Strictly semiprime ideals and nilpotency in near-rings with a defect of distributivity

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A. OSWALD, in chapter V of his *Dissertation* [6], considers strictly prime and strictly semiprime ideals and some of their characteristics in distributively generated (d. g.) near-rings. Also, A. OSWALD [6] considers the conditions for which every nil R-subgroup of a d.g. near-ring R is nilpotent.

The purpose of this paper is to extend certain results of OSWALD [6], to near-rings with a defect of distributivity. In this paper by "near-ring" is meant a zero-symmetric (left) near-ring. A set of generators of the near-ring R is a multiplicative subsemigroup $((S, .)$ of a semigroup $(R, .)$, whose elements generate $(R, +)$. For each $x, y \in R$ and $s \in S$ we can determine the element $d = d(x, y, s) \in R$ so that $(x+y)s = xs + ys + d$. A normal subgroup D of the group $(R, +)$ generated by these elements $d \in R$ is called a defect of distributivity of the near-ring R (see [3]). The near-ring R will be denoted by (R, S) when we wish to stress the set of generators S.

1. Strictly semiprime ideals

An ideal A of a near-ring R is a strictly semiprime ideal, if whenever B is an R-subgroup of R with $B^2 \subseteq A$ then $B \subseteq A$. A strictly semiprime near-ring is a near-ring in which (0) is a strictly semiprime ideal. A. OSWALD [6] characterizes strictly semiprime ideals in terms of a strictly semiprime system denoted as an ssp-system. A subset M of a near-ring R is an ssp-system if $a \in M$ implies that there exists $x \in R$ such that $axa \in M$.

Definition. We recall that the defect D of a near-ring (R, S) has a strictly semiprime property (ssp-property), if whenever A is an ideal of R with $xRx \subseteq A$ $(x \in R)$, then for each $d \in D$ there exists $r \in R$ such that $xr = d$.

The following three theorems generalize the results of A. Oswald ([6], Thms 1.2, 1.5, 2.2, pp. 56—59).

Theorem 1.1. *Let (R, S) be a near-ring whose defect has the ssp-property. The ideal A of R is strictly semiprime if and only if $x \in R$ and $xRx \subseteq A$ implies $x \in A$.*

PROOF. Let A be a strictly semiprime ideal of R and $xRx \subseteq A$ for $x \in R$. It needs to be shown that $x \in A$. Let B an R-subgroup of R generated by x.

We first need to show that B has consists of elements of the form

$$\sum_i (\pm x r_i + m_i x) \quad (r_i \in R,\ m_i\text{-integers}).$$

Evidently, these elements form an additive subgroup of $(R, +)$. If $\sum_i (\pm x r_i + m_i x) = b \in B$ then for all $s \in S$ we have $bs = \sum_i (\pm x r_i s + m_i x s) + d$, $(d \in D)$. Since for each $d \in D$ there exists $r \in R$ such that $xr = d$, we obtain $bs = \sum_i (\pm x r_i s + m_i x s) + xr + ox$. Thus $bs \in B$, i.e. B is an S-subgroup and hence an R-subgroup.

If $b, c \in B$, then $b = \sum_i (\pm x r_i + m_i x)$ and $c = \sum_j (\pm x r'_j + m'_j x)$, $(r_i, r'_j \in R, m_i, m'_j$ integers). On the other hand we have $m'_j x = \sum_k (\pm s_{jk})$ and $x r'_j = \sum_k (\pm s'_{jk})$, $(s_{jk}, s'_{jk} \in S)$. Thus,

$$bc = \sum_i (\pm x r_i + m_i x) \sum_j (\pm x r'_j + x'_j m)$$

$$bc = \sum_j [(\sum_i (\pm x r_i + m_i x))(\pm x r'_j) + (\sum_i (\pm x r_i + m_i x))(m'_j x)]$$

$$bc = \sum_j [\sum_k (\pm \sum_i (\pm x r_i + m_i x) s'_{jk}) + \sum_k (\pm \sum_i (\pm x r_i + m_i x) s_{jk})]$$

$$bc = \sum_j [\sum_k (\pm \sum_i (\pm x r_i s'_{jk} + m_i x s'_{jk})) + \sum_k (\pm \sum_i (\pm x r_i s_{jk} + m_i x s_{jk}))] + d \quad (d \in D).$$

By using the Proposition 2.2a of [3] we have

$$bc = x[\sum_j (\sum_k (\pm \sum_i (\pm r_i s'_{jk} + m_i s'_{jk}))) + \sum_k (\pm \sum_i (\pm r_i s_{jk} + m_i s_{jk}))] + d', \quad (d' \in D).$$

By hypothesis, for any $d' \in D$ there exists $r' \in R$ such that $xr' = d'$. Thus we obtain $bc \in xR$, i.e. $B^2 \subseteq xR$. Since $xR \subseteq A$ it follows that $B^2 \subseteq A$, i.e. $B \subseteq A$. Finally, we have that $x \in A$, because B is an R-subgroup generated by x.

The converse is immediate.

Corollary. *Let (R, S) be a near-ring whose defect has the ssp-property. A near-ring R is strictly semiprime if and only if $xRx = (0)$ implies $x = 0$.*

Theorem 1.2. *Let (R, S) be a near-ring whose defect has the ssp-property. A near-ring R is strictly semiprime if and only if R contains no nonzero nilpotent R-subgroup.*

PROOF. Let R be a strictly semiprime near-ring. Let us suppose that there exists a nilpotent R-subgroup P, i.e. $P^n = (0)$ for some integer n. If $x_1, \ldots, x_{n-1} \in P$ and $u = x_1 \ldots x_{n-1}$ then $uR \subseteq P$. Since $u \in P^{n-1}$ we have $uRu = (0)$. By the Corollary of Theorem 1.1 it follows that $u = 0$. Thus R has no nilpotent R-subgroups.

The converse is immediate.

Theorem 1.3. *Let (R, S) be a near-ring whose defect has the ssp-property. An ideal A of R is strictly semiprime if and only if $C(A) = \{x \in R;\ x \notin A\}$ is an ssp-system.*

PROOF. Let A be a strictly semiprime ideal of R and let $a\in C(A)$, i.e. $a\notin A$. We will show that there exists $x\in R$ such that $axa\in C(A)$, i.e. $axa\notin A$. Let us suppose the opposite: for some $x\in R$, $axa\in A$. From Theorem 1.1, we obtain $a\in A$. This is contradicts the above supposition. Thus $C(A)$ is an ssp-system.

Conversely, let $C(A)$ be an ssp-system and $xRx\subseteqq A$ $(x\in R)$. We will show that $x\in A$. Let us suppose that $x\notin A$, i.e. $x\in C(A)$. Thus, by definition of the ssp-system, there exists $r\in R$ such that $xrx\in C(A)$, hence $xrx\notin A$. This contradicts $xRx\subseteqq A$. Consequently, $xRx\subseteqq A$ implies $x\in A$. By Theorem 1.1, it follows that A is a strictly semiprime ideal of R.

J. C. BEIDLEMAN [1] defines a strictly prime near-ring as follows: R is strictly prime if, whenever C is an R-subgroup of R and B is a right ideal of R with $CB=(0)$, then either $C=(0)$ or $B=(0)$. A. Oswald ([6], Prop. 2.4, p. 60) obtains a corresponding symmetry in the definition of this notion for the class of d.g. near-rings. Thus, a d.g. near-ring R is strictly prime if and only if whenever A, B are R-subgroups of R with $AB=(0)$ then either $A=(0)$ or $B=(0)$. This characteristic of strictly prime d.g. near-rings becomes applicable to near-rings with a defect of distributivity in the light of the following proposition.

Proposition 1.4. *Let (R, S) be a near-ring with a defect such that, for any R-subgroups A and B of R with $AB=(0)$, a normal subgroup of $(R, +)$ generated by B contains a relative defect of the subset B with respect to R. Then, R is strictly prime if and only if $AB=(0)$ implies either $A=(0)$ or $B=(0)$.*

PROOF. Let R be strictly prime in the sense of Beidleman's definition and let us suppose that $AB=(0)$ for R-subgroups A and B. Denote by $\bar{B}$ a right ideal of R which is generated by an R-subgroup B. By Lemma 1.1 of [4] the elements of $\bar{B}$ are of the form

$$\bar{b} = \sum_i (r_i \pm b_i s_i + m_i b_i' - r_i) \quad (r_i\in R,\ s_i\in S,\ b_i,\ b_i'\in B,\ m_i\text{-integers}).$$

If $a\in A$ then $a\bar{b}=\sum_i (ar_i \pm ab_i s_i + m_i ab_i' - ar_i)=0$, because $ab_i s_i, ab_i'\in AB=(0)$. Thus, $A\bar{B}=(0)$ implies either $A=(0)$ or $\bar{B}=(0)$. But, if $\bar{B}=(0)$ then $B=(0)$. Therefore, $AB=(0)$ implies either $A=(0)$ or $B=(0)$.

The converse is immediate.

2. Nilpotent and nil R-subgroups

It is interesting to establish when a nil R-subgroup of a near-ring R is nilpotent. For the class of d.g. near-rings, A. OSWALD ([6], Thms 3.1, 3.2, 3.3, 3.4, pp. 62—65) has considered this problem. Several following theorems refer to near-rings with a defect and generalize the results of Oswald.

Theorem 2.1. *Let (R, S) be a near-ring whose defect has the ssp-property. If R is strictly semiprime and R has the maximum condition on right annihilators, then R contains no nonzero nil R-subgroups.*

PROOF. Let B be a nil R-subgroup of R and $b\in B$, where $b\neq 0$. Because R is strictly semiprime, we have $Rb\neq(0)$. Let us denote by $A(x)$ a right annihilator of $x\in R$. A set of right annihilators of nonzero elements from Rb has a maximal element, say $A(tb)$ $(tb\neq 0)$. For all $y\in R$ and $t\in R$ we obtain

$$(ytb)^m = ytb\cdot ytb \dots ytb = yt\cdot b_1 \dots b_1\cdot b = yt\cdot b_1^{m-1}\cdot b,$$

where $byt=b_1\in B$. Since the elements from B are nilpotent, there exists an integer $k>0$ such that $b_1^k=0$ and $b_1^{k-1}\neq 0$. Hence for $m=k+1$, $(ytb)^m=0$. If $x\in A(c)$ then $x\in A((yc)^{m-1})$, where $c=tb$. Therefore, $A(c)\subseteq A((yc)^{m-1})$. Since the right annihilator is maximal for $c=tb$, we have $A(c)=A((yc)^{m-1})$. On the other hand, $yc\in A((yc)^{m-1})$, i.e. $yc\in A(c)$. Thus, $cyc=0$ for each $y\in R$. Consequently $cRc=(0)$. From the Corollary of Theorem 1.1 it follows that $c=0$, i.e. $tb=0$. This is contradictory to the supposition that $tb\neq 0$ which follows from the supposition that $b\neq 0$. Therefore $b=0$, i. e. $B=(0)$.

Theorem 2.2. *Let (R, S) be a near-ring whose defect has the ssp-property. If a maximal nilpotent right ideal contains all the nilpotent R-subgroups of R and R has the maximum condition on right ideals, then every nil R-subgroup of R is nilpotent.*

PROOF. Let M be a maximal nilpotent right ideal of R. By Theorem 2.6 of [3], $\bar{R}=R/M$ is a near-rings with the defect $\bar{D}=\{d+M: d\in D\}$. Because of the natural homomorphism $h: R\to\bar{R}$, the defect $\bar{D}$ of $\bar{R}$ has an ssp-property. By Theorem 1.2 $\bar{R}$ is a strictly semiprime near-ring, because $\bar{R}$ has no nonzero nilpotent $\bar{R}$-subgroups. From Theorem 2.1 it follows that $\bar{R}$ has no nil $\bar{R}$-subgroups. Therefore, if A is a nil R-subgroup of R then $A\subseteq M$. Thus A is a nilpotent R-subgroup.

Theorem 2.3. *Let R be a near-ring whose defect D is contained in the commutator subgroup of $(R, +)$ and has the ssp-property. If $(R, +)$ is a solvable group and R has maximum condition on right ideals, then every nil R-subgroup of R is nilpotent.*

PROOF. Let R' be a commutator subgroup of $(R, +)$. By Theorem 3.7 of [3], R' is a nilpotent ideal of R. Thus, if M is a maximal nilpotent right ideal of R, then $R'\subseteq M$. Also, we have $D\subseteq R'$. On the other hand, by the Corollary of Theorem 2.6 of [3] and by Theorem 4.4.3 of [5], it follows that $\bar{R}=R/M$ is a ring. Thus, every $\bar{R}$-subgroup of $\bar{R}$ is a right ideal of $\bar{R}$. Let A be a nilpotent R-subgroup of R. Since the ring $\bar{R}$ has no nonzero nilpotent right ideals, it follows that $A\subseteq M$. Therefore, every nilpotent R-subgroup is contained in a maximal nilpotent right ideal of R. Because of Theorem 2.2, every nil R-subgroup of R is nilpotent.

Definition. Let (R, S) be a near-ring with the defect D. A subset $B\subseteq R$ is D-nilpotent in case there is an integer $k>0$ such that $x_1 \dots x_k\in D$ for every sequence $x_1, \dots, x_k$ in B.

Clearly, if a subset is nilpotent then it is D-nilpotent. By using the notion of D-nilpotency, the last two theorems obtain the following form.

Theorem. 2.2'. *Let (R, S) be a near-ring with the defect D. If a maximal D-nilpotent right ideal of R contains all D-nilpotent R-subgroups of R and R has the maximum condition on right ideals of R, then every nil R-subgroup of R is D-nilpotent.*

Proof. If M is a maximal D-nilpotent right ideal of R, then $D \subseteqq M$. By Corollary of Theorem 2.6 of [3] it follows that $\bar{R}=R/M$ is a d.g. near-ring, whereby $\bar{R}$ has no nonzero nilpotent $\bar{R}$-subgroups. From Theorem 1.5 of [6], $\bar{R}$ is a strictly semiprime d.g. near-ring. Also, $\bar{R}$ has the maximum condition on right annihilators. Thus by Theorem 3.1 of [6], (p. 62) $\bar{R}$ has no nonzero nil $\bar{R}$-subgroups. Consequently, if A is a nil R-subgroup of R then $A \subseteqq M$, i.e. A is a D-nilpotent R-subgroup of R.

Corollary. *Let (R, S) be a near-ring with the nilpotent defect D. If a maximal nilpotent right ideal of R contains all nilpotent R-subgroups of R and R has the maximum condition on right ideals, then every nil R-subgroup of R is nilpotent.*

Theorem 2.3′. *Let (R, S) be a near-ring with the defect D. If $(R, +)$ is a solvable group and R has the maximum condition on right ideals of R, then every nil R-subgroup of R is D-nilpotent.*

Proof. If M is a maximal D-nilpotent ideal of R, then $D \subseteqq M$. By Corollary of Theorem 2.6 of [3], $\bar{R}=R/M$ is a d.g. near-ring. Therefore, by Theorem 3.4 of ([6], p. 64) it follows that every nil $\bar{R}$-subgroup of $\bar{R}$ is nilpotent. Thus, every nil R-subgroup of R is D-nilpotent.

The previous theorem and the following corollary generalize the result of Oswald ([6], Theorem 3.4, p. 64).

Corollary. *Let (R, S) be a near-ring with a nilpotent defect. If $(R, +)$ is a solvable group and R has the maximum condition on right ideals of R, then every nil R-subgroup of R is nilpotent.*

We recall that a near-ring R with the defect D is D-distributive, if for all $x, y, z \in R$ there exists $d \in D$ such that $(x+y)z=xz+yz+d$, (see [3]).

Theorem 2.4. *Let R be a D-distributive near-ring, where D is a defect of R. If R has the maximum condition on right ideals of R, then every nil R-subgroup of R is D-nilpotent.*

Proof. If M is a maximal D-nilpotent ideal of R then $D \subseteqq M$. By the Corollary of Theorem 2.6 of [3] it follows that $\bar{R}=R/M$ is a distributive near-ring. Because of Proposition 3.3 of [6], (p. 64) every nil $\bar{R}$-subgroup of $\bar{R}$ is nilpotent. Thus, every nil R-subgroup of R is D-nilpotent.

Corollary. *Let R be a D-distributive near-ring, where D is a nilpotent defect of R. If R has the maximum condition on right ideals of R, then every nil R-subgroup of R is nilpotent.*

An R-subgroup B of a near-ring R is called minimal nonnilpotent if B is nonnilpotent and every proper R-subgroup is nilpotent. The following theorem gives a necessary and sufficient condition that a nonnilpotent R-subgroup be a minimal nonnilpotent R-subgroup and generalizes Theorem 2.8 of Beidleman [2].

Theorem 2.5. *Let A be a nonnilpotent R-subgroup of the near-ring (R, S) with a defect, whereby a normal subgroup of $(A, +)$ generated by any minimal nonnilpotent R-subgroup $B \subset A$, contains a relative defect of subset B with respect to A.*

Let every minimal nonnilpotent R-subgroup of R be a term of a normal series for $(R, +)$ and let R satisfy the descending chain condition on R-subgroups. A nonnilpotent R-subgroup A is minimal nonnilpotent if and only if every proper right ideal of A is a nilpotent R-subgroup of R.

PROOF. Let A be a nonnilpotent R-subgroup of R and every proper right ideal of A be a nilpotent R-subgroup. It needs to be proved that A is a minimal nonnilpotent R-subgroup.

Let us assume that B is a minimal nonnilpotent R-subgroup such that B is a proper R-subgroup of A. By Theorem 3.51a of [7], B contains a left identity e with $eR=B$. Since B is a term of a normal series for $(R, +)$, it follows that there exists a proper normal subgroup $\bar{B}$ of group $(A, +)$ which is generated by B. Thus, the elements of $\bar{B}$ have the form

$$\bar{b} = \sum_i (a_i \pm es_i + m_i e - a_i), \quad (a_i \in A, \; e \in B, \; s_i \in S, \; m_i\text{-integers}).$$

From Lemma 1.1 of [4], it follows that $\bar{B}$ is a right ideal of A. By hypothesis, $\bar{B}$ is a nilpotent R-subgroup of R. But $\bar{B}$ contains the R-subgroup B and this is contradictory to the supposition that B is a minimal nonnilpotent R-subgroup. Therefore $B=A$, i.e. A is a minimal nonnilpotent R-subgroup.

The converse is immediate.

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