Some remarks on D. Borwein's note on Nörlund methods of summability associated with polynomials

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1. Introduction

It is the aim of this paper to generalize several results of the note of BORWEIN [2] just mentioned. BORWEIN remarked that Nörlund methods associated with polynomials are not of most general type. The properties used in his proofs are often only those of Taylor series being absolutly convergent on the closed unit circle and perhaps having some zeros in it. The reader will recognize easily the connections of our theorems with those of [2].

2. Definitions and notations

Let $\mathbf{S}:=\{s|s:\mathbf{N}_0\to\mathbf{C}\}$ be the set of complex sequences. If $s, t\in\mathbf{S}$ $s*t\in\mathbf{S}$ is defined by

$$(s*t)_n:=\sum_{\nu=0}^{n}s_{n-\nu}t_\nu,\quad n\in N_0.$$

We define $\frac{s}{t}\in\mathbf{S}$ if $s,t\in\mathbf{S}$ by

$$\frac{s}{t}n:=\begin{cases}\frac{s_n}{t_n} & \text{if}\quad t_n\neq 0,\\ 0 & \text{if}\quad t_n=0.\end{cases}$$

We often use the following abbreviations

$$1:=\{1,1,1,\ldots\},$$

$$e:=\{1,0,0,\ldots\}$$

for sequences and

$$l:=\Big\{s\in\mathbf{S}\Big|\sum_{\nu=0}^{\infty}|s_\nu|<\infty\Big\},$$

$$o(t):=\{s\in\mathbf{S}|s_n=o(t_n)\}\quad\text{if}\quad t\in\mathbf{S}.$$

Let $\alpha, p \in \mathbf{S}$ be such that $p_0 \neq 0$, $\alpha_n \neq 0$ for $n \in \mathbf{N}_0$ and $(p*\alpha)_n \neq 0$ for almost all n. If $s \in \mathbf{S}$ $\frac{p*\alpha s}{p*\alpha}$ is the generalized Nörlund mean of s. s is said to be generalized Nörlund summable ((N, p, α)-summable) to zero if $p*\alpha s \in o(p*\alpha)$. [1], [3]. We write

$$o(N, p, \alpha) := \{s \in \mathbf{S} \mid p*\alpha s \in o(p*\alpha)\}.$$

The method (N, p, α) reduces to the method (N, p) if $\alpha = 1$ and to the Cesàro method (C, δ), $\delta \geqq 0$, if $p_n = \binom{n+\delta-1}{n}$, $\alpha = 1$.

If $p \in \mathbf{S}$ we write $P(z) := \sum_{\nu=0}^{\infty} p_\nu z^\nu$ and denote the radius of convergence of the power series by $\varrho(p)$. We use similar notations with other letters in place of p.

Let $p \in \mathbf{S}$ be such that $\varrho(p) > 0$ and $P(x) \neq 0$ for $\chi < x < \varrho(p)$. If $PS(z) = \sum p_\nu s_\nu z^\nu$, $\varrho(p \cdot s) \geqq \varrho(p)$ and

$$\lim_{x \to \varrho(p)-} \frac{PS(x)}{P(x)} = 0$$

we write $s \in o(J, p)$. [5, p. 79], [8, p. 186].

We define $o(J, p)(N, q, \alpha) := \left\{ s \in \mathbf{S} \,\middle|\, \frac{q*\alpha s}{q*\alpha} \in o(J, p) \right\}$.

3. Theorems

If $p \in l$ with $P(0) \neq 0 \neq P(1)$ and $\delta \geqq 0$ $o(C, \delta) \not\subset o(N, p, 1)$ remains true. The proof is analogous to the proof of theorem 1 [2] since for z with $|z| = 1$ we have

$$\left\{ \sum_{\nu=n+1}^{\infty} p_\nu z^{-\nu} \right\} \in o(1)$$

and

$$\sum_{\nu=0}^{n} p_\nu z^{n-\nu} = z^n \Big(P(z^{-1}) - \sum_{\nu=n+1}^{\infty} p_\nu z^{-\nu} \Big).$$

The general conditions for $o(C, \delta) \not\subset o(N, p, \alpha)$ are given by a lemma of [4, p. 422].

We prove a theorem in the other direction and consider a summability method related to (C, δ).

Let $\delta \geqq -1$ and let the sequence $\varepsilon^\delta \in \mathbf{S}$ be defined by $\varepsilon_n^\delta := \binom{n+\delta}{n}$. Then $(C, \delta) = (N, \varepsilon^{\delta-1}, 1)$.

Theorem 1. *Let (N, p, α) be defined with p such that $P(\lambda_0) = 0$ for some λ_0 with $0 < |\lambda_0| = \Lambda < 1$ and $\varrho(\alpha) > \Lambda$. If $\delta \geqq 0$ then $o(N, p, \alpha) \not\subset o(N, \varepsilon^{\delta-1}, \alpha)$.*

Proof. Since $p_0 \neq 0$ there exists a $k \in \mathbf{S}$ with $p*k = e$. Hence if $h := e^{\delta-1}$ then $\varrho(h*k) = \varrho(k) \leqq \Lambda$. Since $\varrho(\alpha) > \Lambda$ we have $\varrho(h*\alpha) > \Lambda$ and $h*k \notin o(h*\alpha)$. Now $o(N, p, \alpha) \not\subset o(N, \varepsilon^{\delta-1}, \alpha)$ follows by 4.2 of the lemma 1 of [4, p. 422].

We require the following known lemma [7] for the proof of our other theorems.

Lemma 1. *Let* $\alpha, p \in \mathbf{S}$. *Consider the conditions*

$$\text{(i)} \qquad \lim_{n\to\infty} \frac{(p*\alpha)_{n-1}}{(p*\alpha)_n} = 1.$$

There exists a M *such that for all* $n, \mu \in \mathbf{N}_0$ *with* $n \geqq \mu$

$$\text{(ii)} \qquad \left|\frac{(p*\alpha)_{n-\mu}}{(p*\alpha)_n}\right| < M.$$

Let $r \in l$ *and* $R(1) \neq 0$. *Then*

$$o(p*\alpha) = o(r*p*\alpha)$$

if and only if (i) *and* (ii) *hold. If* (ii) *holds then*

$$o(p*\alpha) \supset \{x \in \mathbf{S} \mid x = g*r,\ g \in o(p*\alpha),\ r \in l\}.$$

The conditions (i) and (ii) are independent of those for the regularity of (N, p, α). If we write $\Delta(p*\alpha) = \{p*\alpha)_n - (p*\alpha)_{n-1}\}$ (i) and (ii) are generally weaker than the regularity of $(N, \Delta(p*\alpha), 1)$. Thus in the case of (ordinary) Nörlund methods $(N, p, 1)$ (i) and (ii) are generally weaker than the regularity of $(N, p, 1)$.

Theorem 2. *Let* (N, p, α) *be defined and* (i), (ii) *hold. For* $\mu = 1, 2$ *let* ${}_\mu r \in l$ *and* ${}_\mu R(0) \neq 0 \neq {}_\mu R(1)$. *Then*

$$o(N, p, \alpha) \subset o(N, {}_\mu r*p, \alpha) \subset o(N, {}_1r*{}_2r*p, \alpha).$$

Proof. It is sufficient to prove $o(N, {}_1r*p, \alpha) \subset o(N, {}_1r*{}_2r*p, \alpha)$. Let $s \in o(N, {}_1r*p, \alpha)$. Then by lemma 1 ${}_1r*p*\alpha s \in o(p*\alpha)$ and ${}_2r*({}_1r*p*\alpha s) \in o(p*\alpha) = o({}_1r*{}_2r*p*\alpha)$. That means $s \in o(N, {}_1r*{}_2r*p, \alpha)$.

Theorem 3. *Let* $r \in l$ *with* $R(0) \neq 0 \neq R(1)$. *Suppose that* R *has an infinity of roots at* $\lambda_\mu, \mu \in \mathbf{N}_0, |\lambda_\mu| < 1$, *with the multiplicities* $\varrho_\mu, \varrho_\mu > 0$. *Let* Ω *be the set of all* $q \in l$ *such that* $\sum q_\nu z^\nu$ *is a polynomial having the roots* $\lambda_{\mu(q)}$ *with the multiplicities* $\pi_{\mu(q)} \leqq \varrho_{\mu(q)}$. *Let* α, p *be such that* (i), (ii) *hold. Then*

$$\bigcup_{q\in\Omega} o(N, q*p, \alpha) \subsetneqq o(N, r*p, \alpha).$$

Proof. Let $q \in \Omega$. Because of lemma 4 of [6] there exists a $f \in l$ such that $q*f = r$. By theorem 2 we have $o(N, q*p, \alpha) \subset o(N, r*p, \alpha)$.

Now consider ${}_0q \in \Omega$. Then there is a ${}_1q \in \Omega$ such that ${}_1Q$ has exactly one more root than ${}_0Q$. Let $h, k \in \mathbf{S}$ be such that $k*p = e$ and $h*{}_1q = e$. Since $\varrho({}_0q*h) < 1$ $({}_0q*p)*(k*h) = {}_0q*h \notin o({}_0q*p*\alpha) = o(p*\alpha)$. Hence by 4.2 of lemma 1 of [4] $o(N, {}_0q*p, \alpha) \neq o(N, {}_1q*p, \alpha) \subset \bigcup_{q\in\Omega} o(N, q*p, \alpha)$. By a known theorem of [8, p. 51] the result of theorem 3 follows.

Theorem 4. *Let for* $\mu = 1, 2, 3$, ${}_\mu r \in l$ *such that* ${}_\mu R(0) \neq 0 \neq {}_\mu R(1)$. *Let* ${}_1r$ *be the "highest common factor" of* ${}_2r$ *and* ${}_3r$ *in the following sense: There are* ${}_\nu a \in l$, $\nu = 1, 2, 3, 4$ *such that* ${}_2r = {}_2a*{}_1r$, ${}_3r = {}_3a*{}_1r$, ${}_1r = {}_1a*{}_2r + {}_4a*{}_3r$. *Let* α, p *be such that* (i), (ii) *hold. Then*

$$o(N, {}_1r*p, \alpha) = o(N, {}_2r*p, \alpha) \cap o(N, {}_3r*p, \alpha).$$

PROOF. Let $s \in o(N, {}_2r * p, \alpha) \cap o(N, {}_3r * p, \alpha)$. ${}_1r * p * \alpha s = {}_1a * {}_2r * p * \alpha s + {}_4a * {}_3r * p * \alpha s \in o(p * \alpha)$ by lemma 1, hence $s \in o(N, {}_1r * p, \alpha)$. The other direction follows by theorem 2.

Theorem 5. *Let ${}_\mu r \in l$, $\mu = 1, 2, 3$, be such that ${}_\mu R(0) \neq 0 \neq {}_\mu R(1)$, ${}_3R(z) \neq 0$ for $|z| \leqq 1$ and*

$$ {}_1R(z) = {}_2R(z) \cdot {}_3R(z) \cdot \prod_{i=1}^{\varkappa} (1 - z/\lambda_i)^{m_i}, $$

$$ \frac{{}_2R(z)}{{}_1R(z)} = C(z) + \sum_{i=1}^{\varkappa} \sum_{j=1}^{m_i} \frac{c_{ij}}{(1 - z/\lambda_i)^j}, $$

where $\varkappa, m_i \geqq 1$, $|\lambda_i| = 1$, $\lambda_i \neq 1$, $c \in l$, $c_{ij} \in \mathbf{C}$. Let $m := \max(m_1, \ldots, m_\varkappa)$ and $o({}_1r * p * \alpha) = o(1)$. Then

$$ \text{(iii)} \qquad o(N, {}_1r * p, \alpha) \subset \{s \in \mathbf{S} \mid {}_2r * p * \alpha s \in o(C, m)\} $$

but if $\eta > 0$

$$ \text{(iv)} \qquad o(N, {}_1r * p, \alpha) \not\subset \{s \in \mathbf{S} \mid {}_2r * p * \alpha s \in o(C, m - \eta)\}. $$

PROOF OF (iii). Let $s \in o(N, {}_1r * p, \alpha)$, then $t := {}_1r * p * \alpha s \in o(1)$. With $w \in \mathbf{S}$ such that $w * {}_1r = e$ and $u := {}_2r * p * \alpha s = {}_2r * w * t$ we have for $|z| < 1$

$$ U(z) = T(z) \left(C(z) + \sum_{i=1}^{\varkappa} \sum_{j=1}^{m_i} \frac{c_{ij}}{(1 - z/\lambda_i)^j} \right). $$

Since $\lambda_i \neq 1$ by lemma 5 of [2] and $t * c \in o(1)$ we get $u \in o(C, m)$. Hence (iii) holds.

PROOF OF (iv). Without loss of generality we assume $m_\varkappa = \max(m_1, \ldots, m_\varkappa) = m$. With $|z| < 1$, $0 < \eta < 1$, and ${}_4R(z) := \prod_{i=1}^{\varkappa-1} (1 - z/\lambda_i)^{m_i}$ we write $B(z) := {}_3R(z) \cdot {}_4R(z) \cdot (1 - z)^{m - \eta}$ and $V(z) := B(z) \cdot (1 - z/\lambda_\varkappa)^{\eta - 1}$. By $\varepsilon_n^{-\eta} \sim \dfrac{n^{-\eta}}{\Gamma(1 - \eta)}$ and $|\lambda_\varkappa| = 1$ $\{\varepsilon_n^{-\eta} \cdot \lambda_\varkappa^{-n}\} \in o(1)$. Since $b \in l$ we have $v \in o(1)$. Because of $\alpha_n \neq 0$, ${}_1r_0 \cdot p_0 \neq 0$ there is a $s \in \mathbf{S}$ such that ${}_1r * p * \alpha s = v$. That means $s \in o(N, {}_1r * p, \alpha)$. Writing $G(z) := (1 - z/\lambda_\varkappa)^{-(m + 1 - \eta)}$, we get for z in a neighbourhood of zero

$$ \sum_{\nu=0}^{\infty} (p * \alpha s)_\nu z^\nu = \frac{V(z)}{{}_1R(z)} = \frac{G(z) \cdot (1 - z)^{m - \eta}}{{}_2R(z)}. $$

Since $\{g_n\} = \{\varepsilon_n^{m - \eta} \lambda_\varkappa^{-n}\} \notin o(\{\varepsilon_n^{m - \eta}\})$ it follows that ${}_2r * p * \alpha s \notin o(C, m - \eta)$. Hence (iv) holds.

By an example we wish to show that $o({}_1r * p * \alpha) = o(1)$ is a necessary condition of theorem 5. Take $p = {}_2r = {}_3r = e$ and ${}_1R(z) = 1 + z$. If $\alpha = \{n + 1\}$ and $s = \{(-1)^n\}$ then $o({}_1r * p * \alpha) \neq o(1)$ and $s \in o(N, {}_1r * p, \alpha)$ but $\alpha s = {}_2r * p * \alpha s \notin o(C, 1)$.

The assumption $\dfrac{{}_2r * p * \alpha s}{{}_2r * p * \alpha} \in o(C, m)$ is wrong even in the case $o({}_1r * p * \alpha) = o(1)$. Take the same ${}_1r$, ${}_2r$ and p as just mentioned but α and s such that $\alpha_{2n} = 2$, $\alpha_{2n+1} = 1$, $s_{2n} = \dfrac{1}{2}$, $s_{2n+1} = -1$. Then $s \in o(N, {}_1r * p, \alpha)$ but $s \to -\dfrac{1}{4}$ in the $(C, 1)$ sence.

Theorem 6. *Let* (N, p, α), (N, q, α), (J, m) *be defined and* $h \in \mathbf{S}$ *such that* $p * h = q$. *If* $\varrho\left(\frac{m \cdot h}{q * \alpha}\right) < \varrho(m)$, *then*

$$o(N, p, \alpha) \not\subset o(J, m)(N, q, \alpha).$$

PROOF. There is a $k \in \mathbf{S}$ with $p * k = e$. Define s by $\alpha s = k$. Then $s \in o(N, p, \alpha)$. Since $q * \alpha s = h$ and $\varrho\left(\frac{m \cdot h}{q * \alpha}\right) < \varrho(m)$ $s \notin o(J, m)(N, q, \alpha)$.

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