On minimax principles and sets with convex sections

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Abstract. In this note a lemma similar to Knaster—Kuratowski—Mazurkiewiez—Fan lemma on Hausdorff topological vector space is obtained. This is used to obtain a Fan's minimax principle wich yields a Von Neumann—Sion minimax principle. It has also been applied to obtain two fixed point theorems. Finally these fixed point theorems are applied to problems relating to sets with convex sections.*)

1. *Introduction.* To begin with let us consider the following lemma due to FAN [3].

Lemma 1.1. *Let X be a nonempty subset of a Hausdorff topological vector space E. For each $x \in X$, let a nonempty closed subset $F(x)$ be given such that* (i) *$F(x_0)$ is compact for some $x_0 \in X$ and* (ii) *for each finite subset $\{x_1, x_2, \dots, x_n\}$ of X the convex hull of $\{x_1, x_2, \dots, x_n\}$ is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. Then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

The above is a generalization of the well known finite dimensional result of KNASTER—KURATOWSKI—MAZURKIEWICZ [6]. In [4] Fan has used his lemma to prove a minimax principle and also applied it to a variety of problems. In [1] BREZIS—NIRENBERG—STAMPACCHIA has extended the above lemma to some extent and obtained a generalized Fan's minimax principle which yields a VON NEUMANN—SION minimax principle slightly stronger than the one indicated by Fan in [4].

In the present note we proved a lemma similar to Lemma 1.1 which admits most of the applications considered in [1].

In particular, we have applied our lemma to obtain a Fan's minimax principle which yields a von Neumann–Sion minimax principle which includes the corresponding result given in [1]. We have also used our lemma to obtain two fixed point theorems, each being dual to the other in the sense of [7]. Finally we have applied these two fixed point theorems on the problems relating to sets with convex sections. Our results in this direction are more general than the corresponding results in [3] (also see [2]). Our approach relies on a fixed point theorem of BROWDER [2]. Very recently in [8] a lemma stronger than ours and distinct from the lemma proved in [1] has been proved.

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2. *For the rest of the paper E will denote a Hausdorff topological vector space.* We first consider the following fixed point theorem of Browder ([2], theorem 1).

Theorem 2.1. *Let K be a nonempty compact convex subset of E. Let $T: K \to 2^K$ be a multi-valued mapping such that*

(i) *for each $x \in K$, $T(x)$ is a nonempty convex subset of K;*

(ii) *for each $x \in K$, $T^{-1}(x) = \{y : x \in T(y)]$ is open in K. Then there is a point $x_0 \in K$ such that $x_0 \in T(x_0)$.*

A theorem equivalent to the above theorem has been given by FAN ([4], theorem 2) with different proof.

Lemma 2.1. *Let X be a nonempty convex subset of E. To each $x \in X$, let a nonempty subset $F(x)$ in E be given such that*

(a) *$x \in F(x)$ for each $x \in X$;*

(b) *$F(x_0)$ is compact for some $x_0 \in X$;*

(c) *for each $x \in X$, the set $A(x) = \{y \in X : x \notin F(y)\}$ is convex;*

(d) *for each $x \in X$, the intersection of $F(x)$ with any finite dimensional subspace of E is closed;*

(e) *for each $x \in X$, $F(x_0) \cap F(x)$ is closed.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

Remark. The above lemma motivates from a lemma of FAN ([4], lemma 4). The conditions (a) and (c) together imply the condition (i) of lemma 1.1. Thus if (d) and (e) are replaced by (ii) of lemma 1.1, the above lemma would be a special case of lemma 1.1.

PROOF OF LEMMA 2.1. In view of (b) and (e) it would suffice to prove that $\bigcap_{i=1}^{n} F(x_i) \neq \emptyset$ for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of X. If possible, let us assume that $\bigcap_{i=1}^{n} F(x_i) = \emptyset$. Then for each $x \in S$, the convex hull of $\{x_1, x_2, \ldots, x_n\}$ the set $B(x) = \{y \in S : x \notin F(y)\}$ is nonempty. Indeed, at least one of the x_i, $i = 1, 2, \ldots, n$ belongs to $B(x)$. Since S is convex, it follows from (c) that $B(x)$ is convex. Let us define a mapping $T: S \to 2^S$ by

$$T(x) = B(x) \text{ for each } x \in S.$$

Now $T^{-1}(x) = \{y \in S : x \in T(y)\} = \{y \in S : x \in B(y)\} = \{y \in S : y \notin F(x)\}$ is open in S by (d). Hence by theorem 2.1 there is a point $x_0 \in S$ such that $x_0 \in T(x_0) = B(x_0)$. But this means that $x_0 \notin F(x_0)$ which contradicts (a). This proves the lemma.

We now prove our fixed point theorems.

Theorem 2.2. *Let K be a nonempty convex subset of E. Let $T: K \to 2^K$ be a multi-valued mapping such that*

(a)′ *for each $x \in K$, $T(x)$ is a nonempty convex subset of K;*

(b)′ *for some $x_0 \in K$, the complement of $T^{-1}(x_0)$ in K, denoted by $[T^{-1}(x_0)^c$ is compact;*

(c)′ *for each* $x\in K$, *the intersection of* $[T^{-1}(x)]^c$ *with any finite dimensional subspace of* E *is closed;*

(d)′ *for each* $x\in K$; $[T^{-1}(x)]^c\cap[T^{-1}(x_0)]^c$ *is closed.*

Then there is a point $x_0\in K$ *such that* $x_0\in T(x_0)$.

PROOF. If possible, let us assume that T has no fixed point, i.e. there is no point $x\in K$ such that $x\in T(x)$. This implies that there is no $x\in K$ such that $x\in T^{-1}(x)$. Thus we have (a) $x\in[T^{-1}(x)]^c$ for each $x\in K$; and (b) $[T^{-1}(x)]^c$ is nonempty for each $x\in K$.

We set $F(x)=[T^{-1}(x)]^c$ for each $x\in K$.

Now $A(x)=\{y\in K: x\notin F(y)\}=\{y\in K: x\notin[T^{-1}(y)]^c\}=\{y\in K: x\in T^{-1}(y)\}=T(x)$ which is convex by (a)′. Thus we have condition (c) of Lemma 2.1. Conditions (b), (d) and (e) of lemma 2.1 follow from assumptions (b)′, (c)′ and (d)′ respectively. Hence there is a point $u\in K$ such that $u\in\bigcap_{x\in K}F(x)$, i.e. $u\in[T^{-1}(x)]^c$ for each $x\in K$ i.e. $u\notin T^{-1}(x)$ for any $x\in K$. However, $u\in K=\bigcup_{x\in K}T^{-1}(x)$ which is a contradiction. This proves the theorem.

The following theorem is dual to the above theorem in the sense of [7].

Theorem 2.3. *Let* K *be a nonempty convex subset of* E. *Let* $T: K\to 2^K$ *be a multi-valued mapping such that*

(1) *for each* $x\in K$, $T(x)$ *is a nonempty subset of* K;

(2) *for some* $x_0\in K$, $[T(x_0)]^c$ *is compact in* K;

(3) *for each* $x\in K$, $T^{-1}(x)$ *is convex (may be empty);*

(4) *for each* $x\in K$, *the intersection of* $[T(x)]^c$ *with any finite dimensional subspace of* E *is closed;*

(5) *for each* $x\in K$, $[T(x)]^c\cap[T(x_0)]^c$ *is closed;*

(6) $\bigcup_{x\in K}T(x)=K$.

Then there is a point $x_0\in K$ *such that* $x_0\in T(x_0)$.

PROOF. As before, let us assume that there is no point $x\in K$ such that $x\in T(x)$. This implies (a) $x\in[T(x)]^c$ for each $x\in K$ and (b) $[T(x)]^c$ is nonempty for each $x\in K$.

We set $F(x)=[T(x)]^c$ for each $x\in K$.

Them $A(x)=\{y\in K: x\notin F(y)\}=\{y\in K: x\in T(y)\}=T^{-1}(x)$ which is convex by (3). Thus we have the condition (c) of lemma 2.1. Conditions (2), (4) and (5) imply respectively conditions (b), (d) and (c) of lemma 2.1. Hence there is a point $u\in K$ such that $u\in\bigcap_{x\in K}F(x)=\bigcap_{x\in K}[T(x)]^c$. This implies that $u\notin\bigcup_{x\in K}T(x)$ which is impossible by (6). Thus the theorem is proved.

Corollary 2.1. *Let* K *be a nonempty convex subset of* E *and* $T: K\to 2^K$ *be a multi-valued mapping such that*

(i) *for each* $x\in K$, $T(x)$ *is a nonempty convex subset of* K;

(ii) *for each* $x\in K$, $T^{-1}(x)$ *is open in* K;

(iii) *for some* $x_0\in K$, $[T^{-1}(x_0)^c$ *is compact in* K.

Then there is a point $x_0\in K$ *such that* $x_0\in T(x_0)$.

PROOF. Corollary follows from theorem 2.2.

Remark. This corollary generalizes the theorem 1.1 of BROWDER ([2], theorem 1).

Corollary 2.2. *Let K be a nonempty convex subset of E and $T: K\to 2^K$ be a multi-valued mapping such that*

(i)′ *for each $x\in K$, $T(x)$ is a nonempty open subset of K;*

(ii)′ *for each $x\in K$, $T^{-1}(x)$ is convex (may be empty);*

(iii)′ *for some $x_0\in K$, $[T(x_0)]^c$ is compact;*

(iv)′ $\bigcup_{x\in K} T(x)=K$.

Then there is a point $x_0\in K$ such that $x_0\in T(x_0)$.

Proof. This follows from theorem 2.3.

3. *Applications on Minimax Principles*

3.1. *(Fan's minimax principle). Let K be a nonempty convex subset of E. Let $f(x, y)$ be a real valued function defined on $K\times K$ such that*

(i) *$f(x, x)\leqq 0$ for $x\in K$;*

(ii) *for every $x\in K$, the set $A(x)=\{y\in K: f(x, y)>0\}$ is convex;*

(iii) *there is a compact subset L of E and $x_0\in L\cap K$ such that $f(x, x_0)>0$ for all $x\in K, x\notin L$;*

(iv) *for every $y\in K$, we have*

(1) *$f(x, y)$ is a lower semicontinuous function of x on the intersection of K with any finite dimensional subspace of E;*

(2) *$f(x, y)$ is also a lower semicontinuous function of x on L.*

Then there exists a point $y_0\in L$ such that $f(y_0, y)\leqq 0$ for all $y\in K$.

Proof. For each $y\in K$, we set $F(y)=\{x: f(x, y)\leqq 0\}$. It is easy to see that conditions (a), (c) and (d) of lemma 2.1 follow from assumption (i), (ii) and (iv) respectively. $F(x_0)$ being a subset of L is compact by (iv) (2). Thus condition (b) of Lemma 2.1 holds. (e) follows also from (iv) (2). Hence there exist a point $y_0\in \bigcap_{x\in K} F(x)$, i.e.

$$f(y_0, x) \leqq 0 \quad \text{for all} \quad x\in K.$$

Remark. This includes Fan's minimax principle [4] and is different from the one given in [1].

Although the following form of von Neumann—Sion minimax principle is a minor generalization of the one given by Brezis—Nirenberg—Stampacchia ([1], proposition 1) we would like to include it as a direct application of the minimax principle 3.1.

3.2. *(Von Neumann—Sion minimax principle). Let F be a Hausdorff topological vector space and G be a vector space; let $A\subseteqq F$ and $B\subseteqq G$ be convex sets. Let $H(u, v)$ be a real valued function defined on $A\times B$ satisfying*

(a)′ *for some $\tilde{v}\in B$ and some $\lambda>\sup_{v\in B}\inf_{u\in A} H(u, v)$, the set $P=\{u\in A: H(u, \tilde{v})\leqq\lambda\}$ is compact;*

(b)′ *for each $v\in B$, $H(u, v)$ is a quasi-convex function of u on A;*

(c)′ *for each $v\in B$, $H(u, v)$ is a lower semicontinuous function vf u on P and also a lower semicontanuous funcsion of u on the intersection of A with any finite dimensional subspace of F;*

(d)′ *for each* $u\in A$, $-H(u, v)$ *is a qasi-convex function of* v *on* B *and a lower semicontinuous function of* v *oo the intersection of* B *with any finite dimensional subspace of* G.

Then $\alpha=\sup_{v\in B}\inf_{u\in A} H(u, v)=\inf_{u\in A}\sup_{v\in B} H(u, v)=\beta$.

PROOF. The same proof given in [1] with slight modification will do. For the sake of completeness we give the proof. We maintain the notations of [1]. Obviously $\inf_{u\in A} H(u, v_0)\leqq H(u_0, v_0)\leqq \sup_{v\in B} H(u_0, v)$ for all $u_0\in A$ and $v_0\in B$. Thus $\alpha\leqq\beta$. If possible, let us assume $\alpha<\beta$. We can choose a real number γ satisfying $\alpha<\gamma<<\beta$, $\gamma\leqq\lambda$.

Let $A(v)=\{u\in A: H(u, v)\leqq\gamma\}$ and $B(u)=\{v\in B: H(u, v)\geqq\gamma\}$. By choice of γ we have (1) $\bigcap_{v\in B} A(v)=\emptyset$ and (2) $\bigcap_{u\in A} B(u)=\emptyset$. We set $\bar{A}(v)=A(v)\cap P$ for each $v\in B$. Then by (c) $\bar{A}(v)$ is a closed subset of the compact subset P for each $v\in B$ and by (1) $\bigcap_{v\in B} \bar{A}(v)=\emptyset$. Hence we can find $v_1, v_2, \ldots, v_n\in B$ such that (3) $\bigcap_{i=1}^{n} \bar{A}(v_i)==\emptyset$. We note that $A(\tilde{v})=\bar{A}(\tilde{v})$ as $\gamma\leqq\lambda$. Consequently as $\gamma>\alpha$, we can assume $\lambda=\gamma$ and $\tilde{v}$ as one of v_i, $i=1, 2, \ldots, n$. Let B' be the convex hull of $\{v_i, v_2, \ldots, v_n\}$. We now set $E=F\times R^n$ and $K=A\times B'$ where R^n is the usual n-dimensional Euclidean space. We define f on $K\times K$ by

$$f(x, y) = \min\{H(u, v')-\gamma,\ -H(u', v)+\gamma\},\quad x=(u, v),\quad y=(u', v').$$

Obviously f satisfies (i) of 3.1. f satisfies (ii) of 3.1 by virtue of the quasiconvexity of H and $-H$ assumed in (b)′ and (d)′. We take $L=A(\tilde{v})\times B'$. The lower semicontinuity of H on P assumed in (c)′ and the lower semicontinuity of $-H$ assumed in (d)′ imply the lower semicontinuity of f on L with respect to y for each fixed $x\in K$, i.e. (iv)(2) of 3.1 holds. For each fixed $x\in K$ the lower semicontinuity of f with respect to y on the intersection of K with any finite dimensional subspace of E follows from the corresponding lower semicontinuities of H and $-H$ assumed in (c)′ and (d)′ (we recollect that minimum of two lower semicontinuous functions is lower semicontinuous), i.e. (iv)(1) of 3.1 holds. Finally we take $x_0=(u_0, \tilde{v})\in L\cap K$ for any $u_0\in A$.

We can easily see that (iii) of 3.1 holds with this x_0.

Hence by 3.1 there is a point $y_0=(u^0, v^0)\in K\cap L$ such that $f(y_0, y)\leqq 0$ for all $y\in K$, i.e. for all $u\in A$, $v\in B'$, either $H(u^0, v)\leqq\gamma$ or $\gamma\leqq H(u, v^0)$. Let v be one of v_i. We can choosee $v=v_i$ such that $u_0\notin\bar{A}(v_i)$. This is possible as $\bigcap_{i=1}^{n}\bar{A}(v_i)=\emptyset$. Thus $H(u^0, v_i)>\gamma$ as $u^0\in P$. Thus it follows $H(u, v^0)\geqq\gamma$ for all $u\in A$, i.e. $v^0\in\bigcap_{u\in A} B(u)$ which contradicts (2).

4. *Applications on Sets with Convex Sextions*

4.1. Let $K_1, K_2, \ldots, K_n$ be $n\geqq 2$ nonempty convex sets, each in a Hausdorff topological vector space, and let $K=\prod_{j=1}^{n} K_j$. Let $S_1, S_2, \ldots, S_n$ be n subsets of K having the following properties:

(a) Let $\hat{K}_j = \prod_{i \neq j} K_i$ and let us denote the points of $\hat{K}_j$ by $\hat{x}_j$. For $j=1, 2, \ldots, n$ and for each $\hat{x}_j \in \hat{K}_j$, the set $S_j(\hat{x}_j) = \{x_j \in K_j\colon [x_j, \hat{x}_j] \in S_j\}$ is a nonempty convex subset of K_j;

(b) For each $j=1, 2, \ldots, n$ and for each point $x_j \in K_j$, the set $S_j(x_j) = \{\hat{x}_j \in \hat{K}_j\colon [x_j, \hat{x}_j] \in S_j\}$ is an open subset of $\hat{K}_j$.

(c) For some point $x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in K$, the complement of the set $\bigcap_{j=1}^{n} \{S_j(x_j^0) \times \times K_j\}$ is compact in K.

Then $\bigcap_{j=1}^{n} S_j \neq \emptyset$.

PROOF. For each $x \in K$, let $A(x) = \prod_{j=1}^{n} S_j(\hat{x}_j)$ where $\hat{x}_j$ is the natural projection of x on $\hat{K}_j$. By (a) $A(x)$ is a nonempty convex subset of K for each $x \in K$. We define a multi-valued mapping T of K into 2^K by $T(x) = A(x)$, $x \in K$. Now $x \in T^{-1}_{(y)}$ if $y \in T(x)$, i.e. $y \in A(x) = \prod_{j=1}^{n} S_j(\hat{x}_j)$, i.e. if $y_j \in S_j(\hat{x}_j)$ for each $j=1, 2, \ldots, n$, i.e., $\hat{x}_j \in S_j(y_j)$ for $j=1, 2, \ldots, n$. Hence $T^{-1}(y) = \bigcap_{j=1}^{n} \{S_j(y_j) \times K_j\}$ which is an open set by (b). Finally by (c) $[T^{-1}(x^0)]^c$ is compact in K. Hence by Corollary 2.1, there is a point $z \in K$ such that $z \in T(z)$, i.e. $z = \prod_{j=1}^{n} S_j(\hat{z}_j)$ i.e. $z_j \in S_j(\hat{z}_j)$ for $j=1, 2, \ldots, n$. Hence $z = [z_j, \hat{z}_j] \in S_j$ for $j=1, 2, \ldots, n$. Thus $z \in \bigcap_{j=1}^{n} S_j$.

4.2. Let $K_1, K_2, \ldots, K_n, K, S_1, S_2, \ldots, S_n$ be as in 4.1 satisfying the following properties:

(i) For each $j=1, 2, \ldots, n$ and each point $x_j \in K_j$, the set $S_j(x_j) = \{\hat{x}_j \in \hat{K}_j\colon [x_j, \hat{x}_j] \in S_j\}$ is a convex subset of $\hat{K}_j$.

(ii) For each $j=1, 2, \ldots, n$ and each $\hat{x}_j \in \hat{K}_j$, the set $S_j(\hat{x}_j) = \{x_j \in K_j\colon [x_j, \hat{x}_j] \in S_j\}$ is a nonempty open subset of K_j.

(iii) For some $x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in K$, the complement of the set $\prod_{j=1}^{n} S_j(\hat{x}_j)$ in K is compact.

(iv) $\bigcup_{x \in K} A(x) = K$, where $A(x) = \prod_{j=1}^{n} S_j(\hat{x}_j)$, $\hat{x}_j$ being the natural projection of x on $\hat{K}_j$.

Then $\bigcap_{j=1}^{n} S_j \neq \emptyset$.

PROOF. By (ii) $A(x)$ is a nonempty open subset of K for each $x \in K$. As before we define $T\colon K \to 2^K$ by $T(x) = A(x)$, $x \in K$. By the same argument as before, for each $y \in K$, $T^{-1}(y) = \bigcap_{j=1}^{n} \{S_j(y_j) \times K_j\}$ which is convex by (i). By (iii) the complement of $T(x_0)$ is compact and by (iv) $\bigcup_{x \in K} T(x) = K$. Hence by Corollary 2.2, there is a point $z \in K$ such that $z \in T(z) = A(z)$, i.e. $z = [z_j, \hat{z}_j] \in S_j$ for all $j=1, 2, \ldots, n$. Hence $z \in \bigcap_{j=1}^{n} S_j$.

Remark. 4.1 generalizes a result of Fan [(3), theorem 1]. 4.2 generalizes theorem 2.1 of [7]. We should point out that instead of using corollaries 2.1 and 2.2 we could have used theorems 2.2 and 2.3 to obtain more general statements of 4.1 and 4.2.

The following result is a generalization of a result of Fan ([3], theorem 3).

4.3. Let $K_1, K_2, \ldots, K_n$ and K be as in 4.1. Let $f_1, f_2, \ldots, f_n$ be n real valued functions defined on K satisfying the following properties:

(a) For each $j=1, 2, \ldots, n$, and for each point $x_j \in K_j$, $f_j(x_j, \hat{x}_j)$ is a lower semicontinuous function of $\hat{x}_j$ of $\hat{K}_j$.

(b) For each $j=1, 2, \ldots, n$ and for each point $\hat{x}_j \in \hat{K}_j$, $f_j(x_j, \hat{x}_j)$ is a quasi-concave function of x_j on K_j (i.e. for each real number t, the set $\{x_j \in K_j : f_j(x_j, \hat{x}_j) > t\}$ is a convex subset of K_j.

(c) Let $t_1, t_2, \ldots, t_n$ be n real numbers such that for each j and each point $\hat{x}_j$ of $\hat{K}_j$, there exists a point $y_j \in K_j$ such that $f_j(y_j, \hat{x}_j) > t_j$.

(d) Let us assume that for some point $x^0 = (x_1^0, x_2^0, \ldots, x_n^0) \in K$, the complement of the set $\bigcap_{j=1}^{n} [\{\hat{x}_j \in \hat{K}_j : f(x_j^0, \hat{x}_j) > t_j\} \times K_j]$ in K is compact.

Then there is a point $u \in K$ such that $f_j(u) > t_j$ for all $j=1, 2, \ldots, n$.

Proof. For each $j=1, 2, \ldots, n$. We define the subsets S_j of K by $S_j = \{x \in K : f_j(x) > t_j\}$ which is nonempty by condition (c). The condition (d) says that the complement of $\bigcap_{j=1}^{n} \{S_j(x_j^0) \times K_j\}$ in K is compact. The rest of the proof follows from 4.1.

4.4. Let $K_1, K_2, \ldots, K_n$ and K be as in 4.1. Let $f_1, f_2, \ldots, f_n$ be n real valued functions defined on K satisfying the following properties:

(i) For each $j=1, 2, \ldots, n$ and each $x_j \in K_j$, $f_j(x_j, \hat{x}_j)$ is a quasi-concave function of $\hat{x}_j$ on $\hat{K}_j$.

(ii) For each $j=1, 2, \ldots, n$ and each $\hat{x}_j \in \hat{K}_j$, $f_j(x_j, \hat{x}_j)$ is a lower semicontinuous function of x_j on K_j.

(iii) Let $t_1, t_2, \ldots, t_n$ be n real numbers such that for each $j=1, 2, \ldots, n$ and for each $\hat{x}_j$ of $\hat{K}_j$, there exists a point $y_j \in K_j$ such that

$$f_j(y_j, \hat{x}_j) > t_j.$$

(iv) For some $x_0 = (x_1^0, x_2^0, \ldots, x_n^0) \in K$, the complement of the set $\prod_{j=1}^{n} \{x_j \in K_j : f(x_j, \hat{x}_j^0) > t_j\}$ in K is compact.

(v) $\bigcup_{x \in K} A(x) = K$ where

$$A(x) = \prod_{j=1}^{n} \{x_j \in K_j : f(x_j, \hat{x}_j) > t_j\},$$

$\hat{x}_j$ being the projection of x on $\hat{K}_j$.

Then $\bigcap_{j=1}^{n} S_j \neq \emptyset$.

PROOF. As in the proof of 4.3 for each $j=1, 2, \ldots, n$ we define the subsets S_j of K by $S_j=\{x\in K: f_j(x)>t_j\}$ which is nonempty by (iii). Now condition (iv) reduces to the condition that the complement of $\prod_{j=1}^{n} S_j(\hat{x}_j^0)=A(x_0)$ in K is compact. Now it is easy to see that 4.4 follows from 4.2.

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