha_n\| \to \infty$, hence by assumption $\|\alpha_n^{-1}\| \to 0$. Therefore $\alpha_n\mu_n \to \nu$ implies $\alpha_n^{-1}(\alpha_n\mu_n) = \mu_n \to \varepsilon_0$, i.e. $\mu = \varepsilon_0$, a contradiction. Conversely, $\|\alpha_n\| \to 0$ implies $\nu = \varepsilon_0$. Hence (C-T) holds for $(\mathcal{B},\ \mathcal{F}_0(\boldsymbol{E}))$. □

The following examples 4.4.a)–e) may be subsumed in 4.4:

**4.4.a) Homothetical transformations** are quasi-contracting.
Hence, as mentioned in 4.3. above, a new proof of 4.3 follows. More generally:

**4.4.b) Contracting one-parameter groups** are quasi-contracting:
Let $A \in \mathbf{Gl}(\boldsymbol{E})$ and assume $\mathrm{Spec}(A) \subseteq \{\lambda \in \boldsymbol{C} : \mathrm{Re}\,\lambda > 0\}$. Then $\mathcal{B} := \{t^A := \exp(\log t)A\}_{t>0}$ and $\mathcal{F}_0(\boldsymbol{E}) := \mathcal{M}^1(\boldsymbol{E})\backslash\{\varepsilon_0\}$ fulfil the convergence of types condition (C-T). $\Big[\!\!\Big[$Indeed, as well known $\lim\limits_{t\to\pm\infty} \|t^A\| = \infty$ resp.$= 0$ iff $\mathrm{Spec}(A) \subset \{\mathrm{Re}\,\lambda > 0\}\Big]\!\!\Big]$.

**4.4.c) Discrete contracting groups** are quasi-contracting:
Let $\alpha \in \mathbf{Gl}(\boldsymbol{E})$ such that $\mathrm{Spec}(\alpha) \subset \{|\ \lambda\ |< 1\}$. Then $\mathcal{B} := \{\alpha^k : k \in \boldsymbol{Z}\}$ and $\mathcal{F}_0(\boldsymbol{E}) = \mathcal{M}^1(\boldsymbol{E})\backslash\varepsilon_0$ fulfil (C-T).
$\Big[\!\!\Big[$As well known, $\lim\limits_{k\to\pm\infty} \|\alpha^k\| = \infty$ resp.$= 0$ iff $\mathrm{Spec}(\alpha) \subset \{|\ \lambda\ |< 1\}.\Big]\!\!\Big]$

**4.4.d) Compact extensions** of quasi-contracting groups are quasi-contracting:
Let $\mathcal{K} \subseteq \mathbf{Gl}(\boldsymbol{E})$ be a compact subgroup, and let $\mathcal{T}$ be quasi-contracting, e.g. $\mathcal{T} := (t^A : t > 0)$ as in 4.4.b), resp. $= \{\alpha^k : k \in \boldsymbol{Z}\}$ as in 4.4.c) such that $\mathcal{T}$ is contained in the normalizer of $\mathcal{K}$. Then the (semidirect) product $\mathcal{B} := \mathcal{T} \cdot \mathcal{K}$ is quasi-contracting and hence $\mathcal{B}$ and $\mathcal{F}_0(\boldsymbol{E})$ fulfil (C-T). $\big[$Indeed, if $\mathcal{T} = \{t^A\}$ then, as well known $\mathcal{B} = \mathcal{T} \cdot \mathcal{K}$ is isomorphic to a direct product.$\big]$

Proof. W.l.o.g. we choose a norm on $\boldsymbol{E}$ such that $\mathcal{K} \subset \mathcal{O}(\boldsymbol{E})$, the orthogonal group, see e.g. [HR] $V$ §22, (22.23). Hence for $\kappa \in \mathcal{K}, t \in \mathcal{T}, x \in \boldsymbol{E}\backslash\{0\}$ we have $\|t\kappa\| \leq \|t\|$ and $\|t\kappa x\|/\|x\| = \|t\kappa x\|/\|\kappa x\|$. Hence $\|t\kappa\| = \|t\|$ for all $t\kappa$, i.e. $\mathcal{B}$ herits the quasi-contracting property from $\mathcal{T}$. □

**4.4.e) Remark**. The examples mentioned in 4.4.a)–4.4.d) appear in a natural way as admissible normalizing operators, e.g. in connection with investigations of domains of normal attraction: Let $\mu$ be (strictly) operator-stable resp. -semistable on $\boldsymbol{E}$, let $(\mu_t)$ be the corresponding c.c.s. such that $\mu = \mu_1$. Then the decomposability group $\mathcal{Z}(\mu) := \{\alpha \in \mathbf{Gl}(\boldsymbol{E}) : \alpha\mu_t = \mu_{ct}$ for all $t > 0$ and some $c = c(\alpha) > 0\}$ is of the form $(t^A)\mathcal{J}(\mu)$ resp. $(\alpha^k)\mathcal{J}(\mu)$. Hence, if $\mu$ is a full measure (i.e. $\mu \in \mathcal{F}_1(\boldsymbol{E})$), then $\mathcal{J}(\mu)$ is compact and hence $\mathcal{Z}(\mu)$ is of the form 4.4.d). ⟦Cf. [S], [Lu], [Ha1], [Ha2].⟧

**Thus we have proved:** Let $\mu$ be a full strictly operator-stable law on $\boldsymbol{E}$ ([Sh]) and let $\mathcal{T} := \{t^A : t > 0\}$, a corresponding one-parameter automorphism group. Then for the group of admissible normalizations $\mathcal{T}$ – more generally for the decomposability group $\mathcal{Z} := \mathcal{J}(\mu) \cdot \mathcal{T}$ – the corresponding class of full measures is just $\mathcal{F}_0(\boldsymbol{E}) = \mathcal{M}^1(\boldsymbol{E}) \backslash \{\varepsilon_0\}$, i.e. $(\mathcal{Z}, \mathcal{F}_0)$ fulfil (C-T). If we allow affine normalizations $x \mapsto t^A x + a$, we have to replace $\mathcal{F}_0$ by $\mathcal{F}_0^s$. An analogous result holds for operator semistable laws.

**4.4.f) Motion groups**.
Special examples of compact extensions of (quasi-)contracting groups $\mathcal{B}$ are motion groups. We start with the following almost obvious example:

Let $\boldsymbol{C} \cong \boldsymbol{R}^2$ the complex plane, and let $\mathcal{B} := \boldsymbol{C}^\times := \boldsymbol{C} \backslash \{0\}$ operate on $\boldsymbol{C} \cong \boldsymbol{R}^2$ canonically. Then $c = |c|.(c/|c|)$ defines a decomposition $\mathcal{B} = \boldsymbol{R}_+^\times \otimes \boldsymbol{T}$ with $\boldsymbol{T} := \{c : |c| = 1\}$. Hence (4.4.d) $\boldsymbol{C}^\times$ acts quasi-contracting on $\boldsymbol{C} \cong \boldsymbol{R}^2$, and thus $\mathcal{F}_0$ is a suitable class of full measures.

More generally: Let $\boldsymbol{E} = \boldsymbol{R}^d$, let $\mathcal{O}$ be the (compact) group of orthogonal transformations and put $\mathcal{B} := \{tU : U \in \mathcal{O}, t > 0\}$. Then the corresponding affine group $\mathcal{B}_{\text{aff}} := \{x \mapsto \text{tUx} + b : t > 0, U \in \mathcal{O}, b \in \boldsymbol{E}\}$ is the *group of Euclidean motions (semidirectly extended)*. Obviously $\mathcal{B}$ fulfils the conditions described in 4.4.d), hence $(\mathcal{B},\ \mathcal{F}_0(\boldsymbol{E}) := \mathcal{M}^1(\boldsymbol{E}) \backslash \{\varepsilon_0\})$ and $(\mathcal{B}_{\text{aff}},\ \mathcal{F}_0^s)$ fulfil (C-T).

Analogously *motion groups of stratified Lie groups* resp. *Lie algebras* are treated: E.g. let $\boldsymbol{E} := \mathcal{H} := (\boldsymbol{R}^d \oplus \boldsymbol{R}^d \oplus \boldsymbol{R}, [\ ,\ ])$ be the $2d+1$-dimensional Heisenberg Lie-algebra, let $\mathcal{T} := \left\{\delta_t = \begin{pmatrix} t & 0 \\ 0 & t^2 \end{pmatrix} : t > 0\right\}$ be the group of dilations and let $\mathcal{S}$ be the group of symplectic orthogonal transformations of $\boldsymbol{R}^d \oplus \boldsymbol{R}^d$. Finally, let $\mathcal{S}^* := \left\{ \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}, s \in \mathcal{S}\right\}$. Then $\mathcal{B} := \{\delta_t U : U \in \mathcal{S}^*, t > 0\}$ is quasi-contracting, hence $(\mathcal{B}, \mathcal{F}_0)$ fulfil (C-T).

**4.5. Diagonal operators.**
Let $\mathcal{B} \subseteq \mathbf{Gl}(\boldsymbol{E})$ be a closed subgroup. Let $\boldsymbol{E}_i$ be $\mathcal{B}$-invariant subspaces, $1 \leq i \leq n$, such that $\boldsymbol{E} = \sum \oplus \boldsymbol{E}_i$. Let $\mathcal{B}_i := \mathcal{B}|_{\boldsymbol{E}_i}$ be the restrictions to $\boldsymbol{E}_i$ such that $\mathcal{B} \subseteq \sum \oplus \mathcal{B}_i$. Assume that for all i there exist auxiliary spaces $\boldsymbol{F}_i$ and $\boldsymbol{H}_i \subseteq \mathbf{Lin}(\boldsymbol{E}_i, \boldsymbol{F}_i)$ such that $(\mathcal{F}_{\boldsymbol{H}_i}, \mathcal{B}_i)$ fulfil (W-C-T).

Define $\boldsymbol{F} := \sum \oplus \boldsymbol{F}_i$. Let $p_i : \boldsymbol{F} \to \boldsymbol{F}_i$ the canonical projections and $\boldsymbol{j}_i : \boldsymbol{F}_i \to \boldsymbol{F}$ the injections, $\boldsymbol{j}_i = p_i^*$. Furthermore, $\boldsymbol{H} := \bigcup \boldsymbol{H}_i (-\boldsymbol{H}_i$ embedded as subspaces of $\mathbf{Lin}(\boldsymbol{E}, \boldsymbol{F}))$. Then $(\mathcal{F}_{\boldsymbol{H}}, \mathcal{B})$ fulfil (W-C-T), and hence also (C-T) according to 2.3 and 3.3.

Proof. Let $\pi_i : \boldsymbol{E} \to \boldsymbol{E}_i$ be the natural projections. Then by definition $\mu \in \mathcal{F}_{\boldsymbol{H}}$ iff $\pi_i \mu \in \mathcal{F}_{\boldsymbol{H}_i}$ for all $i, 1 \leq i \leq n$. Hence the assertion follows. □

**4.5.a) Example**: Let $A_i \in \mathbf{Gl}(\boldsymbol{E}_i)$ with $\mathrm{Spec}(A_i) \subseteq \{\mathrm{Re}\,\lambda > 0\}$, $1 \leq i \leq n$. Let $\boldsymbol{E} := \sum \oplus \boldsymbol{E}_i, \mathcal{B} := \left\{ \begin{pmatrix} S_1 & \cdot & & 0 \\ 0 & & \cdot & \cdot & S_n \end{pmatrix} : S_i := t_i^{A_i}, t_i > 0, \ i = 1, \ldots, n \right\}$. Put $\boldsymbol{H} := \{\pi_i : 1 \leq i \leq n\}$, and hence $\mathcal{F}_{\boldsymbol{H}} := \mathcal{F}_{\mathcal{B}} = \{\mu : \pi_i(\mu) \neq \varepsilon_0 \text{ for } 1 \leq i \leq n\} = \mathbf{Cp} \bigcup \pi_i^{-1}\{\varepsilon_0\} = \{\mu$: not concentrated on $\boldsymbol{E}_i, 1 \leq i \leq n\}$. ($\mathbf{Cp}$ denoting the complement).

Then by 4.5, $\mathcal{B}$ and $\mathcal{F}_{\mathcal{B}}$ fulfil (C-T).

For examples and applications of diagonal norming operators cf. [Sch3] and the literature mentioned there.

**4.6. Non-quasi-contracting one-parameter groups.**
Let $N \in \mathbf{End}(\boldsymbol{E})$ be step$-r+1$- nilpotent, i.e. $N^r \neq 0, N^{r+1} = 0$. Define $\mathcal{B} := \{t^N : t > 0\}$. Then put $\boldsymbol{F} := \boldsymbol{E}$, $\boldsymbol{H} := \{N^r\}$ and hence $\mathcal{F}_{\boldsymbol{H}} := \{\mu : N^r(\mu) \neq \varepsilon_0\}$ is a suitable class of $\mathcal{B}$-full measures, i.e. $(\mathcal{B}, \mathcal{F}_{\boldsymbol{H}})$ fulfil (C-T).

Proof. Since $t^N = \sum_{k=0}^{r} (\log t)^k N^k$ is representable as $(\log t)^r \cdot C(t)$, with $C(t) := \sum_{k=0}^{r-1} (\log t)^{k-r} N^k + N^r$, we obtain that $\|C(t)\|$ is uniformly bounded outside any neighbourhood $(1-\varepsilon, 1+\varepsilon)$ of 1, $\|C(t)\| \leq K_\varepsilon$, say. Assume $\mu_n \to \mu, t_n^N(\mu_n) \to \nu$, i.e. $[(\log t_n)^r \text{ id} \cdot C(t_n)](\mu_n) \to \nu$. Assume $t_n \to \infty [\to 0]$ for some subsequence. Then $C(t_n) \to N^r$ and $|(\log t_n)^r| \to \infty$, and w.l.o.g. $t_n < 1-\varepsilon$ or $> 1+\varepsilon$. Hence $\kappa_n := C(t_n)(\mu_n) \to N^r \mu := \kappa$, and $((\log t_n)^r I)(\kappa_n) = t_n^N \mu_n \to \nu$. And we obtain therefore $\kappa = \varepsilon_0$ (cf. the proof of 4.4), hence $N^r \nu = \varepsilon_0$ in contradiction to the assumption $\mu \in \mathcal{F}_{\boldsymbol{H}}$.

Hence we have proved that $\{t_n\}$ is bounded away from 0 and $\infty$, i.e. (C-T) is fulfilled. □

**4.7. Automorphisms of nilpotent Lie Algebras.**
Let $\boldsymbol{E} \cong \boldsymbol{R}^d$ be endowed with a Lie algebra structure $[\,,\,]$ such that $(\boldsymbol{E}, [\,,\,])$ is nilpotent of step $r$. Define $\mathbf{Aut}(\boldsymbol{E})$ to be the group of Lie algebra automorphisms, $\mathbf{Aut}(\boldsymbol{E}) := \{\alpha \in \mathbf{Gl}(\boldsymbol{E}) : \alpha[x,y] = [\alpha x, \alpha y] \text{ for } x, y \in \boldsymbol{E}\}$. We consider $\mathcal{B} = \mathbf{Aut}(\boldsymbol{E})$ as a closed subgroup of $\mathbf{Gl}(\boldsymbol{E})$.

$\boldsymbol{E}^{(1)} := [\boldsymbol{E}, \boldsymbol{E}]$ is a characteristic ideal in the Lie algebra $\boldsymbol{E}$. Let $\pi_1 : \boldsymbol{E} \to \boldsymbol{E}/\boldsymbol{E}^{(1)} := \boldsymbol{M}_0$ be the canonical projection. Put $\boldsymbol{H} := \{\varphi \circ \pi_1 : \varphi \in \boldsymbol{M}_0'\} \backslash \{0\} = \{\varphi \in \boldsymbol{E}' \backslash \{0\} : \ker \varphi \supseteq \boldsymbol{E}^{(1)}\}$.

**Proposition.** $\mathcal{F}_{\mathcal{B}} := \mathcal{F}_{\boldsymbol{H}} := \{\mu \in \mathcal{M}^1(\boldsymbol{E}) : \pi_1(\mu) \in \mathcal{F}_1(\boldsymbol{M}_0)\}$ *is a suitable class of full measures for the whole group* $\mathcal{B} := \mathbf{Aut}(\boldsymbol{E})$*, i.e.* $(\mathcal{B}, \mathcal{F}_{\mathcal{B}})$ *fulfil* (C-T).

$\big[$*Note that by definition of* $\boldsymbol{M}_0$ *we have* $\mathcal{F}_{\mathcal{B}} = \{\mu$*: not concentrated on a proper ideal*$\}$.$\big]$

Proof. We need some preparations:
The descending central series $\boldsymbol{E}^{(0)} := \boldsymbol{E}, \boldsymbol{E}^{(i+1)} := [\boldsymbol{E}, \boldsymbol{E}^{(i)}], \ldots, \boldsymbol{E}^{(r+1)} = \{0\}$ is a sequence of characteristic ideals. Let $\boldsymbol{E}_{(i)} := \boldsymbol{E}/\boldsymbol{E}^{(i)}$. For $i \geq 1$ let $\boldsymbol{M}_i$ be a vector space complement of $\boldsymbol{E}^{(i+1)}$ in $\boldsymbol{E}^{(i)}$, hence $\boldsymbol{E}^{(i)} \cong \sum_{j=i}^{r} \oplus \boldsymbol{M}_j$, $\boldsymbol{E}_{(i)} \cong \sum_{j=0}^{i-1} \oplus \boldsymbol{M}_j$. Let $\pi_i : \boldsymbol{E} \to \boldsymbol{E}_{(i)}$ be the canonical projection. $\boldsymbol{E}^{(i)}$ being characteristic, we obtain a representation $\pi_i \alpha = \alpha^{(i)} \pi_i$ for $\alpha \in \mathbf{Aut}(\boldsymbol{E})$, $1 \leq i \leq r$, with $\alpha^{(i)} \in \mathbf{Aut}(\boldsymbol{E}_{(i)})$. (Note that $\boldsymbol{E}_{(i)}$ and $\boldsymbol{E}^{(i)}$ are nilpotent Lie algebras). Let $(\mu_n), \mu, \nu \in \mathcal{M}^1(\boldsymbol{E})$, let $(\alpha_n) \subseteq \mathcal{B} = \mathbf{Aut}(\boldsymbol{E})$, such that $\mu_n \to \mu$, $\alpha_n \mu_n \to \nu$ and $\mu, \nu \in \mathcal{F}_{\mathcal{B}}$.

*1.)* $\boldsymbol{i = 1}$. $\pi_1 \mu_n := \mu_n^{(1)} \to \mu^{(1)} := \pi_1(\mu)$ and $\pi_1 \alpha_n \mu_n = \alpha_n^{(1)} \pi_1 \mu_n = \alpha_n^{(1)} \mu_n^{(1)} \to \nu^{(1)} := \pi_1(\nu)$. Since $\pi_1(\mu) \in \mathcal{F}_1(\boldsymbol{E}_{(1)}) = \mathcal{F}_1(\boldsymbol{M}_0)$ we obtain by 4.2. boundedness of $(\alpha_n^{(1)})$.

*2.)* $\boldsymbol{i = 2}$. As before we have $\pi_2 \mu_n := \mu_n^{(2)} \to \mu^{(2)} := \pi_2(\mu)$ and $\alpha_n^{(2)} \mu_n^{(2)} \to \nu^{(2)}$, where $\alpha_n^{(2)}$ has a matrix representation of the form
$\alpha_n^{(2)} = \begin{pmatrix} \alpha_n^{(1)} & 0 \\ b_n & c_n \end{pmatrix}$ with $c_n \in \mathbf{Aut}(\boldsymbol{M}_1)$ and $b_n \in \mathbf{Lin}(\boldsymbol{M}_0, \boldsymbol{M}_1)$.

We have $\boldsymbol{M}_1 \subseteq [\boldsymbol{E}, \boldsymbol{E}]$. Hence to show that $(c_n)$ is bounded it is sufficient to show the boundedness of $(\alpha_n[X,Y])$ for $X, Y \in \boldsymbol{E}$. Indeed, $\boldsymbol{M}_1$ being central in $\boldsymbol{E}_{(2)}$, it is sufficient to consider $X, Y \in \boldsymbol{M}_0$. But then the sequence $(\alpha_n[X,Y] = [\alpha_n^{(1)}X, \alpha_n^{(1)}Y])_{n\geq 1}$ is bounded since $(\|\alpha_n^{(1)}\|)_{n\geq 1}$ is.

Assume $(\|\alpha_n^{(2)}\|)$ to be unbounded. Then accumulation points $\alpha^*$ of $(\alpha_n^{(2)}/\|\alpha_n^{(2)}\|)$ are of the form $\alpha^* = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$ with $b \in \mathbf{Lin}(\boldsymbol{M}_0, \boldsymbol{M}_1)\backslash\{0\}$. As in the proof of Proposition 3.3 we obtain $\alpha^*\mu^{(2)} = \varepsilon_0$, hence $b(\pi_1(\mu)) = \varepsilon_0$, a contradiction. Hence the boundedness of $(\alpha_n^{(2)})_{n\geq 1}$ is proved.

*3.)* $\boldsymbol{2 < i < r}$. Now continue by induction. Assume that $(\alpha_n^{(i)})_{n\geq 1}$ is bounded. Then, as above, $\alpha_n^{(i+1)}$ has a representation $\alpha_n^{(i+1)} = \begin{pmatrix} \alpha_n^{(i)} & 0 & 0 \\ b_n & d_n & c_n \end{pmatrix}$ with $c_n \in \mathbf{Gl}(\boldsymbol{M}_{i+1})$ and $d_n \in \mathbf{Lin}(\sum_2^i \oplus \boldsymbol{M}_j, \boldsymbol{M}_{i+1})$, $b_n \in \mathbf{Lin}(\boldsymbol{M}_0, \boldsymbol{M}_{i+1})$.

As in step $i = 2$ boundedness of $(\alpha_n^{(i)})$ implies boundedness of $(d_n)$ and of $(c_n)$. If $(\alpha_n^{(i+1)})$ were unbounded there exist non-zero accumulation points $\alpha^* = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$ of $\left\{\alpha_n^{(i+1)} \,/\, \|\alpha_n^{(i+1)}\|\right\}$, hence $\alpha^*\pi_1(\mu) = \varepsilon_0$, a contradiction. □

So we obtain a *new proof of the convergence-of-types theorem for nilpotent Lie groups* (Cf. [HN], [N]):

**Theorem 4.8.** *Let* $\boldsymbol{G}$ *be a simply connected nilpotent Lie group with Lie Algebra* $(\mathcal{G}, [\,,\,])$ *and let* $\mathbf{Aut}(G)$ *be the group of topological automorphisms of* $G$*. Then* $G/[G,G] \cong \mathcal{G}/[\mathcal{G},\mathcal{G}] := \mathcal{G}_{(1)}$ *(topologically and algebraically). Let* $\pi : G \to \mathcal{G}_{(1)}$ *be the canonical homomorphism. Let "full measures on the group* $\boldsymbol{G}$*" be defined as* $\mathcal{F} := \{\mu \in \mathcal{M}^1(G) : \pi(\mu) \in \mathcal{F}_1(\mathcal{G}_{(1)})\} = \{\mu \in \mathcal{M}^1(G)$*: For any continuous surjective homomorphism* $\varphi : G \to \boldsymbol{R}$ *we have* $\varphi(\mu) \neq \varepsilon_0\}$.

Then $(\mathbf{Aut}(G), \mathcal{F})$ fulfil the convergence-of-types condition (C-T).

Proof. Being a homeomorphism, $\exp : \mathcal{G} \to G$ defines bijections

$f \longleftrightarrow f^o := f \circ \exp, C^b(G) \longleftrightarrow C^b(\mathcal{G})$, $\mu \longleftrightarrow \mu^o$, $\mathcal{M}^1(G) \longleftrightarrow \mathcal{M}^1(\mathcal{G})$, $\tau \longleftrightarrow \tau^o$, $\mathbf{Aut}(G) \longleftrightarrow \mathbf{Aut}(\mathcal{G})$. ⟦Cf. e. g. [Ha1], [HN]. ⟧

Let $\mathcal{F}^o := \{\mu^o : \mu \in \mathcal{F}\}$. So it is sufficient to show: $(\mathbf{Aut}(\mathcal{G}), \mathcal{F}^o := \{\mu^o : \mu \in \mathcal{F}\})$ fulfil (C-T) on the vector space $\mathcal{G}$. But obviously $\mathcal{F}^o = \{\mu^o \in \mathcal{M}^1(\mathcal{G}) : \pi_1(\mu^o) \in \mathcal{F}_1(\mathcal{G}_{(1)})\}$, hence by 4.7 (C-T) is fulfilled. 4.8 is proved. □

*Remark 4.9.* In most of the concrete examples above we can show that the constructed class of full measures $\mathcal{F}_{\boldsymbol{H}}$ is *maximal w.r.t.* $\mathcal{B}$ (cf. Proposition 2.5). E.g. we can show that for $\mu \notin \mathcal{F}_{\boldsymbol{H}}$ the invariance group $\mathcal{J}_{\mathcal{B}}(\mu)$ is non-compact. (To exclude trivial situations we always assume $\mathcal{B}$ to be non-compact):

Maximality is obvious for the examples 4.2–4.4, i.e. for $\mathcal{F}_0(\boldsymbol{E}) = \mathcal{M}^1(\boldsymbol{E})\backslash\{\varepsilon_0\}$ since we have $\mu \notin \mathcal{F}_0(\boldsymbol{E})$ iff $\mu = \varepsilon_0$ and since $\mathcal{J}_{\mathcal{B}}(\varepsilon_0) = \mathcal{B}$ is non-compact.

On the other hand, since $\mu \notin \mathcal{F}_{\boldsymbol{H}}$ iff $\mu(\ker\varphi) = 1$ for some $\varphi \in \boldsymbol{H}$, $\mathcal{F}_{\boldsymbol{H}}$ is maximal for $\mathcal{B}$ if for any $\varphi \in \boldsymbol{H}$ the set $\{\alpha \in \mathcal{B} : \alpha|_{\ker\varphi} = \mathrm{id}|_{\ker\varphi}\}$ is non-compact.

Hence maximality of $\mathcal{F}_{\boldsymbol{H}}$ also follows in the case of the examples 4.5–4.6, – and for the corresponding groups of affine transformations if $\mathcal{F}$ is replaced by $\mathcal{F}^s$ –, and also for automorphism groups of nilpotent Lie algebras resp. groups. Indeed, maximality can be proved for the group of automorphisms of nilpotent Lie algebras $\mathcal{B} = \mathbf{Aut}(\boldsymbol{E})$ (Example 4.7) along the lines mentioned above. This is just the way Proposition 3.3 and Theorem 3.4 in [HN] are proved.

See also the general discussion in [Ha1], [Ha2], where we suggested for more general convolution structures to *define* fullness of $\mu$ by compactness of the invariance group $\mathcal{J}(\mu)$.

Note that in the situation of 4.9 we gain the following characterization:

**Proposition 4.10.** *Let $\mathcal{F}_{\boldsymbol{H}}$ be maximal w.r.t. $\mathcal{B}$. Then $\mu$ is $\mathcal{B}$-full iff the invariance group $\mathcal{J}_{\mathcal{B}}(\mu)$ is compact.* ⟦*Cf. also* [Sh], [U], [JM], [HN], [Ha1], [Ha2], [T] ⟧

## §5 Infinite dimensional spaces. Concluding remarks.

If $\boldsymbol{E}$ is an (infinite dimensional) Banach space and if $\mathbf{End}(\boldsymbol{E})$ is endowed with the (natural) strong operator topology, then the results of §3 and §4 are no longer true: The action of $\mathbf{Gl}(\boldsymbol{E})$ on $\boldsymbol{E}$ resp. on the probabilities $\mathcal{M}^1(\boldsymbol{E})$ is not simultaneously continuous. Only the restriction to bounded subsets of $\mathbf{End}(\boldsymbol{E})$ of the map $\theta$ (defined in §1) is continuous, and (cf. [LS]) on these sets the weak condition (W-C-T) is fulfilled, if the set of full measures is defined to be $\mathcal{F}_1$. Hence (C-T) is fulfilled for all $(K_{a,b}, \mathcal{F}_1(\boldsymbol{E}))$, where $K_{a,b} := \{\alpha \in \mathbf{Gl}(\boldsymbol{E}) : \|\alpha\| \leq a \text{ and } \|\alpha^{-1}\| \leq b\}$. But in general (C-T) does not hold, see e.g. [LS] or [JM]. This is one of the reasons why in the infinite dimensional setup the relations between limit theorems and stability concepts are less satisfactory than in finite dimensional spaces.

On the other hand, for homothetical transformations (cf. e.g. [CR]) or for $\|\,.\,\|$-contracting one-parameter groups $(T_t)$, continuous w.r.t. the strong operator topology (cf. e.g. [Ja1] "$V$-decomposability", [JM]) the C-T-theorem is valid. Indeed, quasi-contracting groups (cf. 4.4) may be defined in a more general setup:

*Definition 5.1.* A subgroup $\mathcal{B} \subseteq \mathbf{Gl}(\boldsymbol{E})$, endowed with the strong operator topology is called quasi-contracting if the following conditions (5.1) and (5.2) are fulfilled:

$$\|\alpha_n\| \to \infty \quad \text{iff} \quad \|\alpha_n^{-1}\| \to 0 \tag{5.1}$$

$$\mathcal{B}_1 := \{\alpha \in \mathcal{B} : \|\alpha\| = 1\} \text{ is compact.} \tag{5.2}$$

**Proposition 5.2.** *Let $\mathcal{B}$ be quasi-contracting. Then $\mathcal{B}$ and $\mathcal{F}_0 := \mathcal{M}^1(\boldsymbol{E})\backslash\{\varepsilon_0\}$ fulfil* (C-T), *hence $\mathcal{B}$ acts properly on $\mathcal{F}_0$.*

⟦The proof is identical with the proof of 4.4.⟧

**Proposition 5.3.** *Let $\mathcal{B}$ be a subgroup of* $\mathbf{Gl}(\boldsymbol{E})$ *endowed with the strong operator topology. Assume $\mathcal{B}$ to be either* a) *locally compact or* b) *metrizable. Then $\mathcal{B}$ acts as a transformation group on $\mathcal{M}^1(\boldsymbol{E})$.*

Proof. If a) or b) holds it is sufficient to consider the restrictions of $\theta$ to relatively compact subsets of $\mathcal{B}$. But these sets are uniformly $\|.\|$-bounded and on norm-bounded sets the action is simultaneously continuous. (Cf. [LS].) □

**Proposition 5.4.** *Let $\mathcal{B}$ be a subgroup of* $\mathbf{Gl}(\boldsymbol{E})$ *endowed with the strong operator topology and assume condition* (5.1) *to be fulfilled. Then $\mathcal{B}$ and $\mathcal{F}_1(\boldsymbol{E}) := \{\mu$: not concentrated on a proper closed subspace$\}$ fulfil* (C-T).

Proof. Let $(\mu_n) \subseteq \mathcal{M}^1(\boldsymbol{E}), \mu, \nu \in \mathcal{F}_1$, let $(\alpha_n) \subseteq \mathcal{B}, \mu_n \to \mu$ and $\alpha_n\mu_n \to \nu$. Let $\|\alpha_n\| \to 0$. Then, $\{\mu_n\}$ being uniformly tight, $\nu = \varepsilon_0$

follows. A contradiction. Conversely, assume $\|\alpha_n\| \to \infty$, hence by assumption $\|\alpha_n^{-1}\| \to 0$, therefore $\mu = \varepsilon_0$ follows. So $\{\|\alpha_n\|\}$ and $\{\|\alpha_n^{-1}\|\}$ are uniformly bounded and the assertion follows from [LS]. □

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