Note on maximal asymptotic nonbases of order h

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In [3] NATHANSON introduced the notion of a maximal nonbase sequence of order h. In his paper a sequence $\mathscr{A}$ is called an asymptotic nonbase sequence of order h if it has the following two properties:

(1) $\mathscr{A}$ is not an asymptotic base sequence of order h;

(2) if "a" is a nonnegative integer and $a \notin \mathscr{A}$, then $\mathscr{A} \cup \{a\}$ is an asymptotic base sequence of order h.

In [3] NATHANSON constructs maximal asymptotic nonbase sequences with can be represented as a union of residue classes; moreover he poses the problem whether there exist asymptotic maximal nonbase sequence of another type. An affirmative answer to this problem was given by ERDŐS and NATHANSON in [2]. In [6] one finds asymptotic maximal nonbase sequences $\mathscr{A}$ of 2nd order for which $\liminf \frac{A(x)}{x} = 0$ (where $A(x)$ denotes as usual the number of those elements of the sequence which are not greater than x); in [4] and [7] one even finds a sequence for which the order of $A(x)$ is $O(x^{\frac{1}{2}})$. In the case $h > 2$, however, no maximal nonbase sequence of order h and of density O is known.

In the present paper we show that there exist asymptotic nonbases of order h such that the order of their density function is $O(x^{\frac{1}{h}})$. More exactly, we give a procedure by the aid of which we depart from nonbase sequences of order h whose density functions have the order $O(x^{\frac{1}{h}})$; from these sequences we construct sequences whose density functions have the same order but which will be maximal nonbase sequences of order h. The abovementioned procedure has the same main features as the procedure applied in [6] and in [7].

The basic idea of the proof stems from the following analysis of the definition of an asymptotic maximal nonbase sequence of order h.

What does the first condition mean for $\mathscr{A}$? This condition requires the existence of infinitely many natural numbers M_i such that $M_i \notin h\mathscr{A}$ where

$$h\mathscr{A} = \left\{x \mid x = \sum_{i=1}^{h} a_{l_i} \text{ and } a_{l_i} \in \mathscr{A}\right\}.$$

The second condition means that in case $a \notin \mathscr{A}$ ("a" is a natural number) the sufficiently large M_i's belong to $h\{\mathscr{A} \cup \{a\}\}$, that is: those numbers which are not represented by $h\mathscr{A}$, are represented by $h\{\mathscr{A}\cup\{a\}\}$ provided that they are sufficiently large. This can be carried through in such a way that either

$$\begin{aligned} &\{M_i-a\}\in(h-1)\mathscr{A} \quad \text{or}\\ &\{M_i-2a\}\in(h-2)\mathscr{A} \quad \text{or}\\ &\vdots\\ &\{M_i-(h-1)a\}\in\mathscr{A} \quad \text{or}\\ &M_i=la \quad (l=1,2,\dots,h).\end{aligned} \tag{3}$$

Not that the latter relation can be satisfied by only finitely many M_i's.

Assume now $a\notin\mathscr{A}$ and choose "a" from the interval $[0;b]$ then there exists a number m which depends on b and on $\mathscr{A}$ in such a way that in case $M_i>m$ we have $M_i\in h\mathscr{A}\cup\{a\})$ and then at least one of the relations (3) is satisfied. It is more convenient for us to construct $\mathscr{A}$ in such a way that in case of given $b\geqq a\geqq 0$ and $M_i>m$ the relation

$$\{M_i-a\}\in(h-1)\mathscr{A}$$

is satisfied.

We shall construct sequences $\mathscr{A}^*$; $\{m_i\}$; $\{b_i\}$; $\{M_i\}$ for which the following conditions hold:

(4) With the exception of the integers of $\{M_i\}$ the sequence $h\mathscr{A}^*$ contains each nonnegative integer.

(5) If "a" is a nonnegative integer with $a\notin\mathscr{A}^*$, and if $0\leqq a\leqq b$ and $m_i<M_i\leqq M_j$, then

$$\{M_j-a\}\in(h-1)\mathscr{A}^*.$$

(6) $b_i<m_i<M_i$ and $\{b_i\}$; $\{m_i\}$; $\{M_i\}$ are strictly monotonically increasing $(i=1,2,\dots,n,\dots)$.

Condition (4) requires that $\mathscr{A}^*$ is not an asymptotic base of order h. From (5) it follows that $a\notin\mathscr{A}^*$ implies for sufficiently large M_i's the relation $\{M_i-a\}\in\in(h-1)\mathscr{A}^*$, that is, $\{\mathscr{A}^*\cup\{a\}\}$ is an asymptotic basis of order h. By (6) the relations of the orders of the sequences $\{b_i\}$; $\{m_i\}$; $\{M_i\}$ are restricted. This restriction will be sharpened later on by other conditions.

Theorem. *There exists a sequence $\mathscr{A}^*$ which is an asymptotic maximal nonbase sequence whose density function has an order of $O(x^{\frac{1}{h}})$ $(h\geqq 3)$.*

Remark. This result cannot be improved in the following sense: an essentially less dense sequence cannot be an asymptotic maximal nonbase of order h.

PROOF. Let A_0 be a basis sequence of order h. Moreover let $0\in A_0$ and $A(x)=O(x^{\frac{1}{h}})$. A proof of the existence of such a sequence can be found in [1] and [5]. From A_0 we form A_1 by a suitable choice of b_1, m_1, M_1. In general, A_k is formed from A_{k-1} by a suitable choice of b_k, m_k, M_k. If b_k, m_k, M_k and A_{k-1} are already determined, then define

$$(7)\qquad \begin{cases} A_k = A^{\mathrm{I}}_{k-1}\cup A^{\mathrm{II}}_{k-1}\cup A^{\mathrm{III}}_{k-1}\cup A^{\mathrm{IV}}_{k-1} \\ A^{\mathrm{I}}_{k-1} = \{x\,|\,x \leqq m_k \text{ and } x\in A_{k-1}\} \\ A^{\mathrm{II}}_{k-1} = \{x\,|\,x = M_k+a_i+1 \text{ and } a_i\in A_{k-1}\} \\ A^{\mathrm{III}}_{k-1} = \{h(M_k+1)+M_i\}\cup\{sM_k+M_i+s\}\cup\{(s+1)M_k+r\} \end{cases}$$

where $i=1, 2, \dots, k-1$; $s=1, 2, \dots, h$ and $r=0, 1, \dots, h-1$. A^{IV}_{k-1} denotes a set of natural numbers which satisfies the following conditions:

α) if $a\notin A_{k-1}$ and $0\leqq a\leqq b_k$, then $\{M_k-a\}$ is an element of
$$(h-1)\{A^{\mathrm{I}}_{k-1}\cup A^{\mathrm{II}}_{k-1}\cup A^{\mathrm{III}}_{k-1}\cup A_{k-1}\},$$

β) if $d\in A^{\mathrm{IV}}_{k-1}$, then $\frac{1}{3h^2}M_k<d<M_k$,

γ) the number of elements of A^{IV}_{k-1} is $|A^{\mathrm{IV}}_{k-1}|\leqq hb_k$,

δ) $A^{\mathrm{I}}_{k-1}\cap A^{\mathrm{IV}}_{k-1}=\emptyset$ (for certain k's it can also occur that A^{IV}_{k-1} is the empty set).

Knowing the sequences A_k, the sequence $\mathscr{A}^*$ can be defined in the following manner:

$$(8)\qquad x\in\mathscr{A}^* \Leftrightarrow x \leqq M_k \quad\text{and}\quad x\in A_k.$$

If the sequences A_k satisfy the following conditions

(9) With the exception of $M_1, M_2, \dots, M_k$ the set hA_k contains all nonnegative integers;

(10) there exists a constant c such that $A_k(x)\leqq cA_0(x)$ and c does not depend on k,

then the existence of the sequence $\mathscr{A}^*$ mentioned in the theorem is proved.

Indeed, $\mathscr{A}^*(x)=O(x^{\frac{1}{h}})$ holds because of (8) and (10), since in the case $x\leqq M_k$ we have $|\mathscr{A}^*(x)|=A_k(x)$ and $A_k(x)\leqq cA_0(x)$ but $A_0(x)=O(x^{\frac{1}{h}})$. Because of (8) and (9) the set $h\mathscr{A}^*$ contains all nonnegative integers with the exception of the elements of $\{M_k\}$. On the other hand, if x is a nonnegative integer with $x\leqq b_k$ and $x\notin\mathscr{A}^*$, then we have $\{M_l-x\}\in(h-1)\mathscr{A}^*$ for $l\geqq k$ because of (8) and the property (α) of the sets A^{IV}_{l-1}.

It is therefore sufficient to verify (9) and (10), which will be done by induction on k. First we give the numbers b_k, m_k, M_k and assume the existence of A^{IV}_{k-1}. After having seen in this way that (9) and (10) hold, we show that A^{IV}_{k-1} can be constructed for every $k\geqq 1$ according to the corresponding conditions.

By c_0 we denote the least number for which $A_0(x)=c_0x^{\frac{1}{h}}$ holds (provided that $x\geqq 1$). Let $b_1=h^{2h+1}$ and $m_1=a_i$ but $a_i\geqq h^h[c_0b_1h^2]^h$ where a_i denotes an arbitrarily large, but fixed element of A_0. By M_1 we denote that natural number which is not representable as a sum of h members of elements of A_0 which are not greater than a_i, but every natural number less than M_1 is representable as a sum of h members of elements of A_0 which are not greater than a_i. In other words: by M_1 we mean the least positive integer which does not belong to hA^{I}_0. M_1 is clearly greater than m_1, since A_0 was a basis of order h with $0, 1\in A_0$.

First we verify (9). The integers of the interval $[0; M_1-1]$ are contained in hA_0^{I}. The integers of the interval $[hM_1+1; \infty]$ are contained in hA_0^{II}. Namely, we have $M_1+1 \in A_0^{\mathrm{II}}$ since $0 \in A_0$ and so $h(M_1+1) \in hA_0^{\mathrm{II}}$. On the other hand, let $y > h(M_1+1)$ and $0 < y' = y - h(M_1+1)$. Then $y' \in hA_0$, that is,

$$y' = a_{i_1}+a_{i_2}+\ldots+a_{i_h} \quad \text{and} \quad a_{i_j} \in A_0 \quad (j = 1, 2, \ldots, h)$$

but

$$M_1+a_{i_j}+1 \in A_0^{\mathrm{II}} \quad \text{and therefore} \quad y \in hA_0^{\mathrm{II}} \quad \text{since}$$

$$y = M_1+a_{i_1}+1+M_1+a_{i_2}+1+\ldots+M_1+a_{i_h}+1.$$

The integers of the interval $[M_1+1; hM_1+h]$ can be written in the form sM_1+r where s and r are nonnegative integers with $0 < s \leqq h$ and $0 \leqq r < M_1$. We distinguish now two cases according to whether $r-s \geqq 0$ or $r-s < 0$. If $r-s \geqq 0$ then $r-s \in hA_0$, that is, $r-s = a_{r_1}+a_{r_2}+\ldots+a_{r_h}$. In this case we have

$$sM_1+r = M_1+a_{r_1}+1+M_1+a_{r_2}+1+\ldots+M_1+a_{r_s}+1+a_{r_{s+1}}+\ldots+a_{r_h}.$$

If $r-s<0$, then $r < s \leqq h$, that is $r \leqq h-1$, $sM_1 \in A_0^{\mathrm{III}}$ and $1 \in A_0^{\mathrm{I}}$; thus we have

$$sM_1+r = sM_1+\underset{1}{1}+\underset{2}{1}+\ldots+\underset{r}{1},$$

that is,

$$sM_1+r \in h\,\{A_0^{\mathrm{I}} \cup A_0^{\mathrm{II}} \cup A_0^{\mathrm{III}}\}.$$

We ask now whether (10) is satisfied or not. If $x \leqq \frac{1}{3h^2} M_1$, then $A_1(x) = A_0^{\mathrm{I}}(x) = A_0(x)$. If $\frac{1}{3h^2} M_1 < x < m_1$, then $A_0(x) = A_1(x) = A_0(x)+hb_1$, since for A_0^{IV} condition (γ) is satisfied. If $m_1 \leqq x \leqq M_1$, then $A_1(x) = A_0(m_1)+A_0^{\mathrm{IV}}(x) \leqq A_0(x)+hb_1$. If $1+M_1 \leqq x$, then $|A_0^{\mathrm{III}}(x)| \leqq h^2$ and

$$A_1(x) = A_0(m_1)+A_0^{\mathrm{I}}(M_1)+A_0\big(x-(M_1+1)\big) \leqq 3A_0(x)+h^2 = 4A_0(x). \tag{11}$$

Taking into consideration that $A_0^{\mathrm{IV}}(M_1) \leqq hb_1, b_1 = h^3$ and $M_1 > h^h(c_0 b_1 h^2)^h$, it follows that $A_1(x) \leqq 2A_0(x)$ provided that $x \leqq M_1$. On the other hand, (11) yields

$$\lim_{x\to\infty} \frac{A_0(x)}{A_1(x)} = 1.$$

At this place we remark that because of (8) it is appropriate to assign also to m_0 a value; by definition let this value be equal 0. By ε we denote an arbitrarily small but fixed positive number; moreover let c_i be numbers satisfying

$$\prod_{i=1}^{\infty} c_i = 1+\varepsilon \quad \text{and} \quad c_i > 1. \tag{12}$$

Let now for 1 natural numbers which are smaller than k the values of A_l, b_l, m_l, M_l be given. Then $b_k = b_{k-1}+1$ and we choose the m_k's in such a manner that in case $x > m_k$ the relation $A_{k-1}(x)/A_{k-2}(x) < c_{k-1}$ is satisfied.; furthermore let m_k be greater than $h^h(M_{k-1}h^2 b_k)^h$ and let m_k be equal to some

a_i which belongs to A_{k-1}. By M_k we denote that least positive integer which is, on the one hand, greater than M_{k-1}, and which, on the other hand, does not belong to hA^{I}_{k-1}.

First we verify that (9) holds true. With exception of the numbers $M_1, M_2, \ldots, M_{k-1}$ the set hA^{I}_{k-1} contains the integers of the interval $[0; M_{k-1}]$. This follows from the definition of A^{I}_{k-1}, from the fact that (9) is satisfied for A_{k-1} and from the choice of M_k. The integers of the intervallum $[h(M_k+1); \alpha]$ are contained in $h\{A^{\mathrm{II}}_{k-1}\cup A^{\mathrm{III}}_{k-1}\}$. In fact, let $y>h(M_k+1)$ and $0<y'=y-h(M_k+1)$; if y' is equal to some of the numbers $M_1, M_2, \ldots, M_{h-1}$, then $y\in A^{\mathrm{III}}_{k-1}$ and if y' is distinct from each of these numbers, then we have by assumption that $y'\in hA_{k-1}$, that is, $y'=a_{i_1}+a_{i_2}+\ldots+a_{i_h}$ where $a_{i_j}\in A_{k-1}$ $(j=1, 2, \ldots, k)$. It follows that $M_k+a_{i_j}+1\in A^{\mathrm{II}}_{k-1}$ and so $y\in hA^{\mathrm{III}}_{k-1}$ since

$$y = M_k+a_{i_1}+1+M_k+a_{i_2}+1+\ldots+M_k+a_{i_h}+1.$$

The integers of the interval $[M_k+1; (h+1)M_k]$ can be written in the form sM_k+r where s and r are nonnegative integers with $0<s\leqq h$ and $0\leqq r<M_k$. We distinguish now two cases according to whether $r-s\geqq 0$ or $r-s<0$.

If $r-s\geqq 0$ and $r-s\in hA_{k-1}$, that is,

$$r-s = a_{r_1}+a_{r_2}+\ldots+a_{r_s}+a_{r_{s+1}}+\ldots+a_{r_h},$$

then

$$sM_k+r = M_k+a_{r_1}+1+\ldots+M_k+a_{r_s}+s+a_{r_{s+1}}+\ldots+a_{r_h},$$

that is, $sM_k+ r\in h\{A^{\mathrm{II}}_{k-1}\cup A^{\mathrm{I}}_{k-1}\}$. If $r-s\notin hA_{k-1}$, then $r-s=M_i$ and $1\leqq i\leqq k-1$ but $sM_k+M_i+s\in A^{\mathrm{III}}_{k-1}$.

If $r-s<0$, then $r<s\leqq h$, that is, $r\leqq h-1$, $sM_k\in A^{\mathrm{III}}_{k-1}$, $0; 1\in A^{\mathrm{I}}_{k-1}$ and $M_k+1\in A^{\mathrm{II}}_{k-1}$ therefore

$$sM_k+r = sM_k+\underset{1}{1}+\underset{2}{1}+\ldots+\underset{r}{1},$$

that is,

$$sM_k+r\in h\,\{A^{\mathrm{I}}_{k-1}\cup A^{\mathrm{II}}_{k-1}\cup A^{\mathrm{III}}_{k-1}\}.$$

We investigate whether (10) is satisfied for A_k. If $x=\frac{1}{3h^2}M_k$, then $A_{k-1}(x)= =A_k(x)$. If $(3h^2)^{-1}M_k\leqq x\leqq m_k$, then

$$A_{k-1}(x) \leqq A_k(x) \leqq A_{k-1}(x)+hb_k \quad \text{since} \quad |A^{\mathrm{IV}}_{k-1}| \leqq hb_k.$$

If $m_k\leqq x\leqq M_k$, then

$$A_k(x) = A_{k-1}(m_k)+A^{\mathrm{IV}}_{k-1}(x) \leqq A_{k-1}(x)+hb_k = A_{k-1}(x)+h(h+k-1).$$

If $1+M_1\leqq x$, then

$$\begin{aligned}(13)\qquad A_k(x) &= A_{k-1}(m_k)+A^{\mathrm{III}}_{k-1}(x)+A^{\mathrm{IV}}_{k-1}(M_k)+A_{k-1}(x-(M_k+1)) \leqq \\ &\leqq A_{k-1}(m_k)+k-1+h(k-1)+h^2+hb_k+A_{k-1}(x-(M_k+1)) \leqq \\ &\leqq 4A_{k-1}(x).\end{aligned}$$

Here wo used that the number of the elements of the set A^{III}_{k-1} is not greater than $(k-1+h(k-1)+h^2)$ and that $A^{\mathrm{IV}}_{k-1}(M_k)\leqq hb_k=h(h^3+k-1)=h^4+h(k-1)$; besides we took into consideration that $A_{k-1}(M_k)\leqq A_{k-1}(x)$ provided that $x\geqq M_k+1$, and then $M_k^{\frac{1}{h}}\leqq A_{k-1}(M_k)$. Moreover we used that $M_k^{\frac{1}{h}}$ is essentially greater than $h^2+h(k-1)$ and $(h-1)k-1+h^2$ since $M_k>h^h(M_{k-1}h^2b_k)^h$ which implies $M_k^{\frac{1}{h}}>$ $>h(M_{k-1}h^2b_k)$ and $M_{k-1}>h^h(M_{k-2}h^2b_{k-1})^h$. On the other hand (13) immediately yields that

$$\lim_{x\to\infty}\frac{A_{k-1}(x)}{A_k(x)}=1.$$

We show now that there exists a constant c which does not depend on k and for which $A_k(x)\leqq cA_0(x)$. We have seen above that $A_k(x)\leqq 4A_{k-1}(x)$ follows from $x\geqq M_k+1$; because of the choice of m_k we have then $\frac{A_{k-1}(x)}{A_{k-2}(x)}<c_{k-1}$, that is, $A_{k-1}(x)>c_{k-1}A_{k-2}(x)$. Using $x\geqq m_k\geqq m_{k-1}\geqq\ldots\geqq m_1$ we obtain that $A_k(x)\leqq$ $\leqq 4A_{k-1}(x)<4c_{k-1}A_{k-2}(x)<\ldots<4c_{k-2}c_{k-3}\ldots c_1A_0(x)$; because of the choice of the numbers $c_1, c_2, \ldots, c_{k-1}$ we have here $c_1c_2\ldots c_{k-2}<\prod_{i=1}^{\infty}c_i=1+\varepsilon$ and therefore $A_k(x)\leqq 4(1+\varepsilon)A_0(x)$ where the constant $1+\varepsilon$ does not depend on k. If $x<m_k$ and if $m_i\leqq x<M_{i+1}$ for some nonnegative integer, then

$$A_k(x)=A_{i+1}(x)\leqq 4A_i(x)<4c_ic_{i-1}\ldots c_1A_0(x)\leqq$$
$$\leqq 4(1+\varepsilon)A_0(x),$$

where, of course, $i<k$.

It remains now only to show the existence of the sets A^{IV}_{k-1}. First we deal with the case $h=3$. Let a be an integer such that $0\leqq a\leqq b_k$ and $a\notin A_{k-1}$ (if such an a does not exist, then A_{k-1} is the empty set). We show now that there exist integers D_1 and D_2 such that the following conditions are satisfied:

$$\frac{1}{3}M_k<D_1;\quad D_2<\frac{2}{3}M_k \tag{14}$$

and

$$M_k\notin 3A^{\mathrm{I}}_{k-1}\cup\{D_1\}\cup\{D_2\} \tag{15}$$

$$M_k-a=D_1+D_2. \tag{16}$$

It is obvious that D_1 and D_2 have to be determined only if $M_k-a\notin 2A^{\mathrm{I}}_{k-1}$. In particular (15) requires that neither M_k-D_i nor M_k-2D_i belong to $2A^{\mathrm{I}}_{k-1}$ or A^{I}_{k-1}. We look now which numbers of the interval $\left(\frac{1}{3}M_k;\frac{2}{3}M_k\right)$ are at our disposal. In $\left(\frac{1}{3}M_k;\frac{2}{3}M_k\right)$ there are at least $\left[\frac{1}{3}M_k\right]-|2A^{\mathrm{I}}_{k-1}|-1$ integers y for which $M_k-y\notin 2A^{\mathrm{I}}_{k-1}$. We denote the set of these integers y by $\{y\}$. We cancel now those elements of $\{y\}$ for which $M_k-2y\in 2A_{k-1}$. The remaining elements will be denoted by $\{z\}$. For the number of the elements of $\{z\}$ we obviously have the relation

$$|\{z\}| \geqq \frac{1}{3} M_k - 2A^{\mathrm{I}}_{k-1} - 1 - A^{\mathrm{I}}_{k-1} = \frac{1}{3} M_k - 4c^2 M_k - 1 = \frac{3}{4}\frac{1}{3} M_k.$$

Here we took into consideration that $|A^{\mathrm{I}}_{k-1}| \leqq c M_k^{\frac{1}{3}}$ and therefore $|2A^{\mathrm{I}}_{k-1}| = = (cM_k^{\frac{1}{3}})^2 = c^2 M_k^{\frac{2}{3}}$. Among the elements of $\{z\}$ one can choose z_1 and z_2 such that $M_k - \mathrm{a} = z_1 + z_2$. Namely, if the elements of $\{z\}$ are taken away from $M_1 - a$, then we get $\left[\frac{3}{4}\frac{1}{3} M_k\right]$ integers which belong to the interval $\left(\frac{1}{3} M_k - a;\ \frac{2}{3} M_k - a\right)$. Observing now that "$a$" is essentially smaller than M_k, that is, $a \leqq b_k$ and $M_k \geqq \geqq 3^3 (M_{k-1} 3^2 b_k)^3$ and therefore $a \leqq b_k < \frac{1}{9}\frac{1}{4}\frac{1}{3} M_k$, it follows that there exists a z_1 such that $z_2 = M_k - a - z_1 \in \{z\}$. By definition of $\{z\}$, z_1 and z_2 correspond D_1 and D_2, since the conditions (14), (15) and (16) are satisfied. If $D_i \in A^{\mathrm{I}}_{k-1}$, then it will not be considered as an element of A^{IV}_{k-1}; D_i will be considered as an element of A^{IV}_{k-1} only if $D_i \notin A^{\mathrm{I}}_{k-1}$ $(i=1, 2)$. The described procedure can be repeated with an arbitrary a which satisfies $0 \leqq a \leqq b_k$ and $a \notin A^{\mathrm{I}}_{k-1}$; note that then in (15) the set A^{I}_{k-1} has to be replaced by the set $\{A^{\mathrm{I}}_{k-1} \cup \{D_1\} \cup \{D_2\} \cup \ldots \cup \{D_t\}\}$ where the numbers $D_1, D_2, \ldots, D_t$ have been constructed in the preceding step: the corresponding inequalities then preserve their validity. Looking at the conditions for A^{IV}_{k-1} we can see that (α) is satisfied because of (15) and (16), (β) is satisfied because of (14) and that instead of (γ) even $|A^{\mathrm{IV}}_{k-1}| < 2b_k$ is satisfied. Concerning (δ) it suffices to remark that in case $D_j \in A^{\mathrm{I}}_{k-1}$ this element is not taken to the elements of A^{IV}_{k-1}.

If $h > 3$, then the existence of A^{IV}_{k-1} will be proved in the following manner. For arbitrary integers $0 \leqq a \leqq b_k$ with $a \notin A^{\mathrm{I}}_{k-1}$ we determine in the first step an integer F which has the following properties:

$$\frac{M_k}{h^2} - S = F = \frac{M_k}{h^2} + S \quad (i = 1, 2, \ldots, h-3). \tag{17}$$

For reasons of convenience we choose S to be $M_k^{\frac{h-1}{h} + \frac{1}{2h}}$.

$$M_k \notin h\{A_{k-1} \cup \{F\}\}. \tag{18}$$

This means at the same time that $M_k - a - (h-3)F = K \in A^{\mathrm{I}}_{k-1}$ and $a \in 2A^{\mathrm{I}}_{k-1}$ cannot be satisfied simultaneously; therefore we shall require that

$$M_k - a - (h-3)F = K \notin A^{\mathrm{I}}_{k-1}. \tag{19}$$

During the second step we determine to the already existing K numbers D_1 and D_2 such that the following relations are satisfied:

$$\frac{1}{3} M_k < D_i < \frac{2}{3} M_k \quad \text{and} \quad i = 1, 2 \tag{20}$$

$$K = D_1 + D_2 \tag{21}$$

$$M_k \notin h\left\{A^{\mathrm{I}}_{k-1} \cup \{F\} \bigcup_{i=1}^{2} \{D_i\}\right\}. \tag{22}$$

Simply spoken it is here our aim to represent M_1-a as a sum of $h-1$ integers, which fulfil the preceding conditions; the conditions on the order ensure that the density of A_k will not essentially change. For a given "a" the abovementioned procedure has clearly to be applied only if $M_1-a\notin(h-1)A^{\mathrm{I}}_{k-1}$; from the numbers F_j, D_i, of course, only those are taken to be elements of the set A^{IV}_{k-1} which are not elements of A^{I}_{k-1}. Our procedure for the determination of the numbers F and D_i will have the following property: if the numbers $\{F_j\}$ and $\{D_i\}$ have already been determined for some a, then the procedure can automatically be applied for an arbitrary new a satisfying the conditions; to this end we have in (18), (19) and (22) to write $A^{\mathrm{I}}_{k-1}\cup\{F_j\}\cup\{D_i\}$ instead of A^{I}_{k-1} where $\{F_j\}$ and $\{D_i\}$ denotes all until now constructed and in A^{IV}_{k-1} collected numbers. Note that if A^{IV}_{k-1} is constructed in the above described manner, then it satisfies the conditions (α), (β), (γ) and (δ). Indeed, (α) is satisfied since

$$M_k-a\in(h-1)\{A^{\mathrm{I}}_{k-1}\cup A^{\mathrm{IV}}_{k-1}\}\subset(h-1)\{A^{\mathrm{I}}_{k-1}\cup A^{\mathrm{II}}_{k-1}\cup A^{\mathrm{III}}_{k-1}\cup A^{\mathrm{IV}}_{k-1}\}$$

holds because of (19) and (21); moreover we have $M_k\notin hA_k$ because of (7), (18) and (22). (β) holds because of (17) and (20). In detail, we have to show that $F>\dfrac{1}{3h^2}M_k$ if

$$\frac{M_k}{h^2}-S<F \quad\text{and}\quad S=M^{\frac{h-1}{h}+\frac{1}{2h}}.$$

It is obviously sufficient to show that

$$\frac{M_k}{h^2}-M_k^{\frac{h-1}{h}+\frac{1}{2h}}\geqq\frac{M_k}{3h^2},$$

that is,

$$\frac{2}{3}\geqq h^2M_k^{\frac{h-1}{h}+\frac{1}{2h}-1}=h^2M_k^{-\frac{1}{2h}}.$$

But $h^2x^{-\frac{1}{2h}}$ is for positive values of x a monotonically decreasing function and therefore it is sufficient to prove the inequality for M_1. Observing that

$$M_1\geqq h^h[c_0b_1h^2]^h \quad\text{and}\quad b_1=h^3+1, \quad h>3$$

we get

$$h^2M_1^{-\frac{1}{2h}}\leqq h^2(hc_0b_1h^2)^{-\frac{1}{2}}=h^2(h^3c_0h^3)^{-\frac{1}{2}}=\frac{1}{\sqrt{c_0}h}\leqq\frac{2}{3}$$

since $c_0\geqq1$ and $h^h\geqq4$.

From the fact that A^{IV}_{k-1} can have maximally $3b_k$ elements, it follows that even more is true than (γ). Condition (δ) is satisfied since the numbers F_j and D_i with $F_j, D_i\in A^{\mathrm{I}}_{k-1}$ are not taken to be elements of A^{IV}_{k-1}.

We concentrate now on the question which integers of the interval $[h^{-2}M_k-S;\ h^{-2}M_k+S]$ play the role of F. By $\{y\}$ we denote the set of those integers for which

$$h^{-2}M_k-S\leqq y\leqq h^{-2}M_k+S \quad\text{and}\quad M_1-sy\notin(h-1)A^{\mathrm{I}}_{k-1} \quad (s=1,2,\ldots,h)$$

is satisfied. The number of the y's having the preceding property is clearly greater or equal than

$$2[S]-1-h|(h-1)A'_{k-1}| \geqq 2[S]-h\big(A_{k-1}(M_k)\big)^{h-1}-1 \geqq$$

$$\geqq 2[S]-h|A_0(M_k)|^{h-1}\cdot 5^h \geqq 2[S]-1-hM_k^{\frac{h-1}{h}}(5c_0)^h \geqq$$

$$\geqq 2M_k^{\frac{h-1}{h}+\frac{1}{2h}}-hM_k^{\frac{h-1}{h}}(5\cdot c_0)^h-2 > M_k^{\frac{h-1}{h}+\frac{1}{2h}}.$$

From $\{y\}$ we choose now an y such that

$$M_k-a-(h-3)y = K \notin A^{\mathrm{I}}_{k-1}. \tag{23}$$

We denote this element by y_0; this will be our F. This y_0 exists since among the elements of the set $\{y\}$ there is at most a number of $|A^{\mathrm{I}}_{k-1}| \leqq 5c_0M_k^{\frac{1}{h}}$ which does not satisfy (23), whereas the number of the element of y is considerably greater, namely greater than $M_k^{\frac{h-1}{h}+\frac{1}{2h}}$. Knowing F and K the numbers D_1 and D_2 can be constructed similarly as in the case $h=3$.

At this place we also pose the question, how many integers V exist which satisfy (20) and for which

$$M_k \notin hA^{\mathrm{I}}_{k-1} \cup \{F\} \cup \{V\}. \tag{24}$$

This latter condition is necessary because we want to see directly from the determination of D_1, D_2 that (22) holds true. (24) can be written also as

$$M_k-i\cdot V \notin (h-1)\{A^{\mathrm{I}}_{k-1} \cup \{F\}\} \quad \text{and} \quad M_k \neq 2V \quad (i = 1, 2). \tag{25}$$

Here it is obviously sufficient to investigate the case $i=1, 2$ since $\frac{1}{3}M_k < V < \frac{2}{3}M_k$ and $3V > M_k$. At most $2|(h-1)A^{\mathrm{I}}_{k-1}(M_k)+1$ numbers of the interval $\left[\frac{1}{3}M_k; \frac{2}{3}M_k\right]$ does not satisfy condition (25); on the other hand, we have $A_{k-1}(x) \leqq 4(1+\varepsilon)A_0(x) \leqq \leqq 4(1+\varepsilon)c_0x^{\frac{1}{h}}$, and so

$$A^{\mathrm{I}}_{k-1}(M_k) \leqq 5c_0M_k^{\frac{1}{h}},$$

that is,

$$|(h-1)A^{\mathrm{I}}_{k-1}(M_k)| \leqq (5c_0)^{h-1}M_k^{\frac{h-1}{h}}.$$

But in the interval $\left[\frac{1}{3}M_k; \frac{2}{3}M_k\right]$ there are at least $\left[\frac{1}{3}M_k\right]-2$ integers. Hence we have

$$\left[\frac{1}{3}M_k\right]-2-2|(h-1)A^{\mathrm{I}}_{k-1}(M_k)|-1 \geqq \left[\frac{1}{3}M_k\right]-3-2(5c_0)^{h-1}M_k^{\frac{h-1}{h}} > \frac{3}{4}\,\frac{1}{3}M_k$$

possibilities for the choice of the numbers V; subtracting now each of these numbers from K, we obtain at least one among the numbers $K-V$, which likewise satisfies the conditions for the V's; we denote this by V_1 and by V_2 we denote the number subtracted from K, that is, $V_1=K-V_2$. The existence of V_1 is seen from the following fact: if we subtract numbers of type "V" from K, then we get numbers belonging to the interval

$$\left(\frac{1}{3}M_k-a-(h-3)F;\ \frac{2}{3}M_k-a-(h-3)F\right);$$

now the pigeon hole principle can be applied since $a+(h-3)F$ is essentially smaller than $\frac{1}{4}\frac{1}{3}M_k$ which ensures the existence of V_1. It suffices now to remark that V_1 and V_2 satisfy the conditions for D_1y and D_2 because of (25) and $\frac{1}{3}M_k<V_1$; $V_2<\frac{2}{3}M_k$. If we write now instead of A^{I}_{k-1} always the set $\{A'_{k-1}\cup\{F_j\}\cup\{D_i\}\}$ (where $\{F_j\}$ and $\{D_i\}$ denotes the numbers F and D_1, D_2 which have been constructed in the preceding steps), then it was already mentioned that the construction of D_1, D_2 and F is always possible, provided that $0\leqq a\leqq b_k$. This completes the proof.

Remarks

1. Actually we have proved more than the assertion of the theorem, namely: from an arbitrary basis with order h whose density function has an order of $O(x^{\frac{1}{h}})$ we formed a maximal nonbase sequence of order h whose density function has likewise the order $O(x^{\frac{1}{h}})$. Given A_0 and B_0 in case of distinct basis sequences and choosing a suitably large m_1, one sees that the corresponding maximal nonbases $\mathcal{A}^*$; B^* will also be distinct from each other.

2. If we would choose to A_0 a basis sequence of order h whose density function has an order of $O(x^K)$ $\left(\text{where } \frac{1}{h}\leqq K\leqq\frac{1}{h-1}\right)$, then a slight modification of the construction would yield that there exists a maximal nonbase sequence of order h whose density function has an order of $O(x^K)$.

Open problem: Does there exist for an arbitrary K with $h^{-1}<K<1$ an asymptotic maximal nonbase sequence of order h whose density function has an order of $O(x^K)$?

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