On the divergence of Fourier series

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1. Let f be a 2π-periodic and integrable function and let $s_k(x)=s_k(f;x)$ and $\sigma_k(x)=\sigma_k(f;x)$ be the k-th partial sum and the k-th $(C,1)$-mean of its Fourier series, respectively. $m(.)$ denotes the Lebesgue-measure on the line. We say that the matrix $(t_{nk})_{n,k=0}^{\infty}$ is regular if

$$\lim_{n\to\infty}\sum_{k=0}^{\infty} t_{nk}s_k = s$$

whenever $s_k\to s$.

Under "polynomial" we always understand "trigonometric polynomial".

A. N. KOLMOGOROV [1] proved that Fourier series may be divergent everywhere. On the other hand if f is fixed and $\{\mu_k\}$ increases rapidly then $s_{\mu_k}(x)\to f(x)$ almost everywhere (a.e.). Now we prove that here $\{\mu_k\}$ cannot be choosen universally.

Theorem 1. *If $\{\mu_k\}$ is an arbitrary subsequence of the natural numbers then there is a function f for which*

$$\sup_k |s_{\mu_k}(f;x)| = \infty \quad (x\in[-\pi;\pi]),$$

Turning to the strong summation we mention the following result of A. ZYGMUND [2]: if $p>0$ then

$$\frac{1}{n+1}\sum_{k=0}^{n}|s_k(x)-f(x)|^p = o_x(1) \quad \text{(a.e.)}.$$

Naturally arises the question what can we say if we exchange here the sequence $\{s_k\}$ for a subsequence of it. We have

Theorem 2. *If $p>0$ and*

$$1 \leqq \mu_{k+1}-\mu_k \leqq K \quad (k=0,1,\dots)$$

then

(1.1)
$$\frac{1}{n+1}\sum_{k=0}^{n}|s_{\mu_k}(x)-f(x)|^p = o_x(1)$$

almost everywhere.

Furthermore if $\nu_k\to\infty$ arbitrarily and $T=(t_{nk})$ is a non-negative regular matrix then there exist a function f and a sequence $\{\mu_k\}$ for which $\mu_{k+1}-\mu_k\leqq\nu_k$ $(k=1,0,\dots)$

and

$$\sup_n \sum_{k=0}^{\infty} t_{nk} |s_{\mu_k}(f; x)|^p = \infty \tag{1.2}$$

everywhere.

We mention the following consequence: Zygmund's result implies that if $\frac{k}{\mu_k} \geqq \varrho > 0$ then (1.1) is true a.e., while Theorem 2 gives that if $\varrho_n \to 0$ ($\varrho_n \leqq 1$) arbitrarily then there is an f and a sequence $\{\mu_k\}$ for which $\frac{k}{\mu_k} \geqq \varrho_k$ and (1.2) is true everywhere, i.e. divergence may take place on relatively "thick" sequences.

The proofs of these theorems are based on a modification, which was used formerly by K. TANDORI [2], of the original Kolmogorov construction and on the next theorem.

Let $\Phi(x)$ ($x \geqq 0$) be a nondecreasing convex function satisfying $0 \leqq \Phi(2x) \leqq \leqq K\Phi(x)$. Let $L_\Phi = \{f \mid \|f\|_{L_\Phi} < \infty\}$ where $\|f\|_{L_\Phi} = \int_{-\pi}^{\pi} \Phi(|f(x)|)\,dx$ (note that this is not necessarily a norm). Remark that $\sigma_n \to f$ in "L_Φ-norm" i.e. $\|f - \sigma_n(f)\|_{L_\Phi} = o(1)$ $(n \to \infty)$ (this follows easily from the fact that by Jensen inequality $\|\sigma_n(f)\|_{L_\Phi} \leqq \|f\|_{L_\Phi}$).

Theorem 3. *Let Φ be as above, $T = (t_{nk})$ an arbitrary non-negative regular matrix and $p > 0$. There are only two possibilities: either for all $f \in L_\Phi$ we have*

$$\sum_{k=0}^{\infty} t_{n_k} |s_k(f; x) - f(x)|^p = o_x(1) \quad (n \to \infty)$$

almost everywhere, or there is an $f \in L_\Phi$ with

$$\sup_n \sum_{k=0}^{\infty} t_{nk} |s_k(f; x)|^p = \infty \quad (x \in [-\pi; \pi]). \tag{1.3}$$

We remark that similar statement holds for the ordinary means

$$\sum_{k=0}^{\infty} t_{nk}\big(s_k(f; x) - f(x)\big)$$

provided that the (t_{nk}) matrix is row-finite (the proof is almost the same as that of Theorem 3).

2. To prove our theorems we require four lemmas.

Lemma 1. *Let $T = (t_{nk})$ be a non-negative regular matrix and $p > 0$. Let us suppose that there is a $\delta > 0$ such that if M is an arbitrary number then there is a trigonometric polynomial $g(x)$ of L-norm at most 3π and a set $E \subseteq (0; 2\pi]$ of measure at least δ for which*

$$\sup_n \sum_{k=0}^{\infty} t_{nk} |s_k(g; x)|^p > M \tag{2.1}$$

everywhere on E. *Then there is a function* f *for which*

$$\sup_n \sum_{k=0}^{\infty} t_{nk} |s_k(f; x)|^p = \infty \tag{2.2}$$

lamost everywhere.

An interesting consequence is the following. If the (t_{nk}) non-negative regular matrix is such that (2.2) is true for an f on a set E of positive measure then (2.2) is true almost everywhere for an other f.

PROOF OF LEMMA 1. We call a set $E \subseteqq (0; 2\pi]$ a T-set if there is an f satisfying (2.2) almost everywhere on E, and E is called an M-set if for every M there is a polynomial g of L-norm at most 3π for which

$$\sup_n \sum_{k=0}^{\infty} t_{nk} |s_k(g; x)|^p > M$$

everywhere on E, but a set of measure at most $\frac{1}{M}$. Let

$$T_n(f; x) = \sum_{k=0}^{\infty} t_{nk} |s_k(f; x)|^p \quad \text{and} \quad T_n^N(f; x) = \sum_{k=0}^{N} t_{nk} |s_k(f; x)|^p.$$

One can see easily that there is a constant $C_p\,(\geqq 1)$ for which

$$T_n(f+g; x) \leqq C_p\big(T_n(f; x) + T_n(g; x)\big).$$

We prove the statement of Lemma 1 in some steps.

I. *If* E_j $(j=1, 2, \ldots)$ *are* M-*sets then* $\bigcup_{j=1}^{\infty} E_j$ *is a* T-*set.*

Form a new sequence E_j^* from the $E_j's$ in which each E_j occurs infinitely many times.

We shall define five sequences — $\{M_i\}$, $\{\varepsilon_i\}$, $\{g_i(x)\}$, $\{E_i'\}$, $\{N_i\}$ — as follows. Let $M_1 = 1$, $\varepsilon_1 = 1$ and $g_1(x)$ a polynomial for which

$$\int_0^{2\pi} |g_1(x)|\, dx \leqq 3\pi \quad \text{and} \quad \sup_n T_n\left(\frac{1}{2} g_1; x\right) > 1$$

everywhere on E_1^* but a set of measure at most $\frac{1}{2}$. Thus there is an N_1 and a set $E_1' \subseteqq E_1^*$ with $m(E_1') > m(E_1^*) - 1$ and

$$\sup_{N_1 \geqq n \geqq 1} T_n^{N_1}\left(\frac{1}{2} g_1; x\right) > 1 \quad (x \in E_1').$$

Let us suppose that $\{M_i\}$, $\{\varepsilon_i\}$, $\{g_i(x)\}$, $\{E_i'\}$ and $\{N_i\}$ are already known for $i \leqq j$ and

$$\sup_{N_j \geqq n \geqq 1} T_n^{N_j}\left(\frac{\varepsilon_j}{2^j} g_j; x\right) > M_j \quad (x \in E_j).$$

This implies that if $\varepsilon_{j+1}<\varepsilon_j$ is small enough and if

$$\int_0^{2\pi}\left|\frac{\varepsilon_j}{2^j}g_j(x)-h(x)\right|dx<\varepsilon_{j+1}$$

then

(2.3) $$\sup_{N_j\geqq n\geqq 1} T_n(h;x)>\frac{1}{2}M_j \quad (x\in E_j').$$

As $\sum_{i=1}^{j}\frac{\varepsilon_i}{2^i}g_i$ is a polynomial, the sequence $\left\{s_k\left(\sum_{i=1}^{j}\frac{\varepsilon_i}{2^i}g_i;x\right)\right\}_{k=0}^{\infty}$, and together with it the sequence $\left\{T_n\left(\sum_{i=1}^{j}\frac{\varepsilon_i}{2^i}g_i;x\right)\right\}_{n=0}^{\infty}$ is uniformly bounded.

Let now

(2.4) $$M_{j+1}>2C_p\left(\sup_{x,n}T_n\left(\sum_{i=1}^{j}\frac{\varepsilon_i}{2^i}g_i;x\right)+j+1\right).$$

By the assumption there is a g_{j+1} for which

$$\int_0^{2\pi}|g_{j+1}(x)|\,dx\leqq 3\pi,\quad \sup_n T_n\left(\frac{\varepsilon_{j+1}}{2^{j+1}}g_{j+1};x\right)>M_{j+1}$$

for all $x\in E_{j+1}^*$ but a set of measure $\frac{1}{M_{j+1}}$. Thus there is an $E_{j=1}'\subseteqq E_{j+1}^*$ and an N_{j+1} which satisfy

$$m(E_{j+1}')>m(E_{j+1}^*)-\frac{1}{j+1}$$

and

$$\sup_{N_{j+1}\geqq n\geqq 1} T_n^{N_{j+1}}\left(\frac{\varepsilon_{j+1}}{2^{j+1}}g_{j+1};x\right)>M_{j+1}\quad (x\in E_{j+1}').$$

Now the definitions of the above sequences are complet.

Let

$$f(x)=\sum_{j=1}^{\infty}\frac{\varepsilon_j}{2^j}g_j(x).$$

As

$$\sum_{j=1}^{\infty}\frac{\varepsilon_j}{2^j}\int_0^{2\pi}|g_j(x)|\,dx\leqq 3\pi\sum_{j=1}^{\infty}\frac{\varepsilon_j}{2^j}\leqq 3\pi,$$

f is integrable. Furthermore for $x\in E_j'$ we have

$$\sup_n T_n(f;x)=\sup_n T_n\left(\sum_{j=1}^{\infty}\frac{\varepsilon_j}{2^j}g_j;x\right)\geqq$$

(2.5) $$\geqq\frac{1}{C_p}\sup_{N_j\geqq n\geqq 1}T_n\left(\frac{\varepsilon_j}{2^j}g_j+\sum_{i=j+1}^{\infty}\frac{\varepsilon_i}{2^i}g_i;x\right)-\sup_n T_n\left(\sum_{i=1}^{j-1}\frac{\varepsilon_i}{2^i}g_i;x\right)\geqq$$

$$\geqq\frac{1}{2C_p}M_j-\sup_n T_n\left(\sum_{i=1}^{j-1}\frac{\varepsilon_i}{2^i}g_i;x\right)\geqq j$$

where we have used (2.3), (2.4) and the fact that

$$\int_0^{2\pi} \left| \sum_{i=j+1}^{\infty} \frac{\varepsilon_i}{2^i} g_i(x) \right| dx \leqq 3\pi \sum_{i=j+1}^{\infty} \frac{\varepsilon_{j+1}}{2^i} \leqq \varepsilon_{j+1} \quad (j \geqq 4).$$

Now as every E_j occurs infinitely many times in $\{E_j^*\}$ and as $m(E_j') \geqq m(E_j^*) - \frac{1}{j}$, (2.5) implies that $\bigcup_{j=1}^{\infty} E_j$ is really a T-set.

II., *There is a T-set of measure at least* $\delta/2$.

Follow the same line as in I, only do not take into consideration the sets E_j. We obtain a function f and sets E_j' such that $m(E_j') \geqq \frac{\delta}{2}$ and (2.5) is true for them. Let now

$$E = \bigcap_{i=1}^{\infty} \bigcup_{j=i}^{\infty} E_j'.$$

Clearly $m(E) \geqq \frac{\delta}{2}$ and $\sup_n T_n(f; x) = \infty$ everywhere on E, and this was to be proved.

III. *Every T-set is an M-set.*

If $\sup_n T_n(f; x) = \infty$ a.e. on E and if M is arbitrary then there is an $E' \subseteqq E$ and an N such that $m(E') > m(E) - \frac{M}{1}$ and

$$\sup_{N \geqq n \geqq 1} \sum_{k=0}^{N} t_{nk} |s_k(f; x)|^p > 2M \Big(\int_0^{2\pi} |f(\tau)|\, d\tau \Big)^p \quad (x \in E').$$

Now if for the $g(x) \not\equiv 0$ polynomial $\int_0^{2\pi} |f(x) - g(x)|\, dx$ is small enough, then $\frac{g(x)}{\int_0^{2\pi} |g(x)|\, dx}$ and E' satisfy the requirements imposed in the definition of M-sets.

IV. Now everything has settled to prove the statement of Lemma 1. This is the same as to say that $(0; 2\pi]$ is a T-set.

Let

$$\mu = \sup \{m(E) | E \subseteqq (0; 2\pi],\ E \text{ is a } T\text{-set}\}.$$

We have to show $\mu = 2\pi$, namely let E_j be a T-set for which $m(E_j) > \mu - \frac{1}{j}$. By I and III $E^* = \bigcup_{j=1}^{\infty} E_j$ is again a T-set, moreover $m(E^*) \geqq \sup_j m(E_j) \geqq \mu$ and thus $m(E^*) = \mu$.

By II there is an E T-set with $m(E) \geqq \frac{\delta}{2}$, and clearly the sets

$$E^h = \{x \in (0; 2\pi] | x - h \in E \pmod{2\pi}\}$$

are also T-sets.

Now were $\mu<2\pi$ then there would be two points, say x and y, at which the sets $(0;2\pi]\setminus E^*$ and E were of density 1, respectively. But then we would have

$$m(E^*\cup E^{x-y})>m(E^*)=\mu,$$

and this would contradict to the definition of μ, since by I and III the set $E^*\cup E^{x-y}$ is a T-set again.

We have proved Lemma 1.

Before stating Lemma 2 we introduce some notations. Let n and $0\leqq k<2n+1$ be natural numbers and

$$I(k,n;r)=\left\{i\,|\,1\leqq i\leqq n,\ \frac{ki}{2n+1}\in\left[\frac{r}{16};\ \frac{r+1}{16}\right)(\mathrm{mod}\ 1)\right\},$$

$$I^*(k,n;r)=\left\{j\,|1\leqq j<n,\ \sum_{\substack{s=1\\ s+j-1\in I(k,n;r)}}^{n-j+1}\frac{1}{s}\geqq\frac{1}{4\cdot 16}\log n\right\}\quad(0\leqq r<16).$$

The next lemma tells us that one of the $I^*(k,n;r)$ has many elements.

Denote by $|I|$ the number of the elements of the set I.

Lemma 2. *If* $n\geqq 100$ *then for each* $0\leqq k<2n+1$ *there exists an* $r(0\leqq r<16)$ *for which*

$$|I^*(k,n;r)|\geqq\frac{1}{32}n.$$

Proof. If $1\leqq j\leqq n-\sqrt{n}$ then

$$\sum_{s=1}^{n-j+1}\frac{1}{s}\geqq\log(n-j)>\frac{1}{4}\log n$$

and so for some $r=r(j)$

$$\sum_{\substack{s=1\\ s+j-1\in I(k,n;r)}}^{n-j+1}\frac{1}{s}\geqq\frac{1}{4\cdot 16}\log n.$$

Thus

$$\sum_{r=0}^{15}|I^*(k,n;r)|\geqq n-\sqrt{n}-1>\frac{n}{2}$$

from which the statement immediately follows.

Lemma 3. *If for the* (t_{pq}) *non-negative, regular matrix*

$$\lim_{p\to\infty}\max_{0\leqq q<\infty}t_{pq}=0,$$

and if $\{\varphi_q\}$ *is any orthonormal system on* $[0;2\pi]$, *then there is a sequence* p_m *for which*

$$\lim_{m\to\infty}\sum_{q=0}^{\infty}t_{p_m q}\varphi_q(x)=0$$

almost everywhere on $[0;2\pi]$.

PROOF. As

$$\int_0^{2\pi}\left(\sum_{q=0}^{\infty} t_{pq}\varphi_q(x)\right)^2 dx = \sum_{q=0}^{\infty} t_{pq}^2 \int_0^{2\pi} (\varphi_q(x))^2 dx \leqq (\max_q t_{pq}) \sum_{q=0}^{\infty} t_{pq} \leqq$$
$$\leqq K \max_q t_{pq},$$

the hypothesis gives a sequence $\{p_m\}$ such that

$$\sum_{m=1}^{\infty}\int_0^{2\pi}\left(\sum_{q=0}^{\infty} t_{p_m q}\varphi_q(x)\right)^2 dx < \infty.$$

Thus we have also

$$\sum_{m=1}^{\infty}\left(\sum_{q=0}^{\infty} t_{p_m q}\varphi_q(x)\right)^2 < \infty$$

almost everywhere on $[0; 2\pi]$ and this is stronger than the statement of our lemma.

Lemma 4. *Let* $\varepsilon < 2\pi$, η *and* M *be positive numbers and* T *be a non-negative regular matrix. If for a* g *trigonometric polynomial and* N *natural number we have*

$$m\left(\left\{x\in(0; 2\pi] \,\middle|\, \sup_{1\leqq n\leqq N} T_n^N(g; x) \leqq M\right\}\right) < \varepsilon$$

then there are a g' *trigonometric polynomial of L-norm at most* $4M\varepsilon$ *and of* L_Φ*-norm at most* $2\Phi(2M)\varepsilon$ *and a natural number* N' *for which*

$$\sup_{1\leqq n\leqq N'} T_n^{N'}(g+\eta g'; x) > \min\left(\frac{1}{2}\left(\eta\frac{M}{6}\right)^p - 2; M\right)$$

for all x.

PROOF. First of all we may suppose $N > \operatorname{grad} g$ and, by the regularity of T, $\sum_{k=0}^{\infty} t_{nk} \geqq \frac{1}{2}$ $(n=1, 2, \ldots)$, furthermore there is also a number N_1 for which

$$T_n^N(g; x) < 1 \quad (n \geqq N_1).$$

Thus for large $N_2 > N$

$$T_{N_1}^{N_2}(g; x) > \frac{1}{2}\left(\eta\frac{M}{3}\right)^p - 2 \tag{2.6}$$

provided that $|g(x)| > \eta\dfrac{M}{3}$.

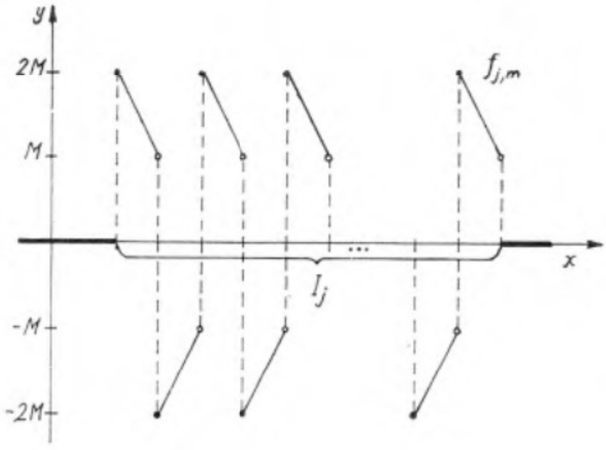
As $\varepsilon < 2\pi$, for some a

(2.7)

$$\left\{x\in(a; a+2\pi] \,\middle|\, |g(x)| \leqq \eta\frac{M}{3}\right\} \cap \left\{x \subseteqq (a; a+2\pi] \,\middle|\, \sup_{1\leqq n\leqq N} T_n^N(g; x) \leqq M\right\} = \bigcup_{j=1}^{l} I_j$$

where $\{I_j\}_1^l$ is a disjoint family of closed intervals. By the assumption $\sum_{j=1}^{l} m(I_j) < \varepsilon$.

For a fixed natural number m let $f_{j,m}$ be the — 2π-periodic — function shown in the figure (the interval I_j is thought to be devided into m parts).



Let

$$f_m(x) = \sum_{j=1}^{k} f_{jm}(x).$$

Clearly $\|f_m\|_L < 2M\varepsilon$ and

$$\|f_m\|_{L_\Phi} \leqq \Phi(2M)\varepsilon.$$

If $n \to \infty$ then $s_n(f_m; x) \to \frac{1}{2}\big(f_m(x-0)+f_m(x+0)\big)$ and, as $\frac{1}{2}\,|(f_m(x-0)+f_m(x+0)| \geqq \frac{M}{2}$ everywhere in $\bigcup_{j=1}^{l} I_j$, for some $N_3 = N_3(m)$

$$\sup_{1\leqq n\leqq N_3} T_n^{N_3}(g+\eta f_m; x) > \frac{1}{2}\left(\eta\frac{M}{6}\right)^p \tag{2.8}$$

at the points of discontinuity of f_m (as there are only finite many such points). It follows that (2.8) is also true on an U open set containing these points. On the other hand on the set $[a; a+2\pi]\setminus U$ $s_n(f_m)$ converges uniformly to f_m, thus for some $N' = N'(m) > \max\big(N, N_1, N_2, N_3(m)\big)$

$$\sup_{1\leqq n\leqq N'} \sum_{k=N_2+1}^{N'} t_{nk}|s_k(g+\eta f_m; x)|^p > \frac{1}{2}\left(\eta\frac{M}{6}\right)^p \quad \left(x\in\bigcup_{j=1}^{l} I_j\right). \tag{2.9}$$

Now if m tends to the infinity then for fixed k

$$\int_{-\pi}^{\pi} f_m(x)\sin kx\,dx \to 0 \quad \text{and} \quad \int_{-\pi}^{\pi} f_m(x)\cos kx\,dx \to 0,$$

from which we get that for large enough m $s_{N_2}(f_m)$ is arbitrary small in absolute value. The Fourier coefficients of $\sigma_n(f_m)$ tend to those of f_m if $n\to\infty$, thus for large enough m and $n=n(m)$ the polynomial $g' = \sigma_n(f_m) - s_{N_2}(\sigma_n(f_m))$ satisfies all the requirements (take into account (2.6), (2.7), (2.9) and that

$$\|\sigma_n(f_m)\|_L \leqq \|f\|_L, \quad \|\sigma_n(f_m)\|_{L_\Phi} \leqq \|f_m\|_{L_\Phi},$$

$$\int_{-\pi}^{\pi} g'(x)\sin kx\,dx = 0, \quad \int_{-\pi}^{\pi} g'(x)\cos kx\,dx = 0 \quad (0 \leqq k \leqq N_2)).$$

3. PROOF OF THEOREM 1. By Theorem 3 it is enough to show the almost everywhere divergence.

We shall show that the conditions of Lemma 1 are fulfilled $\left(\text{with } \delta = \frac{1}{4\cdot 32}\right)$ for the matrix (t_{nk}) where

$$t_{nk} = \begin{cases} 0 & \text{if } k \neq \mu_n \\ 1 & \text{if } k = \mu_n, \end{cases}$$

and thus Theorem 1 will follow from Lemma 1.

Let $n \geqq 100$ be an arbitrary natural number. By cutting-out we can achieve that each $(2\mu_i+1)$ gives the same, say k, remainder if we divide it by $(2n+1)$, i.e. we may suppose that each $(2\mu_i+1)$ is of the form $(2n+1)t+k$.

Let r be the number given by Lemma 2 to the pair n, k; let

$$a_i = \frac{4\pi i}{2n+1} \quad (i = 0, 1, \ldots, n)$$

and

$$g(x) = \frac{1}{n} \sum_{i \in I(k,n;r)} V_{\mu_{k_i}}(x - a_i),$$

where

$$V_l = \frac{1}{2} + \sum_{t=1}^{l} \cos tx + \sum_{t=l+1}^{2l} \left(1 - \frac{t-l}{l+1}\right) \cos tx$$

is the de la Vallée Poussin-kernel, and the numbers μ_{k_i} satisfy the conditions $\mu_{k_1} > n^4$, $\mu_{k_{i+1}} > 2\mu_{k_i}$.

Since V_l is expressible by the Fejér kernels in the form

$$V_l(x) = \frac{2l+1}{l+1} K_{2l}(x) - \frac{l}{l+1} K_{l-1}(x)$$

and since $K_l(x)$ is non-negative, we get

$$\int_0^{2\pi} |g(x)|\,dx \leqq \frac{1}{n} \sum_{\substack{i=1 \\ i \in I(k,n;r)}}^{n} \int_0^{2\pi} |V_{\mu_{k_i}}(x - a_i)|\,dx \leqq \frac{1}{n} \sum_{i=1}^{n} 3 \int_0^{2\pi} \frac{1}{2}\,dt = 3\pi.$$

Let $\Delta_i = (a_{i-1}, a_i)$, $\Delta_i' = (a_i - n^{-2}, a_i + n^{-2})$ $(i = 1, 2, \ldots, n)$. The estimation $K_l(t) = O(l^{-1}t^{-2})$ together with $\mu_{k_i} \geqq n^4$ imply that $V_{\mu_{k_i}}(x - a_i)$ are uniformly bounded outside Δ_i', i.e. there is an $A \geqq 1$ absolute constant for which $|V_{\mu_{k_i}}(x - a_i)| \leqq A$ if $x \notin \Delta_i'$.

From this, taking into account that

$$s_m(V_l; x) = \frac{1}{2} + \cos x + \ldots + \cos mx = D_m(x) \quad (m \leqq l)$$

we get that outside $\Delta' = \bigcup_{i=1}^{n} \Delta_i'$ we have

$$|s_{\mu_{k_j}}(g; x)| \geqq \left| \frac{1}{n} \sum_{\substack{i=j \\ i \in I(k,n;r)}}^{n} D_{\mu_{k_j}}(x-a_i) \right| - A = \frac{1}{n} \left| \sum_{\substack{i=j \\ i \in I(k,n;r)}}^{n} \frac{\sin\left(\mu_{k_j} + \frac{1}{2}\right)(x-a_i)}{2 \sin \frac{1}{2}(x-a_i)} \right| - A.$$

Now

$$\left(\mu_{k_j} + \frac{1}{2}\right) a_i = \frac{2\mu_{k_j}+1}{2n+1} \frac{4\pi i}{2} \equiv \frac{ki}{2n+1} 2\pi \pmod{2\pi},$$

thus for $i \in I(k, n; r)$ we obtain

$$\left(\mu_{k_j} + \frac{1}{2}\right) a_i \in \left[r \frac{2\pi}{16}; (r+1) \frac{2\pi}{16} \right) \pmod{2\pi}.$$

If

$$E_j = \left\{ x \in \Delta_j \setminus \Delta' \middle| \left(\mu_{k_j} + \frac{1}{2}\right) x \in \left[\frac{\pi}{2} - \frac{\pi}{8} + \frac{r2\pi}{16}; \frac{\pi}{2} - \frac{\pi}{8} + \frac{(r+1)2\pi}{16} \right) \pmod{2\pi} \right\}$$

then $m(E_j) > \frac{1}{4n}$ and in the case $x \in E_j$, $i \in I(k, n; r)$ we have

$$\left(\mu_{k_j} + \frac{1}{2}\right)(x-a_i) \in \left[\frac{\pi}{4}; \frac{\pi}{2} \right] \pmod{2\pi},$$

by which, if even $j \in I^*(k, n; r)$,

$$s_{\mu_{k_j}}(g; x) \geqq \frac{\sqrt{2}}{2} \frac{1}{n} \sum_{\substack{i=j \\ i \in I(k,n;r)}}^{n} \frac{1}{a_i - a_{j-1}} - A =$$

$$= \frac{\sqrt{2}}{2} \frac{1}{n} \frac{2n+1}{4\pi} \sum_{\substack{s=1 \\ s+j-1 \in I(k,n;r)}}^{n-j+1} \frac{1}{s} - A \geqq \frac{1}{10} \frac{1}{4 \cdot 16} \log n - A > 10^{-3} \log n$$

provided $n \geqq n_0$.

We have got that

$$\sup_{n \geqq j \geqq 1} |s_{\mu_{k_j}}(g; x)| > 10^{-3} \log n$$

on the set $\bigcup_{j \in I^*(k,n;r)} E_j$. Here — by Lemma 2 — $m\left(\bigcup_{j \in I^*(k,n;r)} E_j\right) \geqq \frac{1}{32} n \cdot \frac{1}{4n} = \frac{1}{4 \cdot 32}$, and thus (t_{nk}) really satisfies the conditions of Lemma 1.

We have proved our theorem.

Proof of Theorem 2. The first statement of Theorem 2 is an immediate consequence of the mentioned Zygmund's result since

$$\frac{1}{n+1}\sum_{k=0}^{\infty}|s_{\mu_k}(x)-f(x)|^p \leqq (\mu_1+K+1)\frac{1}{\mu_1+Kn+1}\sum_{k=0}^{\mu_1+Kn}|s_k(x)-f(x)|^p = o_x(1) \quad \text{(a.e.)}.$$

Concerning the second part first of all we notice that it is enough to prove the statement for the case "almost everywhere" (see Theorem 3), and that this is surely true if

$$\overline{\lim_{p\to\infty}}\sup_q t_{pq} > 0.$$

This follows at once from Theorem 1.

Thus we may suppose

$$\lim_{p\to\infty}\sup_q t_{pq} = 0.$$

We may also suppose $\{v_k\}$ to be increasing and — by Hölder-inequality — $p\leqq 2$. From the regularity of T there is a K with

$$\sum_{q=0}^{\infty} t_{pq} \leqq K \quad (p=1,2,\ldots).$$

We define two sequences $\{\mu_n\}$ and $\{\varrho_n\}$. Let us suppose that $\varrho_1,\ldots,\varrho_n$ and $\mu_0,\ldots,\mu_{\varrho_n}$ are already known. Let $\eta_n>\varrho_n$ be so large that $v_{\eta_n}>2n+1$ be true, let $\mu^*_{\varrho_n+i}=\mu_{\varrho_n}+i$ for $i\leqq\eta_n-\varrho_n$ and if M_n is the smallest natural number not less than $\mu_{\varrho_n}+\eta_n-\varrho_n$ for which $(2M_n+1)$ is divisible by $(2n+1)$ then let

$$\mu^*_{\eta_n+i} = M_n+(2n+1)(i-1) \quad (1\leqq i<\infty).$$

Consider the function

$$g(x)=\frac{1}{n}\sum_{i=1}^{n}V_{m_i}(x-a_i)$$

where we use the notations of the previous proof, and where m_1 be so large that $m_1>n^4$, $\mu^*_{m_1}>M_n$ be valid. Similar computation as in the proof of Theorem 1 gives that in the case $2m_j<k\leqq m_{j+1}$, $(2n+1)|(2k+1)$, $x\in\Delta_j\setminus\Delta'$ $\left(1\leqq j<\frac{n}{2}\right)$, we have $|s_k(g;x)|>\frac{1}{100}\sin\left|\left(k+\frac{1}{2}\right)x\right|\log n-\mathrm{A}$, and thus if $t_j=\left(\left[\frac{2m_j}{2n+1}\right]+1\right)(2n+1)+n$, $t'_j=\left(\left[\frac{m_{j+1}}{2n+1}\right]-1\right)(2n+1)+n$ *), then

$$\sum_{q=t_j}^{t'_j} t_{pq}|s_{\mu^*_q}(g;\, x)^p| \geqq \frac{(\log n)^p}{C_p}\sum_{q=t_j}^{t'_j} t_{pq}\left|\sin\left(\mu^*_q+\frac{1}{2}\right)x\right|^p - KA^p.$$

*) $[y]$ denotes the integral part of y.

As $p\leqq 2$ we have

$$\sum_{q=t_j}^{t_j'} t_{pq}\left|\sin\left(\mu_q^*+\frac{1}{2}\right)x\right|^p \geqq \sum_{q=t_j}^{t_j'} t_{pq}\left(\sin\left(\mu_q^*+\frac{1}{2}\right)x\right)^2 =$$

$$=\frac{1}{2}\sum_{q=t_j}^{t_j'} t_{pq}-\frac{1}{2}\sum_{q=t_j}^{t_j'} t_{pq}\cos(2\mu_q^*+1)x.$$

Now fix m_j. Then t_j is fixed, too. By Lemma 3 there is a sequence $\{p_m\}$ for which

$$\lim_{m\to\infty}\sum_{q=t_j}^{\infty} t_{p_m q}\cos(2\mu_q^*+1)x=0$$

almost everywhere. By the regularity of T if p_m is large enough then $\frac{1}{2}\sum_{q=t_j}^{\infty} t_{p_m q}>\frac{1}{4}$. Thus there is an $E_j\subseteq\Delta_j$ of measure at least $\frac{1}{n}$ and a t_j' for which

$$\text{(3.1)}\qquad \sup_p\sum_{q=t_j}^{t_j'} t_{pq}\left|\sin\left(\mu_k^*+\frac{1}{2}\right)x\right|^p>\frac{1}{5}\quad(x\in E_j).$$

Now after choosing m_1, let m_2 be so large that (3.1) be true for $j=1$, then m_3 be so large that (3.1) be true for $j=2$, etc. up to $j=\left[\frac{n}{2}\right]$. Let $\varrho_{n+1}=\eta_n+t'_{\left[\frac{n}{2}\right]}+1$ and $\mu_i=\mu_i^*$ for $\varrho_n<i\leqq\varrho_{n+1}$.

Clearly $\mu_{i+1}-\mu_i\leqq\nu_i$, and what we have proved above is that

$$\text{(3.2)}\qquad \sup_p\sum_{q=0}^{\varrho_{n+1}} t_{pq}|s_{\mu_q}(g;x)|^p\geqq\frac{1}{5C_p}(\log n)_q-KA^p$$

on the set $\bigcup_{j=1}^{\left[\frac{n}{2}\right]} E_j$ of measure not less than $\left[\frac{n}{2}\right]n>\frac{1}{3}$.

On this way we can define the sequence $\{\mu_k\}$.

Let

$$t'_{mk}=\begin{cases}0 & \text{if}\quad k\notin\{\mu_k\}\\ t_{mq} & \text{if}\quad k=\mu_q\quad(q=0,1,2,\ldots).\end{cases}$$

By (3.2)

$$\sup_m\sum_{k=0}^{\infty} t'_{mk}|s_k(g;x)|^p\geqq\frac{1}{5C_p}(\log n)-KA^p$$

on $\bigcup_{j=1}^{\left[\frac{n}{2}\right]} E_j$, i.e. Lemma 1 is applicable to (t'_{mk}) by which we have for some f

$$\sup_n\sum_{k=0}^{\infty} t_{nk}|s_{\mu_k}(f;x)|^p=\sup_m\sum_{k=0}^{\infty} t'_{nk}|s_k(f;x)|=\infty$$

almost everywhere, and we only have to remark that $\mu_{k+1}-\mu_k\leqq\nu_k$ is satisfied, too.

We have completed our proof.

PROOF OF THEOREM 3. Let us suppose that for some $f_1 \in L_\Phi$

$$\left\{\sum_{k=0}^{\infty} t_{nk} |s_k(f_1; x) - f_1(x)|^p\right\}$$

does not converge to 0 on a set of positive measure. From this one can see easily that suitably choosen polynomials of the form $M\sigma_n(f_1 - \sigma_m(f_1))$ satisfy the assumptions of Lemma 1 as well as the further assumption $\|g\|_{L_\Phi} \leqq 1$ (use the remark on the space L_Φ made in point 1). Similarly as in Lemma 1 we get that for some $f_2 \in L_\Phi$ (1.3) is valid a.e. (the proof is the same as there).

Let $M(\varepsilon)$ $(\varepsilon > 0)$ be such that

$$\Phi\big(M(\varepsilon)\big)\varepsilon = O(1), \quad M(\varepsilon)\varepsilon = O(1) \quad \text{and} \quad M(\varepsilon) \to \infty \quad (\varepsilon \to 0)$$

be satisfied.

Now we define three sequences $\{N'_k\}$, $\{g_k\}$ and $\{\varepsilon_k\}$ where the $g'_k s$ are polynomials. Let us suppose that the members of these are already known for $k \leqq m$ and that

$$\sup_{1 \leqq n \leqq N'_m} T_n^{N'_m}\left(\sum_{k=1}^{n} \frac{\varepsilon_k}{2^k} g_k; x\right) > m \tag{3.3}$$

everywhere. If $\varepsilon_{m+1} < \varepsilon_m$ is small enough and if $\left\|h - \sum_{k=1}^{m} \frac{\varepsilon_k}{2^k} g_k\right\|_L < \varepsilon_{m+1}$ then

$$\sup_{1 \leqq n \leqq N'_m} T_n^{N'_m}(h; x) > m - 1 \tag{3.4}$$

will be satisfied, too. Now

$$\overline{\lim_{n\to\infty}}\, T_n\left(\sum_{k=1}^{m} \frac{\varepsilon_k}{2^k} g_k + \frac{\varepsilon_{m+1}}{2^{m+1}} f_2; x\right) = \infty$$

a.e., thus to every $\varepsilon > 0$ there is an $N_m = N_m(\varepsilon)$ for which

$$\sup_{1 \leqq n \leqq N_m} T_n^{N_m}\left(\sum_{k=1}^{m} \frac{\varepsilon_k}{2^k} g_k + \frac{\varepsilon_{m+1}}{2^{m+1}} f_2; x\right) > 2M(\varepsilon)$$

everywhere in $(0; 2\pi]$ but a set of measure at most ε, and so for large enough l

$$\sup_{1 \leqq n \leqq N_m} T_n^{N_m}\left(\sum_{k=1}^{m} \frac{\varepsilon_k}{2^k} g_k + \frac{\varepsilon_{m+1}}{2^{m+1}} \sigma_l(f_2); x\right) > M(\varepsilon)$$

on the same set.

After that we can apply Lemma 4 with the cast

$$g = \sum_{k=1}^{m} \frac{\varepsilon_k}{2^k} g_k + \frac{\varepsilon_{m+1}}{2^{m+1}} \sigma_l(f_2), \quad N = N_m, \quad M = M(\varepsilon), \quad \eta = \frac{\varepsilon_{m+1}}{2^{m+1}}$$

and obtain a g'_{m+1} polynomial and an N'_{m+1} natural number with $\|g'_{m+L1}\| \leqq$

$\leqq 4M(\varepsilon)\varepsilon = O(1)$, $\|g'\|_{L_\Phi} \leqq 2\Phi(2M(\varepsilon))\varepsilon = O(1)$ and

$$\sup_{1 \leqq n \leqq N'_{m+1}} T_n^{N'_{m+1}} \left(\sum_{k=1}^{m} \frac{\varepsilon_k}{2^k} g_k + \frac{\varepsilon_{m+1}}{2^{m+1}} (\sigma_l(f_2) + g'_{m+1}); x \right) >$$

$$> \min \left(M(\varepsilon); \frac{1}{2} \left(\frac{\varepsilon_{m+1}}{2^{m+1}} \frac{M(\varepsilon)}{6} \right)^p - 2 \right) \quad (x \in [-\pi; \pi])$$

by which, if ε is small enough and $g_{m+1} = \sigma_l(f_2) + g'_{m+1}$, (3.3) is satisfied for $m+1$. Thus we have defined the above sequences.

Now for the function $f = \sum_{k=1}^{\infty} \frac{\varepsilon_k}{2^k} g_k$ we have

$$\int_{-\pi}^{\pi} \Phi(|f(x)|)\,dx \leqq \int_{-\pi}^{\pi} \left(\sum_{k=1}^{\infty} \frac{1}{2^k} \Phi(|\varepsilon_k g_k(x)|) \right) dx = O\left(\sum_{k=1}^{\infty} \frac{1}{2^k} \right) = O(1)$$

(we have used the convexity of Φ), i.e. $f \in L_\Phi$ (and similarly $f \in L$), and as

$$\left\| \sum_{k=m+1}^{\infty} \frac{\varepsilon_k}{2^k} g_k \right\|_L = O(1) \sum_{k=m+1}^{\infty} \frac{\varepsilon_k}{2^k} < \varepsilon_{k+1} \quad (k \geqq k_0),$$

(3.3) and (3.4) show that f satisfies the statement of Theorem 3. We have completed our proof.

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