Topological radicals in semigroups

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A semigroup (i.e., topological semigroup) is a non-empty Hausdorff space together with a continuous associative binary multiplication. Let F be a non-empty subset of a semigroup S. The algebraic radical of F in S is the set $\{x\in S|x^k\in F$ for some integer $k\geqq 1\}$ which is denoted by $R(F)$. Algebraic radical in semigroups was studied by K. P. SHUM and C. S. HOO [2]. By the topological radical of F in S, we mean the set $T(F)=\{x\in S|\Gamma(x)\cap F\neq\emptyset\}$, where $\Gamma(x)$ is the closure of the positive powers of x. The notion of topological radical was initiated by R. J. KOCH [5]. Clearly, $T(F)$ contains $R(F)$ as a subset, and in general these two radicals are not equal. If S is a semigroup with zero, then $T(\{0\})$ is usually called the radical of S. The set $T(\{0\})$ was widely studied by K. NUMAKURA [6], R. J. KOCH [5], K. P. SHUM and C. S. HOO [4]. In case if the semigroup S is commutative and compact, then $T(\{0\})$ is the union of all the nil ideals of S. However, if F is a non-zero subset of S or in case if S is a semigroup without zero, the properties of $T(F)$ are still not well-known.

The aim of this paper is to discuss the properties of $T(F)$ in compact semigroups, an attempt is made to relate the algebraic and topological properties of $T(F)$. Results concerning the properties of $R(F)$ and $T(\{0\})$ obtained in [2], [3], [4], [5] and [6] will be amplified and strengthened.

We shall follow A. B. PAALMAN—DE MIRANDA [8] for all concepts and terminology not defined in this paper.

1. Preliminaries

We shall use the following notations:

Let A be any subset of a semigroup S and x be an element of S. Then

$\bar{A}$ = topological closure of A in S.

$J(A)=A\cup AS\cup SA\cup SAS$, which is the principal ideal of S generated by A.

$J_0(A)$ = The largest ideal contained in A, if $J_0(A)\neq\emptyset$.

E = the set of all idempotents of S.

$E(A)=E\setminus A$.

$\operatorname{tod}_A x=\{s\in S|sx\in A\}$ which is called the right topological zero devisors of x with respect to A. If S is a commutative semigroup, then $\operatorname{tod}_A x=\{s\in S|sx\in A\}=$ $=\{s\in S|xs\in A\}$.

$\Gamma(x)=\overline{\{x^n_\infty\}_{n=1}^\infty}$.

$K(x)=\bigcap_{n=1} \overline{\{x^i\}|i\geqq n\}}$.

It is well-known that if $\Gamma(x)$ is a compact, then it contains an unique idempotent. Moreover, $K(x)$ is a group and $K(x)=e\Gamma(x)e=\Gamma(x)e$ where $e\in\Gamma(x)$ is the unique idempotent of $\Gamma(x)$. (See [8], pages 22—25).

We quote several definitions which will be needed later on.

Lemma 1.1 (NUMAKURA [6]). *The set E of idempotents of a semigroup S is a closed subspace of S which is partially ordered under the relation $e\leqq f$ if $ef=fe=e$, and this partial order is closed.*

Definition 1.2. (SHUM—STEWART [12]. An ideal I of a semigroup S is said to be topologically semiprime if $x\notin I$ implies $\Gamma(x)\cap I=\varnothing$.

Clearly, all open completely semiprime ideals of S are topologically semiprime.

It was proved by R. J. KOCH on [5] that if S is a semigroup with zero and $\Gamma(a)$ is locally compact for each $a\notin T(\{0\})$, then $T(\{0\})$ is topologically semiprime.

Definition 1.3. (KOCH [5]). Let A be a subset of a mob S. An idempotent $e\in S$ is called A-primitive if $e\notin A$ and e is the only idempotent in $eSe\setminus A$. An idempotent $e\in S$ is called A-maximal if $e\in A$ and $f^2=f\in A$, then either $ef\in A$ or $f\leqq e$.

Obviously, A-primitive idempotents are the atoms of the partial ordered set $(E(A), \leqq)$. A-maximal idempotents are the maximal elements in the ordered set $(E(A), \leqq)$.

The set of all A-maximal idempotents of S is denoted by $E(A)^*$.

The set of all A-primitive idempotents of S is denoted by $\widetilde{E(A)}$.

Definition 1.4. A semigroup S is called semi-normal if and only if $eS=Se$ for all idempotents $e\in S$.

Clearly, commutative semigroups are semi-normal, but not conversely. We should note that many results on commutative semigroups can be extended to semi-normal semigroups trivially.

The following results are topological generalization of algebraic radicals which are analogous to those in ring theory [12].

Theorem 1.5. *Let A, B be any two ideals of a semigroup S. The topological radical of ideals in S has the following properties:*

(i) *$A^k\subset B$ for some integer $k>1$ implies that $T(A)\subset T(B)$; moreover, if A is a compact subset of S, then $\bigcap_{n=1}^{\infty} A^n\subset B$ implies that $T(A)\subset T(B)$. ($\bigcap_{n=1}^{\infty} A^n$ is usually denoted by A^∞).*

(ii) $T(AB)=T(A\cap B)=T(A)\cap T(B)$.

(iii) $T\{T(A)\}=T(A)$.

(iv) $T(A\cup B)=T\{T(A)\cup T(B)\}$.

The proofs are straightforward and hence omitted.

Theorem 1.6. *Suppose S and T are semigroups and h is a continuous homomorphism of S into T. We have*

(i) *If A is an ideal of S, then $h(T(A))\subset T(h(A))$.*

(ii) *If S is compact and B is an ideal of T, then $h^{-1}(T(B))=T(h^{-1}(B))$.*

PROOF. We only prove (ii). Let $x \in h^{-1}(T(B))$, then $h(x) \in T(B)$ and $\Gamma(h(x)) \cap \cap B \neq \emptyset$. Hence there exists $b \in B$ such that $b = \lim h(x)^{\alpha}$ for some net α. Since B is compact, there exists a subnet β of α such that x^{β} converges. Then $b = \lim (h(x))^{\beta} = h(\lim x^{\beta})$ and so $\Gamma(x) \cap h^{-1}(B) \neq \emptyset$, that is $x \in T(h^{-1}(B))$. On the other hand, let $x \in T(h^{-1}(B))$, then $\Gamma(x) \cap h^{-1}(B) \neq \emptyset$. Hence there is an element $y \in h^{-1}(B)$ such that $y = \lim x^{\alpha}$ for some net α. Thus, $\lim h(x)^{\alpha} = h(y) \in B$ and so $\Gamma(h(x)) \cap B \neq \emptyset$. It then follows that $h(x) \in T(B)$, whence $x \in h^{-1}(T(B))$.

Remark. Topological radicals and algebraic radicals are in general not equal as illustrated by the following example.

Example 1.7. Let S be the subset of the plane defined by $S = \{0\} \times \underline{[0, 1]} \cup \cup \left\{ \frac{1}{n} \middle| n = 1, 2, 3, \ldots \right\} \times \{0\}$, where the underline brackets denote the intervals. The multiplication in S is the coordinatewise usual multiplication. Then S is a semigroup. Let A be the set $\{0\} \times \underline{[0, 1/2)}$ which is an ideal of S. Obviously, $R(A) = \{0\} \times \underline{[0, 1)}$, $T(A) = \{0\} \times \underline{[0, 1)} \cup \left\{ \frac{1}{n} \middle| n = 2, 3, \ldots \right\} \times \{0\}$. So, $A \subsetneqq R(A) \subsetneqq T(A) \subsetneqq S$.

2. Topological radicals

Let A be an ideal of a semigroup S. If S is a Γ-compact semigroup with zero and A is the zero ideal, then the topological radical $T(\{0\})$ is the set $\{x \in S \mid x^n \to 0\}$ which is the set of all nilpotent elements of S and is denoted by N. The properties of the set N were widely studied by K. NUMAKURA [6], R. J. ROCH [5], and SHUM—HOO [4]. In this section, we study $T(A)$ instead of N. Generalized results concerning the set N are obtained in compact semigroups without zero.

Theorem 2.1. *Let A be an open subset of a semigroup S, then $T(A)$ is open.*

PROOF. Let $x \in T(A)$, then $\Gamma(x) \cap A \neq \emptyset$. Since A is an open subset of S, there exists an integer n_0 such that $x^n \in A$ for $n \geqq n_0$. Let V be an open neighbourhood of x^n such that $V \subset A$. Then by the continuity of multiplication, there exists an open neighbourhood U of x such that $U^n \subset V \subset A$. Hence $\Gamma(U) \cap A \neq \emptyset$, which means that $U \subset T(A)$. Thus $T(A)$ is open.

In view of theorem 2.1, we obtain generalized versions of the following theorems.

Theorem 2.2. (NUMAKURA [6]). *Let S be a compact semigroup and A be an open ideal of S such that its topological radical is a proper subset of S. Then there is a closed ideal M minimum with respect to not being in $T(A)$. M has the form $M = SeS$ with $e \in \widetilde{E(A)}$. Furthermore, $J_0\{T(A)\}_M = M \cap J_0\{T(A)\}$ is the topological radical of A in the subsemigroup M and $J_0\{T(A)\}_M$ is a maximal proper ideal of M with $M/J_0\{T(A)\}_M$ completely 0-simple.*

The proof of the above theorem is an application of Zorn's lemma. In the process of proof, we need to use the facts that $E(A) = E\{T(A)\}$ and $T(A)$ is open. The reader is referred to [6] for details.

Corollary. *Let A be an open ideal of a compact semigroup S, then S contains a A-primitive idempotent if and only if there is an idempotent $e\in E(A)$ such that $eSe\setminus T(A)$ is closed.*

Theorem 2.3. (R. J. Koch [5]). *Let S be a compact semigroup and A be an ideal of S. Let $e\in\widetilde{E(A)}$, then $Se\setminus T(A)$ and $Se\cap T(A)$ are subsemigroups and $Se\setminus T(A)$ is the disjoint union of the maximal groups $e_\alpha Se_\alpha\setminus T(A)$, where e_α runs over the set $E(A)\cap Se$.*

Theorem 2.4. (R. J. Koch [5]). *Let S be a compact semigroup and A be an ideal of S with $e\in E(A)$. Then the followings are equivalent:*

(i) $e\in\widetilde{E(A)}$.
(ii) *$eSe\setminus T(A)$ is a group.*
(iii) *Se is a closed left ideal minimum with respect to not being contained in $T(A)$.*
(iv) *SeS is a closed (two sided) ideal minimum with respect to not being in $T(A)$.*
(v) *Every idempotent in $SeS\setminus T(A)$ is in $\widetilde{E(A)}$.*

Remarks: If A is an ideal of a semigroup S, then $T(A)$ is not necessarily an ideal of S.

Example 2.5. Let $S=\{(x, y)|x, y\geqq 0\}$ with the usual topology inherited from the plane and with the multiplication defined by $(x_1, y_1)(x_2, y_2)=(x_1x_2, x_1y_2+y_1)$ and let $A=\{(0, y)|y\geqq 0\}$. Then S is a semigroup and A is an ideal of S. Clearly, $T(A)=\{(x, y)|0\leqq x\leqq 1, y\geqq 0\}$ which is not an ideal of S.

We now give conditions for $T(A)$ to be ideals.

Theorem 2.6. *If A is an ideal of a Γ-compact semi-normal semigroup, then $T(A)$ is a topological semiprime ideal of S.*

Proof. Let $a\in T(A)$ and $x\in S$, then $\Gamma(a)\cap A\neq\emptyset$. Since S is Γ-compact, each of the subsemigroup $\Gamma(a)$ and $\Gamma(x)$ is compact and hence each of which contains idempotent $e\in\Gamma(a), f\in\Gamma(x)$ respectively. By the semi-normality of S and a well-known theorem of Schwarz [8; theorem 3.2.3 page 116], we have $ef\in\Gamma(ax)\cap A$, whence $ax\in T(A)$. Similarly, $xa\in T(A)$. Hence $T(A)$ is a two-sided ideal of S. Clearly, $T(A)$ is topologically semi-prime.

Note. The semi-normality of a Γ-compact semigroup S is not sufficient to ensure $T(A)$ be an ideal for any ideal A of S.

Example 2.7. Let S be the set $\left\{\begin{pmatrix}0&0\\x&x\end{pmatrix}, \begin{pmatrix}0&x\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\0&x\end{pmatrix}, \begin{pmatrix}x&x\\0&0\end{pmatrix}\right\}$ with $0\leqq x\leqq {}^1/_2$. The topology on S is the topology that S inherits when S is considered as a subspace of Euclidean 4-space. The operation on S is the usual matrix multiplication. Then S is a compact semi-normal semigroup. Let $A=\left\{\begin{pmatrix}0&0\\0&0\end{pmatrix}\right\}$ which is an ideal of S. The algebraic radical $R(A)$ is the set $\left\{\begin{pmatrix}0&x\\0&0\end{pmatrix}\middle|\, 0\leqq x\leqq {}^1/_2\right\}$. $R(A)$ is not an ideal of S, for if $y\neq 0\neq x$, then $\begin{pmatrix}0&0\\y&y\end{pmatrix}\begin{pmatrix}0&x\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&xy\end{pmatrix}\notin R(A)$.

Let $e\in E(A)$. The set of all e-non-orthogonal idempotents with respect to A is the set $\{e_\beta\in E|e_\beta e\notin A\}$ which is denoted by e_A. The set e_A is always non-empty since $e\in e_A$.

Theorem 2.8. *Let A be an ideal of a compact semi-normal semigroup S and e an idempotent non in A. Then $J_0(S\setminus e_A)=\operatorname{tod}_{T(A)}e$.*

PROOF. Let $t\in e_A$. Then t is an idempotent of S such that $et\in A$. Since $E(A)=E(T(A))$, therefore the idempotent $et\notin T(A)$. This implies that $t\notin \operatorname{tod}_{T(A)}e$. Thus, $\operatorname{tod}_{T(A)}e\subset S\setminus e_A$. As $\operatorname{tod}_{T(A)}e$ is an ideal of S, $\operatorname{tod}_{T(A)}e\subset J_0(S\setminus e_A)$. On the other hand, let $x\in J_0(S\setminus e_A)$. Then $f^2=f\in\Gamma(x)\subset xS\subset J_0(S\setminus e_A)$. Consequently, $f\notin e_A$, which means that $fe\in A\subset T(A)$. Thus, $f\in\operatorname{tod}_{T(A)}e$, i.e., $\Gamma(x)\cap\operatorname{tod}_{T(A)}e\neq\emptyset$. Since $T(A)$ is topologically semiprime and S is semi-normal, $\operatorname{tod}_{T(A)}e$ is topologically semiprime. Therefore, $x\in\operatorname{tod}_{T(A)}e$, and so $J_0(S\setminus e_A)\subset\operatorname{tod}_{T(A)}e$. Our proof is completed.

Theorem 2.8 includes a result of Shum—Stewart [11; lemma 2 (iv); page 212] as a particular case. For if $e\in E(A)$, then $e_A=\{e\}$. Thus, when A is a topologically semiprime ideal of S, we have $\operatorname{tod}_A e=J_0(S\setminus e)$ with $e\in\widetilde{E(A)}$.

Theorem 2.9. *Let A be an ideal of a compact semi-normal semigroup S, then the followings are equivalent:*

(i) $T(A)$ *is topologically semiprime ideal.*
(ii) $T(A)=J_0[S\setminus E(A)]$.
(iii) $T(A)=\cap\{J_0(S\setminus e)|e\in E(A)\}$.
(iv) $T(A)=\cap\{J_0(S\setminus e\in E(A)\}$.
(v) $T(A)=\cap\{\operatorname{tod}_{T(A)}e|e\in E(A)\}$.
(vi) $T(A)=\cap\{\operatorname{tod}_{T(A)}e|e\in E(A)\}$.
(vii) $T(A)=\cap\{\operatorname{tod}_{T(A)}e|e\in E(A)^*\}$.

PROOF. (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v) follows verbtim from the proof of proposition 7 and theorem 11 of Shum—Stewart [11]. It was proved in [4] that if e_1 and e_2 are in $E(A)$ such that $e_1\leqq e_2$, then $\operatorname{tod}_{T(A)}e_2\subseteqq\operatorname{tod}_{T(A)}e_1$. Whence (v)$\Leftrightarrow$(vi)$\Leftrightarrow$(vii) follows immediately. The cycle will be completed if we prove (v)$\Rightarrow$(iv) and (iv)$\Rightarrow$(i).

(v)$\Rightarrow$(iv): For if $e\in\widetilde{E(A)}$, then $J_0(S\setminus e)=\operatorname{tod}_{T(A)}e$ [4; lemma 4.2]. For the converse part, we recall a theorem of SCHWARTZ [8; page 119] that if $e\in A$, then $J_0(S\setminus e)=\cup\{P_\beta|ee_\beta\neq e,\ e_\beta\in E(A)\}$ where $P_\beta=\{x\in S|e_\beta\in\Gamma(x)\}$. Now, suppose $J_0(S\setminus e)=\operatorname{tod}_{T(A)}e$. Then, by using Theorem 2.8, we can easily show that $e\in\widetilde{E(A)}$. The proof is straightforward.

(iv)$\Rightarrow$(i): In view of theorem 11 of SHUM—STEWART [11]. We only need to show that each $J_0(S\setminus e)$, with $e\in\widetilde{E(A)}$, is an open completely prime ideal of S. Let $ab\in J_0(S\setminus e)$. Then since $e(a)^2=e(a)\in\Gamma(a)\subset S$; $e(b)^2=e(b)\in\Gamma(b)$, we have $e(a)e(b)=e(ab)\in\Gamma(ab)\subset abS\subset J_0(S\setminus e)$. If $e(a)$ and $e(b)$ are both in $S\setminus J_0(S\setminus e)$, then by Schwarz's theorem mentioned above, we have $e\cdot e(a)=e$ and $e\cdot e(b)=e$. Hence $e\cdot e(ab)=e[e(a)e(b)]=[e\cdot e(a)]e(b)=e\cdot e(b)=e$ and so $e(ab)\in S\setminus J_0(S\setminus e)$, a contradiction. Therefore, either $e(a)\in J_0(S\setminus e)$ or $e(b)\in J_0(S\setminus e)$. Apply Schwarz's theorem once again, we have either $a\in J_0(S\setminus e)$ or $b\in J_0(S\setminus e)$. Thus, $J_0(S\setminus e)$ is indeed an open completely prime ideal.

Remark. The statement of Theorem 3.2 in [4; page 544] is wrong. The set E^* there should be the set of all non-zero maximal idempotents of S instead of the set of all non-minimal idempotents of S.

In view of Theorem 2.9, we obtain a characterization of topologically semiprime ideals in compact semi-normal semigroups.

Theorem 2.10. *Let S be a compact semi-normal semigroup. An ideal Q of S is topologically semiprime if and only if Q is the intersection of open completely prime ideals.*

3. Topological-radically stability

Let S be a semigroup. SHUM and HOO [2] have introduced the notion of stability of algebraic radicals on S, that is, a non-empty subset F of S is said to be radically stable if and only if $R(F)=R(\bar{F})$. It is recognized in [2] that the notion of stability of algebraic radical could be used to determine whether some properties of an ideal are unchanged under the closure operation. In this section, we extend the stability of algebraic radicals to topological radicals.

Definition 3.1. Let A be a non-empty subset of a semigroup S. Then A is said to be topologically stable if and only if $T(A)=T(\bar{A})$.

The stability of algebraic radical clearly implies the stability of topological radical, but the converse is generally not true.

Example 3.2. Let $S=\overline{[0,1]}\times\overline{[0,1]}$ with the coordinatewise usual multiplication. Then S is a semigroup. Let $A=\overline{[0,1)}\times\overline{[0,{}^1/_2]}\cup\{(1,0)\}$. Then $\bar{A}=\overline{[0,1]}\times\overline{[0,{}^1/_2]}$. Clearly $T(A)=T(\bar{A})$ but $\overline{R(A)\neq R(\bar{A})}$.

Theorem 3.3. *Let S be a compact semigroup and A an ideal of S. Then $T(A)=T(\bar{A})$ if and only if $A\cap E=\bar{A}\cap E$.*

PROOF. The conditions is necessary since if $A\cap E\subsetneq\bar{A}\cap E$, then there exists an idempotent $e\in E$ such that $e\in\bar{A}\setminus A$. Hence $e\in T(\bar{A})=T(A)$ and so $\Gamma(e)\cap A\neq\varnothing$. As $\Gamma(e)=\{e\}$, it follows that $e\in A$, a contradiction. To prove the sufficiency, let $x\in T(\bar{A})$, then $\Gamma(x)\cap\bar{A}\neq\varnothing$ and $e\in A$ where $e^2=e\in\Gamma(x)$. The hypothesis now implies that $e\in A$, whence $\Gamma(x)\cap A\neq\Phi$. Thus $x\in T(A)$.

Parallel to theorem 3.3, we obtain a modified version of Theorem 3.2 of SHUM—HOO [2].

Theorem 3.4. (SHUM—HOO [2]). *Let A be an open ideal of a compact commutative semigroup S. Then A is algebraic-radically stable if and only if $A\cap E=\bar{A}\cap E$.*

PROOF. If A is algebraic-radically stable, then it is trivial that $A\cap E=\bar{A}\cap E$. To prove the sufficiency, let $x\in R(\bar{A})$. Then there exists an integer $n\geqq 1$ such that $x^n\in\bar{A}$. Since $\bar{A}$ is an ideal of S, we have $\overline{\{x^n,x^{n+1},\ldots\}}\subset\bar{A}$, hence $e\in\bar{A}$ where $e^2=e\in\Gamma(x)$. By hypothesis, we have $e\in A$. Now A is open, so $x^k\in A$ for some integer $k\geqq 1$. Thus $x\in R(A)$.

It was seen in [2] that if A is an ideal of S which is algebraic-radically stable in a compact commutative semigroup S, then A is a primary ideal if and only if

$\bar{A}$ is a primary ideal [2]. The proof of this result is trivial. Naturally, we wish to obtain similar results on topological-radically stable ideals in compact semi-normal semigroups. As we shall see that it is far more complicated than one expected to obtain such an analogy.

Definition 3.5. Let A be an ideal of a semigroup S. A is called a Γ-primary ideal of S if the following conditions hold simultaneously.

(i) *$xy\in A$, $x\notin A$ implies that $\Gamma(y)\cap A\neq\varnothing$*
and

(ii) *$xy\in A$, $y\notin A$ implies that $\Gamma(x)\cap A\neq\varnothing$,*
where x, y are arbitrary elements of S.

Example 3.6. Let $S=\{0\}\times\underline{[0, 1/2]}\cup\left\{\frac{1}{n}\middle|n=2, 3, \dots\right\}\times\{0\}$ with the coordinate-wise usual multiplication. Then S is a semigroup and $A=\{0\}\times\underline{[0, 1/4]}$ is a Γ-primary ideal of S.

Definition 3.7. Let S be a semigroup. S will be called topological-radically complete if $T(A)=S$ implies $A=S$ for every ideal A of S. In other words, S is topological-radically complete if S is the only ideal of S with topological radical S.

We now give a characterization on Γ-primary ideals in compact semi-normal semigroups.

Theorem 3.8. *Let S be a compact semi-normal semigroup with $S^2=S$. If A is a proper ideal of S which contains all idempotents in $\Phi(S)$, the intersection of all maximal ideals of S, then A is Γ-primary if and only if $T(A)$ is a proper prime ideal of S.*

The following lemma is needed to prove theorem 3.8.

Lemma 3.9. *Let S be a compact semigroup, then S is topological-radically complete if and only if $S^2=S$.*

Proof. The necessity is trivial by $T(S^2)=S$. For the sufficiency, let us suppose that $T(A)=S$ for some proper ideal A of S. As $T(A)$ is the set $\{x\in S|\Gamma(x)\cap\cap A\neq\Phi\}$ and S is compact, we have $E\subset A$. Since $S^2=S$, by Koch and Wallace (see [8; page 44]), we have $S=SES\subset A$ which is a contradiction. Hence S is topological-radically complete.

We only need to prove the sufficiency for Theorem 3.8 for the necessity is trivial.

Suppose $T(A)$ is a proper prime ideal of S. Since $E\cap\Phi(S)\subset A$, we have $\Phi(S)\subset T(A)$. Then by Schwarz [10], each prime ideal of S containing $\Phi(S)$ and different from S is a maximal ideal. Thus $T(A)$ is a maximal ideal of S. If A is not a Γ-primary ideal of S, then there exists elements x and y such that $xy\in A$, but $x\notin A$ and $y\notin T(A)$. [The proof is similar if $y\in A$ and $x\in T(A)$.] Now let e be the unique idempotent of $\Gamma(y)$. Then $e\notin T(A)$ for $T(A)$ is topologically semi-prime. Consequently, $T(A)\subsetneqq T(A\cup eS)\subseteqq S$ and by the maximality of $T(A)$, we have $T(A\cup eS)=S$. So $A\cup eS=S$ by lemma 3.9. Since $x\notin A$ and $y\notin T(A)$, we have $x\in eS$ and $y\in eS$. These imply that $x=es=xe$ and $y=ey=ye$, whence

$\Gamma(y)$ is a group. Therefore, there is an element $y^{-1}\in\Gamma(y)$ such that $y^{-1}y=yy^{-1}=e$. Hence, $x=xe=(xy)y^{-1}\in A$, a contradiction.

Corollary. *Let S be as in Theorem 3.8. Suppose furthermore that $\Phi(S)\cap E=K(S)\cap E$, where $K(S)$ is the minimal ideal of S. Then an ideal A of S is Γ-primary if and only if $T(A)$ is prime.*

In view of the above corollary, we obtain an analogous result of [2] as follows:

Theorem 3.10. *Let S be a compact semi-normal semigroup with $S^2=S$. If A is a proper ideal of S such that:*
(i) *$K(S)\cap E=\Phi(S)\cap E\subset A\cap E$,*
(ii) *A is topological-radically stable*
then A is Γ-primary if and only if $\bar{A}$ is primary.

4. P-semigroups

In this section, we introduce a special kind of semigroups, namely p-semigroups which is inspired by H. S. Butts and R. Gilmer [1]. We shall show that the algebraic radical and topological radical of ideas in p-semigroups always coincide.

Definition 4.1. *A* p-semigroup is a semigroup in which every ideal is a finite intersection of powers of topological semiprime ideals.

Let S be the set $\left\{\left(\frac{1}{n}, 0\right)\middle| n=1, 2, 3, \ldots\right\}\cup\left\{\left(0, \frac{1}{n}\right)\middle| n=1, 2, 3, \ldots\right\}\cup\{(0, 0)\}$ with usual topology inherited from the plane and usual coordinatewise multiplication, then S is a p-semigroup.

Theorem 4.2. *Let S be a compact semi-normal semigroup in which every ideal can be written as a finite intersection of Γ-primary ideals and the cardinality of E is finite. Then the followings are equivalent:*
(i) *S is a p-semigroup.*
(ii) *Every Γ-primary ideal of S is a power of its topological radical.*
(iii) *Every ideal of S is a finite intersection of power of open ideals.*
For proving Theorem 4.2, we need the following lemma.

Lemma 4.3. *Let P be a topologically semiprime ideal of a compact semigroup S, then $T(P)=T(p^{l_i})=P$ for all integers $1\leqq l_i\leqq\infty$, where P^∞ is the set $\bigcap_{n=1}^{\infty} P^n$.*

Proof. Obviously, $T(p^{l_i})\subset T(P)$ $(1\leqq l_i\leqq\infty)$. Now, let $x\in T(P)$, then $\Gamma(x)\cap P\neq\emptyset$. Let $y\in\Gamma(x)\cap P$. Since S is compact, so is $\Gamma(x)$. Hence there exists an unique idempotent $e\in\Gamma(y)\subset\Gamma(x)$ and $e\in\Gamma(y)\subset P$, which implies that $e\in P^{l_i}$ for all $1\leqq l_i\leqq\infty$. Thus $\Gamma(x)\cap P^{l_i}\neq\emptyset$ and $x\in T(P^{l_i})$.
We now prove theorem 4.2.
(i)$\Rightarrow$(ii). Let Q be a Γ-primary ideal of S. Since S is a p-semigroup, $Q=\bigcap_{1}^{n} p_i^{l_i}$ $(1\leqq l_i\leqq\infty)$ where p_i are distinct topologically semiprime ideals of S. Hence $Q\subset p_i^{l_i}$ for each i. Let $P=T(Q)$, then for each i, $P\subset T(P_i^{l_i})=P_i$. On the

other hand, since $P\supset\bigcap_1^n P_i^{l_i}$, so by lemma 4.2, $P=T(P)\supset T\left(\bigcap_1^n P_i^{l_i}\right)=\bigcap_1^n T(P_i^{l_i})=$ $=\bigcap_1^n P_i\supset\prod_1^n P_i$. Since P is topologically semiprime and E is finite, by theorem 2.9 (iii), P must be an open prime ideal of S. Hence $P\supset P_i$ for some i. Thus, $P=P_i$. Clearly $P_1\subsetneqq P_i$ for $i\geqq 2$, so $\bigcap_2^n P_i^{l_i}\not\subset P_1$. Let $y\in\bigcap_2^n P_i^{l_i}$ with $y\notin P_1$. Then for any $x\in P_1^{l_1}$, we have $xy\in P_1^{l_1}\left(\bigcap_2^n P_i^{l_i}\right)\subset Q$, thus $x\in Q$ since $\Gamma(y)\cap Q=\emptyset$. Therefore $Q=P_1^{l_1}$, i.e., any Γ-primary ideal of S is a power of its topological radical.

(ii)$\Rightarrow$(iii). Let A be an ideal of S. Then by hypothesis, $A=\bigcap_1^n Q_i$, where Q_i are Γ-primary ideals of S. Let $P_i=T(Q_i)$ for each i. By assumption that E is finite and theorem 2.9 (iii), we know that each P_i is an open prime ideal. Also by (ii), each $Q_i=P_i^{l_i}$ for some $1\leqq l_i\leqq\infty$. Thus $A=\bigcap_1^n P_i^{l_i}\,(1\leqq l_i\leqq\infty)$.

(iii)$\Rightarrow$(i). Each open prime ideal of a compact semi-normal semigroup is an open completely prime ideal of S, (see Theorem 2.9) and hence topologically semiprime.

If the condition of finite intersection is weakened, we have the following theorem.

Theorem 4.4. *Let S be a compact semigroup in which every ideal is an intersection (not necessarily finite) of powers of open prime ideals. Then the followings are true and equivalent.*

(i) $(A^\infty)^2=A$ *for every closed ideal* A *of* S.

(ii) $(A_1\cap\ldots\cap A_n)^\infty\subset A_1^\infty\ldots A_n^\infty$ *for any finite number of closed ideals* A_i *of* S, *with equality if* S *is commutative.*

PROOF. Let A be an arbitrary closed ideal of S. Then $(A^\infty)=\bigcap_i P_i^{l_i}$ for some open prime ideals P_i and $1\leqq l_i\leqq\infty$. We have $(A^\infty)^2\subset P_i^{l_i}\subset P_i$, hence $A^\infty\subset P_i$. Since S is compact, it follows that $A^n\subset P_i$ for some n, so $A\subset P_i$ for each i. Now $A^{l_i}\subset P_i^{l_i}$ for each i. Thus $A^\infty=\bigcap_1^\infty A^n\subset\bigcap_i A^{l_i}\subset\bigcap_i P_i^{l_i}=(A^\infty)^2$. The equivalence of (i) and (ii) is immediate.

In Theorem 4.4, if all powers of open prime ideals are equal to 1, then we obtain the following generalized result of M. SATYANARAYANA [9].

Theorem 4.5. *Let S be a compact semigroup. Then the following are equivalent*

(i) *Every proper closed ideal is an intersection of open prime ideals.*

(ii) $A^2=A$ *for every closed ideal* A *of* S.

(iii) *For every* $a\in S$. *there exists* x, y, z *such that* $a=zayaz$.

(iv) *The product of any order of a finite number of closed ideals is equal to their intersection.*

Finally, we remark that the algebraic and topological radicals of an ideal always coincide in p-semigroups.

Theorem 4.6. *Let S be a p-semigroup. Then the algebraic and the topological radicals of ideal in S coincide. As a consequence, an ideal A of S is topologically semiprime if and only if it is completely prime.*

PROOF. Let A be any ideal of S. Then $A=\bigcap_1^n P_i^{l_i}$, where P_i are distinct topologically semiprime ideals, $1\leqq l_i\leqq\infty$. Clearly $R(A)=\bigcap_1^n R(P_i)=\bigcap_1^n P_i$. Since $\bigcap_1^n P_i$ is topologically semiprime, so $R(A)=\bigcap_1^n P_i=T\left(\bigcap_1^n P_i\right)=\bigcap_1^n T(P_i)=\bigcap_1^n T(P_i^{l_i})=$ $=T\left(\bigcap_1^n P_i^{l_i}\right)=T(A)$.

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