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A notion of compactness in topological categories

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Abstract. In this paper, the notion of compactness as well as the notion of compact pairs for an arbitrary topological category is introduced. Furthermore, various generalizations of Tychonoff (completely regular T_1-objects) objects for an arbitrary topological category is given and the relationships among these various forms are investigated. Finally, closed and proper (perfect) morphisms are defined in an arbitrary topological category and some well-known results in the category of topological spaces are proved.

1. Introduction

The notion of a compact topological space is an abstraction of certain important properties of the set of real numbers. The classic theorem of Heine-Borel asserts that every open cover of a closed and bounded subset of the space of real numbers has a finite subcover. This theorem has an extraordinarily profound consequences and, like most good theorems, its conclusion has become a definition.

There is a criterion for a topological space to be compact, a criterion that is formulated in terms of closed sets. It does not look very natural or very useful at first glance, but it in fact proves to be useful on a number of occasions. For example, it is useful when proving the uncountability of the set of real numbers [20] p. 176 and when proving the Tychonoff theorem and the Baire category theorem [20] p. 232 and 294.

One of the well-behaved classes of topological spaces to deal with in mathematics is the compact Hausdorff spaces. Such spaces have many useful properties, which one can use in proving theorems and making constructions and the like.

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One of the other important classes of topological spaces to deal with is Tychonoff spaces, i.e. completely regular T_1 spaces which are identical with the class of all subspaces of compact Hausdorff spaces [20] p. 237. In particular, there are nonconstant continuous functions of every Tychonoff space into real numbers with the usual topology. Tychonoff spaces are, moreover, the most general topological spaces that can be guaranteed to have this property.

Let a and b denote infinite cardinal numbers with $a \leq b$. The notion of $[a, b]$-compact spaces is introduced and investigated in Smirnov [22] and Vaughan [23]. Also Vaughan introduced in [24] the notion of Ω-compact space and gave a characterization of these spaces in terms of closed projections.

The notions of compact class $F(A)$ and discrete class $G(A)$ relative to a fixed class A of (Hausdorff) spaces as well as a relative notion of properness (perfectness) is introduced and investigated in Giuli [11]. The notion of S-perfect maps for an arbitrary epireflective subcategory S of Tychonoff spaces is introduced in Hager [12]. The classes of H-closed spaces (see Ishii [16]) and w-compact spaces (see Joseph [15]) are compact classes with respect to suitable closure operators different from the ordinary closure.

A compactness notion in categories of convergence spaces, which depends on a closure operator, is introduced and investigated in [8] (see also [14] and [18]). Compactness and properness (perfectness) in a transportable construct E, depending on a closure operator C and on a class (full subcategory) A of E-objects ((C, A)-compactness and (C, A)-perfectness), is introduced and investigated in [9].

In [1] and [4], there are various ways to define "Hausdorff" objects in an arbitrary topological category. In [4], the notion of closed subobjects of an object in an arbitrary topological category is given. By using this idea and the closed set formulation of compactness, we define the notion of compactness for an arbitrary topological category over Set, the category of sets.

Furthermore, we have shown the following:

1. To define closed and proper morphisms in an arbitrary topological category over Set and prove some results concern these concepts.
2. To define various forms of Tychonoff (completely regular T_1, i.e. ($T_{7/2}$) objects in topological categories and investigate the relationships among them.
3. To introduce the notion of compact pairs and generalize some results of Giuli [11].

4. To prove that some well-known results, in general topology, that are related to compactness and separation properties are not valid, in general, in an arbitrary topological category over Set.

Let B be a set and p a point in B. The infinite wedge product, $\bigvee_p^\infty B$ is formed by taking countably many distinct copies of B and identifying them at the point p. Define $A_p^\infty : \bigvee_p^\infty B \to B^\infty = B \times B \times \dots$ the countable cartesian product of B, by $A_p^\infty(x_i) = (p, p, \dots, x_i, p, \dots)$ where x_i is in the i^{th} component of the infinite wedge and x_i is in the i^{th} place in $(p, p, \dots x_i, p, \dots)$ Baran [4]. Define $\nabla_p : \bigvee_p^\infty B \to B$ by $\nabla_p(x_i) = x$ for all i. Let $B^2 \bigvee_\Delta B^2$ be the wedge product of B^2, i.e. two disjoint copies of B^2 identified along the diagonal, Δ. A point (x, y) in $B^2 \bigvee_\Delta B^2$ will be denoted by $(x, y)_1$ (resp. $(x, y)_2$) if (x, y) is in the first (resp. second) component of $B^2 \bigvee_\Delta B^2$ Baran [1]. Recall the principal axis map $A : B^2 \bigvee_\Delta B^2 \to B^3$ is given by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map $S : B^2 \bigvee_\Delta B^2 \to B^3$ is given $S(x, y)_1 = (x, y, y)$and $S(x, y)_2 = (x, x, y)$ and, the fold map, $\nabla : B^2 \bigvee_\Delta B^2 \to B^2$ is given by $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$ Baran [1].

Let E be a category and Set be the category of sets. Let $U : E \to \text{Set}$ be a topological functor [13], $f : X \to Y$ a morphism in E, and 1 the terminal object of E. Let $q : U(X) = B \to B/F$ be the identification map identifying the nonempty subset F of B to a point $*$ Baran [4]. Let p be a point in $U(X) = B$.

1.1. *Definitions.*

1. p is closed iff the initial lift of the U-source

$$\left\{ A_p^\infty : \bigvee_p^\infty B \to U(X^\infty) = B^\infty \quad \text{and} \quad \nabla_p : \bigvee_p^\infty B \to U(DB) = B \right\}$$

is discrete, where DB is the discrete structure on B [4] p. 386.

2. $F \subset X$ is closed iff $*$ is closed in X/F [4] p. 386.
3. $F \subset X$ is strongly closed iff X/F is T_1 at $*$ [4] p. 386.
4. X is $\text{Pre}\, T_2'$ iff the initial lift of the U-source

$$\left\{ S : B^2 \bigvee_\Delta B^2 \to U(X^3) = B^3 \right\}$$

and the final lift of the U-sink

$$\left\{ i_1, i_2 : U(X^2) = B^2 \to B^2 \bigvee_\Delta B^2 \right\}$$

agree [1] p. 338.

5. X is T_2' iff X is T_0' and Pre T_2' [1] p. 338.
6. X is ΔT_2 iff the diagonal, Δ, is closed in X^2 [4] p. 387.
7. X is ST_2 iff Δ is strongly closed in X^2 [4] p. 387.
8. X is ST_3' iff X is T_1 and X/F is Pre T_2' for all strongly closed $F \neq \emptyset$ in $U(X)$ [1] p. 340.
9. X is ST_4' iff X is T_1 and X/F is ST_3' for all strongly closed $F \neq \emptyset$ in $U(X)$ [1] p. 340.

1.2. *Remark.* (1) For the category TOP of topological spaces, all of $\Delta T_2, T_2'$, and ST_2 reduce to the usual T_2, the Hausdorff condition. ST_3' and ST_4' reduce to the usual T_3 (regular) and T_4 (normal), respectively [1] p. 339.

(2) In TOP if X is T_1, then the notion of closedness and strongly closedness are equivalent. Further, the notion of closedness reduces to the usual one.

(3) If $U : E \to B$ is topological, where B is a topos with infinite products and infinite wedge products, then Definition 1.1 makes sense.

Let ConLFCO (see [4] or [19]), Prord (see [3], [19] or [21]), and PBorn(Born) (see [4] or [19]) be the categories of Constant Local Filter Convergence Spaces, Preordered Spaces, and Prebornological (Bornological) Spaces, respectively.

2. Compact objects

In this section, we introduce (strongly) closed and (strongly) proper morphisms, and the notion of (strongly) compact objects in topological categories over Set. Also, we give the characterizations of these notions for the above mentioned categories, and we use this to show that some well-known important results in general topology involving compactness and separation properties are not valid, in general, in an arbitrary topological category over Set. Further, we generalize some results to a topological category and prove them.

Let $U : E \to$ Set be topological, 1 be the terminal object, and $f : X \to Y$ be a morphism in E.

2.1. *Definitions.*

1. f is said to be closed iff the image of each closed subobject of X is a closed subobject of Y (subobject means an initial mono lift).
2. f is said to be strongly closed iff the image of each strongly closed subobject of X is a strongly closed subobject of Y.

3. f is said to be proper (or perfect) iff the morphism $f \times \mathrm{id} : X \times Z \to Y \times Z$ is closed, for every object Z in E where id is the identity morphism of Z onto itself.
4. f is said to be strongly proper iff the morphism $f \times \mathrm{id} : X \times Z \to Y \times Z$ is strongly closed, for every object Z in E where id is the identity morphism of Z onto itself.
5. X is compact iff the morphism $X \to 1$ is proper.
6. X is strongly compact iff the morphism $X \to 1$ is strongly proper.

2.2. *Remarks.* (1) For the category TOP of topological spaces, each of closed and proper morphisms, and compactness reduces to the usual ones [7] p. 97 and 103.

(2) In TOP if X is T_1, then the notion of closedness and strongly closedness are equivalent and consequently proper and strongly proper maps agree.

(3) If $U : E \to B$ is topological, where B is a topos with infinite products and infinite wedge products, then Definition 2.1 make sense.

(4) Since the notions of closedness and strongly closedness are, in general, different notions (see [4] p. 393), it follows that the notion of compactness and strongly compactness are different, in general.

(5) For an arbitrary topological category, in general, it is not known whether the used closure in 1.1 is a closure operator in the sense of Dikranjan and Giuli [10] or not. However, it is shown, in [6], that the notions of closedness and strongly closedness that are defined in 1.1 form appropriate closure operators in the sense of Dikranjan and Giuli [10] in the case the category is one of the categories of convergence spaces ([4] or [21]), ConLFCO, Born, and Limit spaces.

Let $U : E \to \mathrm{Set}$ be topological and 1 be the terminal object in E. Then $1 \times X$ is isomorphic to X for all objects X in E.

Thus, we get:

2.3. Lemma. (1) *Every proper morphism in E is a closed morphism.*

(2) *Every strongly proper morphism in E is a strongly closed morphism.*

Proof. In Definition 2.1, let Z be the terminal object and the result follows.

2.4. Lemma. (1) *If $f : X \to Y$ and $g : Y \to W$ are (strongly) closed, so is $g \circ f$.*

(2) *If $f : X \to Y$ and $g : Y \to W$ are (strongly) proper so is $g \circ f$.*

Proof. (1) follows immediately from Definition 2.1. To prove (2) let Z be any object in E. Note that $(g \circ f) \times \mathrm{id} = (g \times \mathrm{id}) \circ (f \times \mathrm{id})$ and by (1), the result follows.

2.5. Lemma. *Let I be a finite set and for $i \in I$ let $f_i : X_i \to Y_i$ be a morphism in E. Let $X = X_1 \times X_2 \times \cdots \times X_n$, $Y = Y_1 \times Y_2 \times \cdots \times Y_n$, and let $f : X \to Y$ be the product morphism $(x_i) \to (f(x_i))$. Then if each of the f_i is (strongly) proper, then f is (strongly) proper.*

Proof. By induction it is enough to consider the case where $I = \{1, 2\}$. Suppose that f_1 and f_2 are (strongly) proper, and let Z be any object in E. Note that $f_1 \times f_2 \times \mathrm{id}_Z$ is the composition of $\mathrm{id}_{X_1} \times f_2 \times \mathrm{id}_Z$ and $f_1 \times \mathrm{id}_{X_2} \times \mathrm{id}_Z$. These two morphisms are (strongly) closed by hypothesis. Hence, by 2.4 and 2.1, $f = f_1 \times f_2$ is (strongly) proper.

2.6. Lemma. *If X_i, $i = 1, 2, \ldots, n$, is (strongly) compact, then so is $X = X_1 \times X_2 \times \cdots \times X_n$.*

Proof. X_i is (strongly)compact implies the family of morphism $X_i \to 1$ is (strongly) proper, $i = 1, 2, \ldots, n$. By 2.5, we get the result.

2.7. Lemma. *If X is (strongly) compact and Y is any object in E, then the projection $\pi_2 : X \times Y \to Y$ is (strongly) proper.*

Proof. We may identify Y with $1 \times Y$ and π_2 with the product of $X \to 1$ and $\mathrm{id}_Y : Y \to Y$, both of which are (strongly) proper. The result follows from 2.5.

2.8. Lemma. (1) *Let $X = (B, K)$ be in ConFCO. $\emptyset \neq F \subset B$ is (str.) closed iff for each $a \notin F$ and for any $\alpha \in K$, $\alpha \bigcup [F]$ is improper or $\alpha \not\subset [a]$* [4] *p. 391.*

(2) *Let $X = (B, \mathcal{F})$ be in ConLFCO, PBorn or Born. $\emptyset \neq F \subset B$ is closed iff $F = B$* [4] *p. 391–392.*

(3) *Let $X = (B, \mathcal{F})$ be in ConLFCO, PBorn or Born. $\emptyset \neq F \subset B$ is always strongly closed* [4] *p. 391–392.*

(4) *Let $X = (B, R)$ be in Prord. $\emptyset \neq F \subset B$ is closed iff for any $x \in B$ if there exist $a, b \in F$ such that xRa and bRx, then $x \in F$. The proof follows from the Definition 1.1 and* [3].

(5) *Let $X = (B, R)$ be in Prord. $\emptyset \neq F \subset B$ is stongly closed iff foreach $x \in B$ if there exists $a \in F$ such that xRa or aRx, then $x \in F$. It follows from the Definition 1.1 and* [3].

(6) *Let $X = (B, \mathcal{F})$ be in ConLFCO, PBorn or Born. X is ΔT_2 iff B is a point or the empty set* [4] *p. 392.*

(7) *Let $X = (B, \mathcal{F})$ be in ConLFCO, PBorn or Born. X is always ST_2* [4].

2.9. Lemma. *Let $X = (B, K)$ and $Y = (A, L)$ be in ConFCO or Prord. If $f : X \to Y$ is epi and initial, then f is (strongly) closed.*

Proof. Let X and Y be in ConFCO. Suppose $\emptyset \neq F \subset B$ is closed in X. Suppose also for any $a \notin f(F)$ and for any $\alpha \in L$, $\alpha \subset [a]$. It follows that there exists $b \in B$ such that $f(b) = a$ and $f^{-1}\alpha \subset f^{-1}[a] = [f^{-1}f(b)] \subset [b]$. Since f is epi, $f(f^{-1}\alpha) = \alpha \in L$ and so $f^{-1}\alpha \in K$ (since f is initial). Since F is (strongly) closed, by 2.6, $f^{-1}\alpha \bigcup [F]$ is improper. Note that by [2] p. 98, $f(f^{-1}\alpha \bigcup [F]) \supset f(f^{-1}\alpha) \bigcup [f(F)] \supset \alpha \bigcup [f(F)]$ and consequently $\alpha \bigcup [f(F)]$ is improper. Hence, by 2.8, $f(F)$ is (strongly) closed in Y. The proof for Prord follows easily.

2.10. Lemma. *Let $X = (B, \mathcal{F})$ and $Y = (A, \mathcal{G})$ be in PBorn or Born. Then,*

(1) *$f : X \to Y$ is closed iff f is epi.*
(2) *$f : X \to Y$ is always strongly closed.*

Proof. It follows from 2.1 and 2.8.

Let 1 be the terminal object in E. Then $1 \times X$ is isomorphic to X for all objects in the above categories, by 2.9 and 2.10, the projections are closed. Hence, in view of this and 2.1, we have

2.11. Corollary. *In these categories, we have*

(1) *$f : X \to 1$ is (strongly) proper.*
(2) *X is always (strongly) compact.*
(3) *Let $f : X \to Y$ be in ConLFCO, PBorn or Born. Then f is proper iff f is closed.*

2.12. Lemma. *Let $X = (B, \mathcal{F})$ be in PBorn. Then X is T_2' iff $X \in$* Born.

Proof. Suppose X is T_2' and $V \subset U$ with $U \in \mathcal{F}$. If $V = U$, then $V \in \mathcal{F}$. If $V \neq U$, then let $W = V^2 \bigvee (U - V)^2$. Since X is T_2', by 2.1, it follows easily that $V \in \mathcal{F}$, i.e., $\mathcal{F}$ is hereditary closed. Hence $X \in$ Born. Suppose $X \in$ Born. By Lemma 1.6 of [4] p. 385, X is T_0'. By 2.1, it remains to show that X is $\mathrm{Pre}\,T_2'$. This can be done easily by using the assumption, initial and final lifts in PBorn (see [4]), and 2.1).

2.13. Proposition. *For $E = TOP$, it is well-known that:*

(1) *Every compact subset of a Hausdorff space is closed* [7] *or* [20].

(2) *Let X be a topological space and R be an equivalence relation on X. If the canonical $f : X \to X/R$ is proper, then R is closed and X/R is T_2* [7] *p. 105.*

(3) *Every continuous function f of a compact space into T_2 space is closed* [7] *p. 87.*

(4) *Every compact Hausdorff space is T_3 and T_4* [7] *or* [20].

(5) *Every normal space, i.e., T_4, is a Tychonoff space* [20].

(6) *Every Tychonoff space is T_3* [20].

We now show that the above well-known results are not valid, in general, in an arbitrary topological category over Set.

2.14. *Remarks.* (1) Let $B = \{a, b\}$ be a two-point set and $\mathcal{F}$ be the discrete structure on B. Note that $X = (B, \mathcal{F}) \in$ PBorn is ST_2 by 2.8 and T_2' by 2.12. By 2.11, a subset $\{a\}$ of X is compact but it is not, by 2.8, closed. This shows 2.13 (1) does not hold in PBorn.

(2) Let $\mathbb{Z}$ be the set of integers, $(\mathbb{Z}, \mathcal{F})$ be a discrete object in (PBorn) or Born, i.e. $\mathcal{F} = \{C : C$ is a (nonempty) finite subset of $\mathbb{Z}\}$, R be the equivalence relation on $\mathbb{Z}$ defined by xRy iff $x \equiv y \pmod 2$, and f the canonical morphism $(\mathbb{Z}, \mathcal{F}) \to (\mathbb{Z}/R, \mathcal{F}')$, $\mathcal{F}'$ the quotient stucture. By 2.11 (3) f is proper but $(\mathbb{Z}/R, \mathcal{F}')$ is not ΔT_2 by 2.8, and R is not, by 2.8, closed in X. This shows 2.13 (2) does not hold in PBorn and Born.

(3) In PBorn, by 2.8 and 2.12, ΔT_2 implies T_2' and T_2' implies ST_2 but the converse of each implication is not true, in general. For example, $(\mathbb{Z}, \mathcal{F})$ in (2) is T_2' by 2.12 but it is not ΔT_2 by 2.5. Let $\mathcal{B} = \{C, \{1, 2, 3, \dots\} : C$ is a nonempty finite subset of $\mathbb{Z}\}$. Then by 2.8, $(\mathbb{Z}, \mathcal{B})$ in PBorn is ST_2 but it is not T_2' by 2.12 since the set $\{3, 4, 5, \dots\} \subset \{1, 2, 3, \dots\}$ and $\{3, 4, \dots\}$ is not in $\mathcal{B}$. Let LFCO denote the category of local filter convergence spaces ([4], [19] or [21]). Then by 3.14 and 3.15 of [4] and 2.9 of [5], we have T_2' implies ST_2 and ST_2 implies ΔT_2 but the converse of each implication is not true, in general.

(4) Let $\mathbb{R}$ be the set of real numbers, $(\mathbb{R}, \mathcal{F})$ be a discrete object in PBorn, and $(\mathbb{Z}, \mathcal{F})$ be in (2). By 2.8 (7), $(\mathbb{R}, \mathcal{F})$ is ST_2 and by 2.11 (2) $(\mathbb{Z}, \mathcal{F})$ is compact. However, the inclusion morphism $f : (\mathbb{Z}, \mathcal{F}) \to (\mathbb{R}, \mathcal{F})$ is not closed by 2.10 (1).This shows 2.13 (3) is not valid in PBorn.

2.15. Lemma. *Let $X = (A, K)$ be in ConLFCO.*

(1) *X is T_2' iff X is discrete* [5].

(2) *X is ST_3' iff for any filter $\alpha \in K$ either $F \in \alpha$ for all nonempty subset F of A or α contains a finite subset, U of A* [5].

(3) *X is ST_4' iff for any filter $\alpha \in K$ either $F \bigcup F' \in \alpha$ for any disjoint subsets F and F' of A or α contains a finite subset U of A* [5]

2.16. Lemma. *Let $X = (B, R)$ be in Prord. X is ΔT_2, ST_2, T_2', ST_3' or ST_4' iff X is discrete, i.e., if xRy, then $x = y$.*

Proof. If X is discrete, then clearly, by 2.8, Δ is (strongly) closed and by 1.1, X is ST_2 and ΔT_2. Suppose X is ΔT_2 and xRy. Then clearly $(x, y)R^2(y, y)$ and $(x, x)R^2(x, y)$, where R^2 is the product strucrure on B. Since Δ is closed, by 2.6, $(x, y) \in \Delta$, i.e., $x = y$. Similarly, if X is ST_2, then X is discrete. The proof for T_2' follows from 1.1 and Theorem 2.10 of [3] p. 198. The proof for ST_3' and ST_4' follows easily from 1.1 and Theorems 2.3 and 2.8 of [3] p. 196–197.

2.17. *Remark.* Let $\mathbb{R}$ be the set of real numbers and $K = F(\mathbb{R})$, the set of all filters on $\mathbb{R}$, and $X = (\mathbb{R}$,K) be in ConLFCO. By 2.10 and 2.8, X is compact and ST_2 but X, by 2.12, is not ST_3' and ST_4'. This shows 2.13 (4) does not hold in ConLFCO.

3. Tychonoff objects

We now define three various forms of Tychonoff objects for an arbitrary topological category over Set. Furthermore, we characterize each of them for the categories that are mentioned in Section 1 and investigate the relationships among them.

3.1. *Definitions.*

1. X is $\Delta T_{7/2}$ iff X is a subspace of a compact ΔT_2.
2. X is $ST_{7/2}$ iff X is a subspace of a compact ST_2
3. X is $T_{7/2}'$ iff X is a subspace of a compact T_2'

3.2. *Remark.* For the category TOP of topological spaces, all of $\Delta T_{7/2}$, $ST_{7/2}$, and $T_{7/2}'$ are equivalent and reduce to the usual $T_{7/2}$ Tychonoff, i.e., completely regular T_1, spaces [20], 1.2, and 2.2.

3.3. Theorem. (1) *Let $X = (B,K)$ be in ConLFCO.*

(a) *X is $\Delta T_{7/2}$ iff B is a point or the empty set.*

(b) *X is always $ST_{7/2}$.*

(c) *X is $T'_{7/2}$ iff X is discrete.*

(2) *Let $X = (B, \mathcal{F})$ be in PBorn.*

(a) *X is $\Delta T_{7/2}$ iff B is a point or the empty set.*

(b) *X is always $ST_{7/2}$.*

(c) *X is $T'_{7/2}$ iff X is hereditary closed, i.e., $X \in$ Born.*

(3) *Let $X = (B, R)$ be in Prord. Then X is $\Delta T_{7/2}, ST_{7/2}$ or $T'_{7/2}$ iff X is discrete.*

Proof. (1) Combine 3.1, 2.8, 2.10, and 2.14. (2) It follows from 3.1, 2.8, 2.11, and 2.12. (3) It follows from 3.1, 2.11, and 2.15.

3.4. *Remarks.* (1) Let $X = (B, K)$ be in ConLFCO. By Theorem 2.25 of [5] if X is T'_2, then the quotient space X/F is also T'_2 for all nonempty subsets F of B. Let $B = \{a, b\}$ be a two-point set and K be the discrete structure on B, i.e., $K = \{[a], [b], [a] \cap [b], [\emptyset]\}$. By 2.14 of [5], it follows that $X = (B, K)$ is ST'_3 and ST'_4 but by 3.3(1) X is not $\Delta T_{7/2}$. This shows that 2.13 (5) does not hold in ConLFCO.

(2) Let $X = (\mathbb{R}, K)$ be the example in 2.17. By 3.3 (1), X is $ST_{7/2}$ but it is not ST'_3. This shows that 2.13 (6) does not hold in ConLFCO.

(3) Generally speaking, $\Delta T_{7/2}, ST_{7/2}$, and $T'_{7/2}$ are independent of each other. See 2.14 (3) and 3.3 (2).

(4) In all of the above categories, compactness and Tychonoff spaces are hereditary, productive and divisible (i.e. the quotient of compact or Tychonoff space is a compact or Tychonoff space, respectively).

3.5. *Remark.* Let $X = (B, K)$ be any object in ConLFCO, PBorn, Born or Prord. If X is $\Delta T_{7/2}, ST_{7/2}$ or $T'_{7/2}$ and F be a nonempty closed subspace of X, then every morphism $f : F \to (\mathbb{R}, L)$, where $\mathbb{R}$, is the set of real numbers and $\mathcal{L}$ is any constant local filter convergence structure, (pre)bornology or preordered relation on R, respectively, has a continuous extension $g : X \to (\mathbb{R}, L)$. The proof for ConLFCO, PBorn and Born follows easily from 2.8 and 3.3. Let X be in Prord and define $g(x) = f(x)$ if $x \in F$ and $g(x) = 0$ if $x \notin F$. By 3.3, the result follows.

4. Compact pairs

In this section, we introduce the notion of compact pairs in an arbitrary topological category over Set which generalizes the notion of compact pairs that is given by Giuli [11] in TOP. Furthermore, some results concerning this concept are proved.

Let E be a topological category over Set, and "T_2" denote any of T_2', ST_2, and ΔT_2 that are defined in 1.1. T_2E, CT_2E, and SCT_2E will denote the category of T_2-objects (spaces), compact T_2-objects, and strongly compact T_2-objects, respectively.

4.1. *Definition.* Let K and D be two non-empty full subcategories of T_2E. We say that K and D form a compact pair (K, D) if

(i) An object X is in K iff, for each object Y in D, the projection $\pi_2 : X \times Y \to Y$ is closed;

(ii) An object Y is in D iff, for each object X in K, the projection $\pi_2 : X \times Y \to Y$ is closed.

Then K is called the compact class and D the discrete class of the compact pair (K, D).

4.2. *Remark.* (1) For E = TOP, the category of topological spaces, we get Definition 2.1 of Giuli [11].

(2) Let E be a topological category. By 2.3 and 2.7, both of (CT_2E, T_2E) and (SCT_2E, T_2E) are compact pairs, where T_2 is $\Delta T_2, ST_2$ or T_2'. Note that this generalizes the well-known compact pair (CT_2TOP, T_2TOP).

4.3. Theorem. *Let (K, D) be a compact pair.*

(i) *If D is finitely productive, then $X \times Y$ is in K whenever X is in K and Y is in both K and D;*

(ii) *If X is in CT_2E, and Y is in K, then $X \times Y$ is in K;*

(iii) *If X is in SCT_2E and Y is in K, then $X \times Y$ is in K.*

Proof. Suppose that X and Y are in K, and Y and Z are in D. Since D is finitely productive, $Y \times Z$ is in D. It follows that the projections $\pi_2 : X \times Z \to Z$ and $\pi_{23} : X \times (Y \times Z) \to Y \times Z$ are closed.

By 2.4, $\pi_3 : X \times Y \times Z \to Z$, which is the composition of π_2 and π_{23}, is closed, where $\pi_{23}(a, b, c) = (b, c)$. Hence, $X \times Y$ is in K.

(ii) If X is in CT_2E, then, by 2.7, $\pi_{23} : X \times (Y \times Z) \to Y \times Z$ is closed. If Y is in K and Z is in D, then, by 4.1, $\pi_2 : Y \times Z \to Z$ is closed. It follows from 2.4 that the composition $\pi_2 \circ \pi_{23}$ is closed and concequently $X \times Y$ is in K. The proof for (iii) is similar to the proof of (ii).

4.4. Corollary. *In Theorem 4.3, if $E = TOP$, then we get Proposition 2.11 of* Giuli [11].

Let $(\mathcal{A}, \leq)$ be the conglomerate of all non-empty subclasses (full subcategories) of T_2E ordered by inclusion. For each A and B in $\mathcal{A}$ set $F(A) = \{X \in T_2E$: for each Y in A, π_2: $X \times Y \to Y$ is closed$\}$ and $G(B) = \{Y \in T_2E$: for each X in $B, \pi_2 : X \times Y \to Y$ is closed$\}$.

4.5. Theorem. *$((\mathcal{A}, \leq), F, G)$ is a Galois connection (cf.* [17] *p. 93). Moreover, $F(A)$ is a compact class and $G(B)$ is a discrete class for each A and B in $\mathcal{A}$.*

Proof. It is easy to see that the assignments $A \to F(A)$ and $B \to G(B)$ are order reversing functions and $A \leq G(F(A))$ and $B \leq F(G(B))$ for each A and B in $\mathcal{A}$, i.e., F and G form a Galois connection. The fact that $F(A)$ is a compact class and $G(B)$ is a discrete class follow from 4.1.

4.6. Corollary. *In Theorem 4.5, if $E = TOP$, then we get Theorem 2.2 (a) of* Giuli [11].

4.7. Lemma. *Let E be any of ConFCO, ConLFCO, PBorn, Born or Prord. Then every compact class K and every discrete class D are both reflective and coreflective in T_2E, where T_2 is T_2', ST_2 or ΔT_2.*

Proof. It follows easily from 2.9, 2.10, 2.12, 2.15, 2.16, and 4.1 that both K and D are closed under the formation of "T_2" quotients, subspaces, coproducts, and products. Also, K and D are isomorphism-closed. By Theorem 1.11 and Lemma 1.14 of [21], we get the result.

4.8. *Definition.* Let A be a class (full subcategory) of E containing the terminal object 1. A morphism $f : X \to Z$ is said to be A-proper if $f \times \mathrm{id} : X \times Y \to Z \times Y$ is closed for each Y in A.

4.9. *Remark.* In 4.8, if $E = \mathrm{TOP}$, then we get Definition 3.1 of [11].

4.10. Lemma.

(1) *The composition of A-proper morphisms is A-proper.*

(2) *X is in $F(A)$ iff the morphism $f : X \to 1$ is A-proper.*

Proof. (1) follows from 2.4. Note that $1 \times Y$ is isomorphic to Y for each Y in E and $f \times \mathrm{id}_Y = \pi_2$. By 4.1 and 4.8, we get (2).

4.11. Corollary. *In 4.10, if $E = TOP$, then we get Proposition 3.3 (b) and (c) of* Giuli [11].

Following Giuli [11] we say that a class H of T_2E is (A)-left-fitting if whenever $f : X \to Z$ is an A-proper morphism and Z is in H, then X is in H.

4.12. Lemma. *Every compact class K is $G(K)$-left-fitting.*

Proof. If $f : X \to Z$ is $G(K)$-proper, then, by 4.8, $f \times \mathrm{id}_Y$ is closed whenever Y is in $G(K)$. If Z is in K, then, by 4.1, $\pi_2 : Z \times Y \to Y$ is closed for each Y in $G(K)$. Hence, for each Y in $G(K)$, $\pi_2 \circ (f \times \mathrm{id}_Y) = \pi_2 : X \times Y \to Y$ is closed, i.e., X is in K.

4.13. Corollary. *In 4.12, if $E = TOP$, then we get Proposition 3.8 of* Giuli [11].

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