On some classification of the linear connection in the small-dimensional space L_n

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§ 1. Introduction

Let be given at point $\overset{0}{\xi}$ of differentiable manifold X_n an abstract geometric object, special, purely differential in with the space of coordinates (fibre) $\mathfrak{M}$ and transformation rule

$$\omega' = F(\omega, L), \quad \omega \in \mathfrak{M}, \quad L \in L_n^s, \tag{1.1}$$

where L_n^s is n-dimensional differential group of rank s, [1].

The set of all elements of space $\mathfrak{M}$ obtaining for fixed $\omega_0 \in \mathfrak{M}$ by action of all elements $L \in L_n^s$:

$$\omega = F(\omega_0, L), \quad L \in L_n^s, \quad \omega_0 \in \mathfrak{M} \tag{1.2}$$

is called an orbit of space $\mathfrak{M}$ and is denoted by symbol $\mathfrak{M}_{\omega_0}$, [1].

Obviously we have distribution

$$\mathfrak{M} = \bigcup_{\omega_0 \in \mathfrak{M}} \mathfrak{M}_{\omega_0}. \tag{1.3}$$

The subset $\overline{\mathfrak{M}} \subset \mathfrak{M}$ is called an invariant set of an abstract object ω if there is satisfied implication:

$$\text{if} \quad \omega_1 \in \overline{\mathfrak{M}}, \quad \text{then} \quad \omega_2 = F(\omega_1, L) \in \overline{\mathfrak{M}} \quad \text{for every} \quad L \in L_n^s. \tag{1.4}$$

The invariant set $\overline{\mathfrak{M}}$ is also the sum of some number of orbits.

The space with linear connection L_n is the differential manifold X_n provided with a field of linear connection $L_{ij}{}^k$ satisfying the following transformation rule:

$$L_{i'j'}{}^{k'} = A_{i'}{}^i A_{j'}{}^j A_k{}^{k'} L_{ij}{}^k + A_l{}^{k'} \partial_{i'} A_{j'} \qquad \begin{array}{l} i, j, k, l = 1, \dots, n \\ i', j', k' = 1', \dots, n', \end{array} \tag{1.5}$$

where

$$A_k{}^{k'} = \frac{\partial \varphi^{k'}(\xi^k)}{\partial \xi^k}, \quad A_{i'}{}^i = \frac{\partial \psi^i(\xi^{i'})}{\partial \xi^{i'}} \tag{1.6}$$

are partial derivatives of transformation of coordinate system $(\varkappa) \to (\varkappa')$ determined in some neighbourhood of every point $\xi \in X_n$ [3].

Moreover the formula is true

$$A_{i'}{}^{i} = \frac{\operatorname{minor} A_{i}{}^{i'}}{J}, \quad J = \det(A_i{}^{i'}) \neq 0, \tag{1.7}$$

where minor is understood in algebraical sense.

We take denotations

$$L_{ij}{}^{k} = \Gamma_{ij}{}^{k} + S_{ij}{}^{k}, \tag{1.8}$$

where

$$\Gamma_{ij}{}^{k} = L_{(ij)}{}^{k}, \quad S_{ij}{}^{k} = L_{[ij]}{}^{k}. \tag{1.9}$$

On base of [3] it is known that $\Gamma_{ij}{}^{k}$ is linear symmetric connection and torsion $S_{ij}{}^{k}$ is antisymmetric tensor field.

In view of papers [4], [5] at every fixed point $\overset{0}{\xi} \in X_n$ are known the canonical forms and the orbits (the families of orbits) of the coordinate space $\mathfrak{M}$ of antisymmetric tensor $S_{ij}{}^{k}$ for $n=2$ and $n=3$. Basing on this we shall carry out the classification of linear connection $L_{ij}{}^{k}$ in next sections of the paper.

Definition 1. Linear connection $L_{ij}{}^{k}$ of space L_n is said to be of type $M_{\underset{0}{S}}$ corresponding with canonical form $\underset{0}{S}$ of the torsion tensor $S_{ij}{}^{k}$, where $\mathfrak{M}_{\underset{0}{S}}$ is an orbit (a family of orbit) of the coordinates space of torsion $S_{ij}{}^{k}$ generated by $\underset{0}{S}$, if at every point $\overset{0}{\xi} \in L_n$, $[S_{ij}{}^{k}] \in \mathfrak{M}_{\underset{0}{S}}$.

Basing on definition 1 we can say that space L_n is of type $\mathfrak{M}_{\underset{0}{S}}$.

§ 2. Classification of space L_2

In space L_2 tensor $S_{ij}{}^{k}$ of type (1.9) has only two essential coordinates $S_{12}{}^{1}$, $S_{12}{}^{2}$ and the matrix of its coordinates is of the form

$$S = \begin{bmatrix} 0 & 0 \\ S_{12}{}^{1} & S_{12}{}^{2} \\ -S_{12}{}^{1} & -S_{12}{}^{2} \\ 0 & 0 \end{bmatrix}. \tag{2.1}$$

Taking denotations

$$S^{k} = \frac{1}{2} S_{ij}{}^{k} \varepsilon^{ij} \quad k, i, j = 1, 2 \tag{2.2}$$

we obtain the relation

$$S_{ij}{}^{k} = S^{k} \varepsilon_{ij}, \tag{2.3}$$

where in formulas (2.2) and (2.3) ε^{ij} and ε_{ij} are Ricci symbols of space X_2.

On base of [2], [3] the abstract object (2.2) has the following transformation rule

$$S^{k'} = J^{-1} A_{k}{}^{k'} S^{k}, \quad k = 1, 2; \quad k' = 1', 2'. \tag{2.4}$$

Object (1.9) is vector $-G$-density of weight (1). In virtue of (2.2) and (2.3) we can see that abstract objects $S_{ij}{}^k$ and S^k are strictly equivalent, therefore they have the same number of homeomorfic in pairs orbits, [1].

Object S^i has two orbits:

$$\mathfrak{N}_1 = (0, 0), \quad \mathfrak{N}_2 = R^2 - \{(0, 0)\}. \tag{2.5}$$

The represant of orbit $\mathfrak{N}_2$ is element (0, 1) determining the second canonical form of object S^i.

Really, for $(S^1, S^2) \neq (0, 0)$ the following nonsingular matrix [5]

$$[A_i^{i'}] = \begin{bmatrix} S_{12}{}^2 & -S_{12}{}^1 \\ S_{12}{}^1 & S_{12}{}^2 \end{bmatrix} \tag{2.6}$$

transformates with the help of rule (2.4) an arbitrary point $S_1, S_2 \neq (0, 0)$ into point (0, 1).

In view of the above tensor $S_{ij}{}^k$ has the following canonical forms:

$$\underset{1}{S} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underset{2}{S} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}. \tag{2.7}$$

In view (2.1) and (2.6) tensor $S_{ij}{}^k$ has two orbits determined below by conditions:

$$\mathfrak{M}_{S_1}\colon (S^1)^2 + (S^2)^2 = 0; \tag{2.8}$$

$$\mathfrak{M}_{S_2}\colon (S^1)^2 + (S^2)^2 > 0. \tag{2.9}$$

Directly from definition 1 (§ 1) and from formulas (2.8) and (2.9) results the following

Theorem 1. *Space L_2 is of the type $\mathfrak{M}_{\underset{1}{S}}$ or $\mathfrak{M}_{\underset{2}{S}}$ when at every point $\xi \in L_2$ matrix S of the form* (2.1) *satisfies the condition: $S \in \mathfrak{M}_{\underset{1}{S}}$ or $S \in \mathfrak{M}_{\underset{2}{S}}$.*

§ 3. Orbits of tensor $S_{ij}{}^k$ in space X_3.

In space X_3 tensor $S_{ij}{}^k$ of the type (1.9) has nine essential coordinates.

Introducing denotations

$$S^{ij} = \frac{1}{2} S_{pq}{}^i \varepsilon^{pqj} \tag{3.1}$$

we get the relations

$$S_{ij}{}^k = S^{kl} \varepsilon_{ijl}, \quad i, j, k, l, p, q = 1, 2, 3 \tag{3.2}$$

where ε^{ijk}, ε_{ijk} are Ricci symbols of space X_3 [2].

Objects $S_{ij}{}^k$ and S_{ij} are strictly equivalent.

On base of [1] the abstract object (1.3) has the transformation rule

$$S^{i'j'} = J^{-1} A_i^{i'} A_j^{j'} S^{ij}. \tag{3.3}$$

In paper [4] is proved the following

Theorem 2. *In space X_3 tensor $-G$-density* (3.3) *has eighteen following canonical forms (congruent) which we will write in an integrate form:*

$$
(3.4)\quad
\left\{
\begin{array}{ll}
S_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & S_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
S_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}, & S_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}, \\
\text{a) } \varepsilon = 1, \text{ b) } \varepsilon = -1; & \text{a) } \varepsilon = 1, \text{ b) } \varepsilon = -1; \\
S_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, & S_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \\
S_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, & S_8 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1+\alpha \\ 0 & 1-\alpha & 0 \end{bmatrix}, \quad \alpha \neq 0, \\
S_9 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & \beta \\ 0 & -\beta & 1 \end{bmatrix}, & S_{10} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \beta \neq 0, \\
S_{11} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}, & S_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1+\alpha \\ 0 & 1-\alpha & 0 \end{bmatrix}, \\
\text{a) } \varepsilon = 1, \text{ b) } \varepsilon = -1; & \alpha \neq 0 \\
S_{13} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & \beta \\ 0 & -\beta & 1 \end{bmatrix}, & S_{14} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}. \\
\text{a) } \varepsilon = 1, \text{ b) } \varepsilon = -1; &
\end{array}
\right.
$$

Theorem 3. *At every point* $\overset{0}{\xi}\in X_3$ *the orbits* $\mathfrak{M}_{S_i}$, $i=1, \ldots, 5$ *of the tensor-density* (3.1) *can be characterizated in the following way:*

$$(3.5)\qquad \begin{cases} \mathfrak{M}_{S_1}\colon\ p_1 = p_2 = p_3 = 0; \\ \mathfrak{M}_{S_2}\colon\ p_1 \neq 0, \quad p_2 = p_3 = 0; \\ \mathfrak{M}^a_{S_3}\colon\ p_1 \neq 0, \quad p_2 > 0, \quad p_3 = 0; \\ \mathfrak{M}^b_{S_3}\colon\ p_1\text{-arbitrary}, \quad p_2 < 0, \quad p_3 = 0; \\ \mathfrak{M}^a_{S_4}\colon\ p_1 \neq 0, \quad p_2 < 0, \quad p_3 \neq 0; \\ \mathfrak{M}^b_{S_4}\colon\ \begin{cases} p_1 \neq 0, \quad p_2 \geqq 0, \quad p_3 \neq 0; \\ \text{or} \\ p_1\text{-arbitrary}, \quad p_2 < 0, \quad p_3 \neq 0; \end{cases} \\ \mathfrak{M}_{S_5}\colon\ S^{(ij)} = 0, \quad S^{[ij]} \neq 0, \end{cases}$$

where $-p_1, p_2, -p_3$ *are the coefficientes of the characteristic polynom of matrix* $S=[S^{ij}]$:

$$(3.6)\qquad \varphi(\lambda) = \det(\lambda E - S) = \lambda^3 - p_1\lambda^2 + p_2\lambda - p_3.$$

To characterize the other orbits (families of orbits) of tensor -G-density (3.1) we will define at first some auxiliary geometric objects. Let

$$(3.7)\qquad h^{ij} = S^{(ij)}, \quad k^{ij} = S^{[ij]}, \quad i, j = 1, 2, 3.$$

On base of (3.1) and (3.3) we observe that objects h^{ij} and k^{ij} are also tensor $-G$-densities of wight (1).

Let us the tensor bundle $\lambda h^{ij}+k^{ij}$ or corresponding with it the matrix bundle $\lambda H+K$, where

$$(3.8)\qquad H = [h^{ij}], \quad K = [k^{ij}].$$

For matrix bundle $\lambda H+K$ let us form a compound matrix of degree 2, ..., where $\underset{2}{S}_{ij}(\lambda)$ is minor of degree 2 of matrix $\lambda H+K$ arised by drawing out of i-th now and j-th column; $i, j=1, 2, 3$.

On base of immediately calculations we obtain the following distribution:

$$(3.9)\qquad \underset{2}{S}_{ij}(\lambda) = \lambda^2 \underset{2}{H}_{ij} + \lambda \underset{2}{M}_{ij} + \underset{2}{K}_{ij}, \quad i, j = 1, 2, 3$$

or in matrix form

$$(3.10)\qquad \underset{2}{S}(\lambda) = \lambda^2 \underset{2}{H} + \lambda \underset{2}{M} + \underset{2}{K},$$

where

$$(3.11)\qquad \underset{2}{H} = [\underset{2}{H}_{ij}], \quad \underset{2}{K} = [\underset{2}{K}_{ij}], \quad \underset{2}{M} = [\underset{2}{M}_{ij}],$$

while in view of (3.7) and (3.10) for $\lambda=1$

$$(3.12)\qquad \underset{2}{M} = \underset{2}{S} - \underset{2}{H} - \underset{2}{K}.$$

Matrices $\underset{2}{H}, \underset{2}{K}$ and $\underset{2}{M}$ satisfy the following conditions of symmetry or anti-symmetry:

(3.13)
$$\underset{2}{H}^t = \underset{2}{H}, \quad \underset{2}{K}^t = \underset{2}{K}, \quad \underset{2}{M}^t = -\underset{2}{M}.$$

In paper [4] is proved that geometric objects $\underset{2}{S}_{ij}(\lambda), \underset{2}{H}_{ij}, \underset{2}{M}_{ij}, \underset{2}{K}_{ij}$ defined by distribution (3.9) have the transformation rules of following type:

(3.14)
$$\underset{2}{M}_{i'j'} = B^i_{i'} B^j_{j'} \underset{2}{M}_{ij}, \quad i, j = 1, 2, 3; \quad i', j' = 1', 2', 3'$$

where

(3.15)
$$B^i_{i'} = (-1)^{i+i'} A^i_{i'}, [A^i_{i'}] \in GL(3, R).$$

They are the geometric linear homogeneous objects therefore their vanishing or non-vanishing has invariant character.

Now we can formulate the following

Theorem 4. *At every point* $\xi \in X_3$ *the orbits (families of orbits)* $\mathfrak{M}_{S_l}$, $l=6, \ldots, 14$ *of tensor-G-density* (3.1) *can be characterizated by the following systems of conditions:*

(3.16)
$$\begin{cases} \mathfrak{M}_{S_6}: H \in \mathfrak{M}^b_{S_3}, \quad K \neq 0, \quad \underset{2}{M} \neq 0, \quad \det S = 0; \\ \mathfrak{M}_{S_7}: H \in \mathfrak{M}_{S_2}, \quad K \neq 0, \quad \underset{2}{M} = 0; \\ \mathfrak{M}_{S_8}: H \in \mathfrak{M}^b_{S_3}, \quad K \neq 0, \quad \underset{2}{M} = 0, \quad \underset{2}{K} = -\alpha^2 \underset{2}{H}, \quad \alpha \neq 0; \\ \mathfrak{M}_{S_9}: H \in \mathfrak{M}^a_{S_3}, \quad K \neq 0, \quad \underset{2}{M} = 0, \quad \underset{2}{K} = \beta^2 \underset{2}{H}, \quad \beta \neq 0; \\ \mathfrak{M}_{S_{10}}: H \in \mathfrak{M}_{S_2}, \quad K \neq 0, \quad \det S \neq 0; \\ \mathfrak{M}^a_{S_{11}}: H \in \mathfrak{M}^a_{S_3}, \quad K \neq 0, \quad \det S \neq 0; \\ \mathfrak{M}^b_{S_{11}}: H \in \mathfrak{M}^b_{S_3}, \quad K \neq 0, \quad \det S \neq 0; \\ \mathfrak{M}_{S_{12}}: H \in \mathfrak{M}^b_{S_4}, \quad \det S \neq \det H, \quad \alpha^2 = \dfrac{\det H - \det S}{\det H}; \\ \mathfrak{M}^a_{S_{13}}: H \in \mathfrak{M}^a_{S_4}, \quad \det S \neq \det H, \quad \beta^2 = \dfrac{\det S - \det H}{\det H}; \\ \mathfrak{M}^b_{S_{13}}: H \in \mathfrak{M}_{S_4}, \quad \det S \neq \det H, \quad \beta^2 = \dfrac{\det S - \det H}{\det H}; \\ \mathfrak{M}_{S_{14}}: H \in \mathfrak{M}^b_{S_4}, \quad K \neq 0, \quad \det S = \det H. \end{cases}$$

§ 4. Classification of space L_3

In view of the results of section 3 we can make the classification of the linear connection $L_{ij}{}^k$ of space L_3.

In virtue of formulas (3.1) or (3.2) we can see that tensor $S_{ij}{}^k$ and G-density S^{ij} are the abstract geometric objects strictly equivalent [1]. In that case both geometric objects $S_{ij}{}^k$ and S^{ij} have the same homeomorphic in pairs orbits (families of orbits) what permits to do the mentioned below classification.

According to formula (3.2) matrix $\hat{S}$ of coordinates of tensor $S_{ij}{}^k$ can be expressed with the help of coordinates of tensor-G-density S^{ij} in the following way:

$$\hat{S} = \begin{bmatrix} 0 & 0 & 0 & -S^{13} & -S^{23} & -S^{33} & S^{12} & S^{22} & S^{32} \\ S^{13} & S^{23} & S^{33} & 0 & 0 & 0 & -S^{11} & -S^{21} & -S^{31} \\ -S^{12} & -S^{22} & -S^{32} & S^{11} & S^{21} & S^{31} & 0 & 0 & 0 \end{bmatrix}. \tag{4.1}$$

Definition 2. The canonical form of tensor $S_{ij}{}^k$ we call the form $\underset{0}{\hat{S}}$ of matrix (4.1) corresponding with canonical form of coordinate matrix $[\underset{0}{S}{}^{ij}]$ [of types (3.5)] of tensor-G-density S^{ij} determined by formulas (3.1) and (3.3).

On base of formulas (3.4) and (4.1) we have the following

Corollary. *Antisymmetric tensor $S_{ij}{}^k$ has at every point $\xi \in X_3$ eighteen canonical forms $\underset{0}{\hat{S}}$ of type* [(4.1), (3.4)].

In accordance with definition 1 from section 1 and with theorems 3 and 4 from section 3 is true the following

Theorem 5. *In space L_3 we can distinguish eighteen types of linear connection $L_{ij}{}^k(\xi^l)$ in dependity of the fact that matrix $\underset{0}{\hat{S}}$ of the tensor of torsion $S_{ij}{}^k$ belongs at each point $\overset{0}{\xi} \in L_3$ to the some orbit (family of orbits) $\mathfrak{M}_{S_1}, \ldots, \mathfrak{M}_{S_{14}}$ defined by formulas* (3.5) *or* (3.16) *respectively.*

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