Hausdorff topologies for the multiplier extensions of admissible vector modules

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1. Notation and introduction

Let $\mathscr{B}$ be an admissible Hausdorff vector-topological (resp. locally convex) $\mathscr{A}$-vector module [14], and denote $\mathfrak{M}=\mathfrak{M}(\mathscr{A}, \mathscr{B})$ its multiplier extension which is an $\mathfrak{N}=\mathfrak{N}(\mathscr{A}, \mathscr{B})$-vector module [12].

Let $\mathscr{I}$ be the family of all ideals I in $\mathscr{A}$ such that I is not a divisor of zero in $\mathscr{B}$, and for $I\in\mathscr{I}$, define

$$\mathfrak{M}_I = \{F\in\mathfrak{M}\colon\ I\subset D_F\} \quad \text{and} \quad \mathfrak{N}_I=\{\Phi\in\mathfrak{N}\colon\ I\subset \Phi^{-1}(\mathscr{A})\}.$$

Moreover, equip the subspaces $\mathfrak{M}_I$ and $\mathfrak{N}_I$ with the corresponding topologies of pointwise convergences on I as in [14]. Then, $\mathfrak{M}_I$ and $\mathfrak{N}_I$ become Hausdorff topological vector (resp. locally convex) spaces and moreover, we have

$$\mathfrak{M} = \bigcup_{I\in\mathscr{I}} \mathfrak{M}_I \quad \text{and} \quad \mathfrak{N} = \bigcup_{I\in\mathscr{I}} \mathfrak{N}_I.$$

In [14], we have considered the finest vector (resp. locally convex) topologies on $\mathfrak{M}$ and $\mathfrak{N}$ for which the identity mappings of the spaces $\mathfrak{M}_I$ and $\mathfrak{N}_I$ into $\mathfrak{M}$ and $\mathfrak{N}$, respectively, are continuous. These topologies make $\mathfrak{M}$ into an admissible vectortopological (resp. locally convex) $\mathfrak{N}$-vector module, and are, to some extent, compatible with the Mikusiński-type convergences too [13]. However, in general, they are not T_1, and even their convergences can not be described explicitly.

In [13], we have considered the finest topologies on $\mathfrak{M}$ and $\mathfrak{N}$ for which the identity mappings of the spaces $\mathfrak{M}_I$ and $\mathfrak{N}_I$ into $\mathfrak{M}$ and $\mathfrak{N}$, respectively, are continuous. These topologies make $\mathfrak{M}$ into an admissible T_1 semitopological $\mathfrak{N}$-vector module, and are compatible with the Mikusiński-type convergences too. However, in general, they are not Hausdorff, and even their convergences can not be described explicitly.

Now, we shall consider the coarsest topologies on $\mathfrak{M}$ and $\mathfrak{N}$ for which the identity mappings of the spaces $\mathfrak{M}_I$ and $\mathfrak{N}_I$ into $\mathfrak{M}$ and $\mathfrak{N}$, respectively, are open. These topologies are finer than the former ones. Moreover, they are Hausdorff, and have easily describable convergences. However, they are not, in general, compatible with the algebraic structures.

The present investigations were motivated by the paper [1] of J. M. ANTHONY, and some of the results of [1] are extended here to a more general setting.

2. Hausdorff topologies for $\mathfrak{M}$ and $\mathfrak{N}$

Definition 2.1. Equip $\mathfrak{M}$ and $\mathfrak{N}$ with the coarsest topologies for which the identity mappings of the spaces $\mathfrak{M}_I$ and $\mathfrak{N}_I$ into $\mathfrak{M}$ and $\mathfrak{N}$, respectively, are open for all $I\in\mathscr{I}$ [16].

Notation 2.2. If X is a topological space, then its topology will be denoted by $\mathscr{T}_X$. Moreover, if $Y\subset X$, then $\mathscr{T}_X|Y$ will denote the restriction of $\mathscr{T}_X$ to Y.

Theorem 2.3. (i) *The topologies* $\mathscr{T}_{\mathfrak{M}}$ *and* $\mathscr{T}_{\mathfrak{N}}$ *are Hausdorff.*
(ii) *We have*

$$\mathscr{T}_{\mathfrak{M}_I}\subset\mathscr{T}_{\mathfrak{M}}|\mathfrak{M}_I,\quad \mathscr{T}_{\mathfrak{N}_I}\subset\mathscr{T}_{\mathfrak{N}}|\mathfrak{N}_I,$$

for all $I\in\mathscr{I}$, *and*

$$\mathscr{T}_{\mathfrak{M}_{\mathscr{A}}}=\mathscr{T}_{\mathfrak{M}}|\mathfrak{M}_{\mathscr{A}},\quad \mathscr{T}_{\mathfrak{N}_{\mathscr{A}}}=\mathscr{T}_{\mathfrak{N}}|\mathfrak{N}_{\mathscr{A}}.$$

(iii) *The topologies* $\mathscr{T}_{\mathfrak{M}}$ *and* $\mathscr{T}_{\mathfrak{N}}$ *are finer than the finest topologies on* $\mathfrak{M}$ *and* $\mathfrak{N}$, *respectively, for which the identity mappings of the spaces* $\mathfrak{M}_I$ *and* $\mathfrak{N}_I$ *into* $\mathfrak{M}$ *and* $\mathfrak{N}$, *respectively, are continuous for all* $I\in\mathscr{I}$.

PROOF. This follows immediately from Theorems 2.9, 2.3—2.5 and 1.4 of [16], and from (ii) in Remarks 4.3 and (iii) in Theorem 4.2 of [14].

Remark 2.4. Note that

$$\mathfrak{M}_{\mathscr{A}}=\mathrm{Hom}_{\mathscr{A}}(\mathscr{A},\mathscr{B})\quad\text{and}\quad \mathfrak{N}_{\mathscr{A}}=\mathrm{Hom}_{\mathscr{A}}(\mathscr{A},\mathscr{A}).$$

Corollary 2.5. *We have*

$$\mathscr{T}_{\mathfrak{M}}|\mathscr{B}\subset\mathscr{T}_{\mathscr{B}},\quad \mathscr{T}_{\mathfrak{M}}|\mathbf{K}\subset\mathscr{T}_{\mathbf{K}}\quad\textit{and}\quad \mathscr{T}_{\mathfrak{N}}|\mathscr{A}\subset\mathscr{T}_{\mathscr{A}},\quad \mathscr{T}_{\mathfrak{N}}|\mathbf{K}\subset\mathscr{T}_{\mathbf{K}}.$$

PROOF. This can be derived easily from the last two assertions in (ii) of Theorem 2.3 by using (i) in Theorem 4.2 of [14].

Remark 2.6. The topologies $\mathscr{T}_{\mathfrak{N}}$ and $\mathscr{T}_{\mathfrak{M}}|\mathfrak{N}$ seem to be incomparable.

Corollary 2.7. *If* φ *is a mapping of* $\mathfrak{M}$ *(resp. of* $\mathfrak{N}$*) into a topological space such that* $\varphi|\mathfrak{M}_I$ *(resp.* $\varphi|\mathfrak{N}_I$*) is continuous for all* $I\in\mathscr{I}$, *then* φ *is continuous.*

PROOF. This is an immediate consequence of (iii) in Theorem 2.3.

Theorem 2.8. (i) *For a net* (F_α) *in* $\mathfrak{M}$ *and an* $F\in\mathfrak{M}$, *we have* $\lim_\alpha F_\alpha=F$ *in* $\mathfrak{M}$ *if and only if there exists* α_0 *such that* $D_F\subset\bigcap_{\alpha\geqq\alpha_0}D_{F_\alpha}$ *and* $\lim_{\alpha\geqq\alpha_0}F_\alpha(\varphi)=F(\varphi)$ *in* $\mathscr{B}$ *for all* $\varphi\in D_F$.

(ii) *For a net* (Φ_α) *in* $\mathfrak{N}$ *and an* $\Phi\in\mathfrak{N}$, *we have* $\lim_\alpha\Phi_\alpha=\Phi$ *in* $\mathfrak{N}$ *if and only if there exists* α_0 *such that* $\Phi^{-1}(\mathscr{A})\subset\bigcap_{\alpha\geqq\alpha_0}\Phi_\alpha^{-1}(\mathscr{A})$ *and* $\lim_{\alpha\geqq\alpha_0}\Phi_\alpha(\varphi)=\Phi(\varphi)$ *in* $\mathscr{A}$ *for all* $\varphi\in\Phi^{-1}(\mathscr{A})$.

PROOF. This follows immediately from Theorem 2.10 of [16] and (iii) in Theorem 4.2 of [14], since the topologies of $\mathfrak{M}_I$ and $\mathfrak{N}_I$ given in Definition 4.1 of [14] have the pointwise convergences on I as convergences.

Remark 2.9. The above convergences are very unnatural since they are very much stronger than the Mikusiński-type convergences [13]. However, for instance, they still suit well to Proposition 2.2 and Remark 2.3 of [5].

Theorem 2.10. (i) *If* $\mathscr{C}$ *is a compact subset of* $\mathfrak{M}$, *then* $I=\bigcap_{F\in\mathscr{C}} D_F$ *is not a divisor of zero in* $\mathscr{B}$, *and* $\mathscr{C}$ *is a compact subset of* $\mathfrak{M}_I$.

(ii) *If* $\mathscr{C}$ *is a compact subset of* $\mathfrak{N}$, *then* $I=\bigcap_{\Phi\in\mathscr{C}} \Phi^{-1}(\mathscr{A})$ *is not a divisor or zero in* $\mathscr{B}$, *and* $\mathscr{C}$ *is a compact subset of* $\mathfrak{N}_I$.

PROOF. This follows at once from Theorem 2.11 of [16].

3. The question of compatibility in $\mathfrak{M}$ and $\mathfrak{N}$

Remark 3.1. In general, the algebraic and topological structures on $\mathfrak{M}$ and $\mathfrak{N}$ are not compatible. However, the question that at which points the algebraic operations are continuous or separately continuous may be investigated.

Theorem 3.2. (i) *If* $F, G\in\mathfrak{M}$ *such that* $D_{F+G}=D_F\cap D_G$, *then the addition* $+:\mathfrak{M}\times\mathfrak{M}\to\mathfrak{M}$ *is continuous at the point* (F, G).

(ii) *If* $0\neq\lambda\in\mathbf{K}$ *and* $F\in\mathfrak{M}$, *then the scalar multiplication* $\cdot:\mathbf{K}\times\mathfrak{M}\to\mathfrak{M}$ *is continuous at the point* (λ, F).

(iii) *If* $F\in\mathfrak{M}$ *and* $\Phi\in\mathfrak{N}$ *such that* $D_{F*\Phi}=D_F*\Phi^{-1}(\mathscr{A})$, *then the multiplication* $*:\mathfrak{M}\times\mathfrak{N}\to\mathfrak{M}$ *is separately continuous at the point* (F, Φ).

PROOF. (i) and (ii) follow immediately from Theorem 3.2 of [16], namely if F and G are as in (i), then we have

$$\mathfrak{M}_{D_F},\quad \mathfrak{M}_{D_G}\subset\mathfrak{M}_{D_F\cap D_G}=\mathfrak{M}_{D_{F+G}}.$$

To prove (iii), let F and Φ be as in (iii), and let W be an open subset of $\mathfrak{M}$ such that $F*\Phi\in W$. By Theorem 1.2 of [16], we may suppose that W is an open subset of $\mathfrak{M}_I$ for some $I\in\mathscr{I}$. Then, by (iii) in Theorem 4.2 of [14], $W\cap\mathfrak{M}_{D_{F*\Phi}}$ is an open subset of $\mathfrak{M}_{D_{F*\Phi}}$. Thus, by (ii) in Theorem 4.2 of [14], there are open subsets U and V of $\mathfrak{M}_{D_F}$ and $\mathfrak{N}_{\Phi^{-1}(\mathscr{A})}$, respectively, such that $F\in U$, $\Phi\in V$, $U*\Phi\subset W$ and $F*V\subset W$. Now, since U and V are also open in $\mathfrak{M}$ and $\mathfrak{N}$, respectively, the proof of (iii) is complete.

Theorem 3.3. (i) *If* $\Phi, \Psi\in\mathfrak{N}$ *such that* $(\Phi+\Psi)^{-1}(\mathscr{A})=\Phi^{-1}(\mathscr{A})\cap\Psi^{-1}(\mathscr{A})$, *then the addition* $+:\mathfrak{N}\times\mathfrak{N}\to\mathfrak{N}$ *is continuous at the point* (Φ, Ψ).

(ii) *If* $0\neq\lambda\in\mathbf{K}$ *and* $\Phi\in\mathfrak{N}$, *then the scalar multiplication* $\cdot:\mathbf{K}\times\mathfrak{N}\to\mathfrak{N}$ *is continuous at the point* (λ, Φ).

(iii) *If* $\Phi, \Psi\in\mathfrak{N}$ *such that* $(\Phi*\Psi)^{-1}(\mathscr{A})=\Phi^{-1}(\mathscr{A})*\Psi^{-1}(\mathscr{A})$, *then the multiplication* $*:\mathfrak{N}\times\mathfrak{N}\to\mathfrak{N}$ *is separately continuous at the point* (Φ, Ψ).

PROOF. The proof parallels that of Theorem 3.2.

Remark 3.4. The separability, restriction, convergence and compatibility properties of the topologies defined in [13], [14] and in the present paper suggest some mixed procedures in topologizing $\mathfrak{M}$ and $\mathfrak{N}$. The following is an example of this type.

Let $\mathcal{T}'_{\mathfrak{M}}$ (resp. $\mathcal{T}'_{\mathfrak{N}}$) be the finest topology or vector topology on $\mathfrak{M}$ (resp. on $\mathfrak{N}$) for which the identity mapping of the space $\mathfrak{M}_I$ (resp. of $\mathfrak{N}_I$) into $\mathfrak{M}$ (resp. into $\mathfrak{N}$) is continuous for all $I\in\mathcal{I}$. Moreover, let $\mathcal{T}''_{\mathfrak{M}}=\mathcal{T}_{\mathfrak{M}_{\mathcal{A}}}\cup\{\mathfrak{M}\}$ and $\mathcal{T}''_{\mathfrak{N}}=\mathcal{T}_{\mathfrak{N}_{\mathcal{A}}}\cup\{\mathfrak{N}\}$. Then, we may consider the topologies

$$\mathcal{T}^{\#}_{\mathfrak{M}}=\sup\{\mathcal{T}'_{\mathfrak{M}},\mathcal{T}''_{\mathfrak{M}}\}\quad\text{and}\quad\mathcal{T}^{\#}_{\mathfrak{N}}=\sup\{\mathcal{T}'_{\mathfrak{N}},\mathcal{T}''_{\mathfrak{N}}\}$$

on $\mathfrak{M}$ and $\mathfrak{N}$, respectively. These topologies also have some advantageous and some disadvantageous properties.

4. An important particular case

Definition 4.1. Let $\mathcal{S}$ be the family of all elements of $\mathcal{A}$ which are not divisors of zero in $\mathcal{B}$, and suppose that $\mathcal{S}\cap I\neq\emptyset$ for all $I\in\mathcal{I}$.

Remark 4.2. In this case, we have

$$\mathfrak{M}=\bigcup_{\varphi\in\mathcal{S}}\mathfrak{M}_\varphi\quad\text{and}\quad\mathfrak{N}=\bigcup_{\varphi\in\mathcal{S}}\mathfrak{N}_\varphi,$$

and the subspaces

$$\mathfrak{M}_\varphi=\mathfrak{M}_{I_\varphi}\quad\text{and}\quad\mathfrak{N}_\varphi=\mathfrak{N}_{I_\varphi},$$

where I_φ denotes the ideal of $\mathcal{A}$ generated by $\varphi\in\mathcal{S}$, are algebraically and topologically isomorphic to $\mathcal{B}$ and $\mathcal{A}$, respectively.

Definition 4.3. Denote $\mathcal{T}^*_{\mathfrak{M}}$ and $\mathcal{T}^*_{\mathfrak{N}}$ the coarsest topologies on $\mathfrak{M}$ and $\mathfrak{N}$, respectively, for which the identity mappings of the spaces $\mathfrak{M}_\varphi$ and $\mathfrak{N}_\varphi$ into $\mathfrak{M}$ and $\mathfrak{N}$, respectively, are open for all $\varphi\in\mathcal{S}$.

Remark 4.4. It is clear that we have

$$\mathcal{T}^*_{\mathfrak{M}}\subset\mathcal{T}_{\mathfrak{M}}\quad\text{and}\quad\mathcal{T}^*_{\mathfrak{N}}\subset\mathcal{T}_{\mathfrak{N}},$$

where $\mathcal{T}_{\mathfrak{M}}$ and $\mathcal{T}_{\mathfrak{N}}$ denotes the topologies of $\mathfrak{M}$ and $\mathfrak{N}$ given in Definition 2.1. However, in some cases the above topologies may coincide.

Theorem 4.5. *Suppose that each $I\in\mathcal{I}$ is generated by a finite subset of $\mathcal{S}$. Then, we have $\mathcal{T}^*_{\mathfrak{M}}=\mathcal{T}_{\mathfrak{M}}$ and $\mathcal{T}^*_{\mathfrak{N}}=\mathcal{T}_{\mathfrak{N}}$.*

PROOF. If $I\in\mathcal{I}$, then by the assumption, there exists $\{\varphi_k\}_{k=1}^n\subset\mathcal{S}$ such that $I=\sum_{k=1}^n(\mathbf{K}\varphi_k+\mathcal{A}*\varphi_k)$. Hence, it is clear that

$$\mathfrak{M}_I=\bigcap_{k=1}^n\mathfrak{M}_{\varphi_k},$$

and the topology of $\mathfrak{M}_I$ is the coarsest topology on $\mathfrak{M}_i$ for which the identity mapping of $\mathfrak{M}_I$ into $\mathfrak{M}_{\varphi_k}$ is continuous for all $k=1,2,\ldots,n$. Thus, the family

$$\{V\cap\mathfrak{M}_I\colon V\in\mathcal{T}_{\mathfrak{M}_{\varphi_k}}\text{ for some }k=1,2,\ldots,n\}$$

is a subbase of $\mathscr{T}_{\mathfrak{M}_I}$. Hence, by Theorem 1.2 of [16], it is clear that $\mathscr{T}_{\mathfrak{M}_I}\subset\mathscr{T}^*_{\mathfrak{M}}$ since $\mathfrak{M}_I=\bigcap_{k=1}^{n}\mathfrak{M}_{\varphi_k}\in\mathscr{T}^*_{\mathfrak{M}}$. Thus, again by Theorem 1.2 of [16], we have $\mathscr{T}_{\mathfrak{M}}\subset\mathscr{T}^*_{\mathfrak{M}}$.

The corresponding assertion for $\mathfrak{N}$ can be proved quite similarly.

Remark 4.6. The condition that each $I\in\mathscr{I}$ is generated by a finite subset of $\mathscr{S}$ surely fails to hold in most of the applications.

Remark 4.7. The topologies $\mathscr{T}^*_{\mathfrak{M}}$ and $\mathscr{T}^*_{\mathfrak{N}}$ seem to be more natural than $\mathscr{T}_{\mathfrak{M}}$ and $\mathscr{T}_{\mathfrak{N}}$, and they were first used by J. M. Anthony [1] in the Mikusiński operational calculus.

However, in the following assertions, it is no matter that which of the above topologies is assigned to $\mathfrak{M}$ and $\mathfrak{N}$, respectively.

Theorem 4.8. (i) *For each* $\varphi\in\mathscr{S}$, *the mapping*

$$f\to\frac{f}{\varphi}\quad(f\in\mathscr{B})$$

is an open mapping of $\mathscr{B}$ *into* $\mathfrak{M}$.

(ii) *For each* $\varphi\in\mathscr{S}$, *the mapping*

$$\psi\to\frac{\psi}{\varphi}\quad(\psi\in\mathscr{A})$$

is an open mapping of $\mathscr{A}$ *into* $\mathfrak{N}$.

Proof. If $\varphi\in\mathscr{S}$ and $V\in\mathscr{T}_{\mathscr{B}}$, then since the mapping $f\to f/\varphi$ $(f\in\mathscr{B})$ is a topological isomorphism of $\mathscr{B}$ onto $\mathfrak{M}_\varphi$, we have $V/\varphi=\{f/\varphi: f\in V\}\in\mathscr{T}_{\mathfrak{M}_\varphi}$, and thus $V/\varphi\in\mathscr{T}^*_{\mathfrak{M}}$. This proves (i).

The assertion (ii) can be proved quite similarly.

Corollary 4.9. *The quotient mappings* $(\varphi, f)\to f/\varphi$ *from* $\mathscr{S}\times\mathscr{B}$ *onto* $\mathfrak{M}$, *and* $(\varphi,\psi)\to\psi/\varphi$ *from* $\mathscr{S}\times\mathscr{A}$ *onto* $\mathfrak{N}$ *are open.*

Proof. If $U\subset\mathscr{S}$ and $V\in\mathscr{T}_{\mathscr{B}}$, then

$$\frac{V}{U}=\left\{\frac{f}{\varphi}\colon (\varphi, f)\in U\times V\right\}=\bigcup\left\{\frac{V}{\varphi}\colon \varphi\in U\right\}\in\mathscr{T}^*_{\mathfrak{M}}$$

by (i) in Theorem 4.8, and hence the first assertion is quite obvious.

The second assertion can be proved in the same way.

Remark 4.10. It is also worth mentioning that for a net (F_α) in $\mathfrak{M}$ and an $F\in\mathfrak{M}$, by Theorem 2.10 of [16], we have $\lim_\alpha F_\alpha=F$ in $(\mathfrak{M}, \mathscr{T}^*_{\mathfrak{M}})$ if and only if $D_F\subset\underline{\lim}_\alpha D_{F_\alpha}=\bigcup_\alpha\bigcap_{\beta\geqq\alpha}D_{F_\beta}$ and $\lim_\alpha F_\alpha(\varphi)=F(\varphi)$ in $\mathscr{B}$ for all $\varphi\in D_F$.

To formulate the corresponding assertion for $\mathfrak{N}$ is left to the reader.

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