On the position of roots of solutions of the differential equation $y''+qy=0$ with $[q(x)]^\nu$ $(\nu<0)$ concave

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Let $y=y(x)$ be a solution of the differential equation

(1) $$y''+qy=0 \quad ('=d/dx)$$

where $q=q(x)$ is continuous on the interval (a, b), and $q(x)>0$ $(a<x<b)$. Denote by $x_0, x_1, \dots, x_n$; $x_0', x_1', \dots, x_n'$ the roots of the equations $y(x)=0$, $y'(x)=0$ respectively, where $(x_0<)x_0'<x_1<\dots<x_n(<x_n')$ $x_0, x_0'\geqq a$, $x_n, x_n'<b$.

Definition. A function $q(x)$ belongs to the class $C_\nu[a, b]$ if it is positive on the interval (a, b) and $[q(x)]^\nu$ is concave.

For example the function $x^{1/\nu}$ belongs to this class if $\nu\neq 0$ is real, $a\geqq 0$, $b\leqq +\infty$.

In [1] Á. ELBERT estimated the functionals

$$\int_{x_0}^{x_n}\sqrt{q}\,dx, \quad \int_{x_0'}^{x_n}\sqrt{q}\,dx, \quad \int_{x_0}^{x_n'}\sqrt{q}\,dx, \quad \int_{x_0'}^{x_n'}\sqrt{q}\,dx$$

provided $q\in C_\nu$, $\nu>0$. In the present paper we shall deal with the same functionals for $q\in C_\nu[a, b]$, $\nu<0$. It is easy to see that in this case the above integrals exist.

We shall assume that $q(x)$ is twice continuously differentiable on (a, b). This does not mean any restriction, since each $q\in C_\nu[a, b]$ can be approximated by

$$q_\varepsilon(x)=\left(\frac{1}{\varepsilon^2}\int_x^{x+\varepsilon}\Big(\int_u^{u+\varepsilon} q^\nu(v)\,dv\Big)\,du\right)^{1/\nu} \quad (a<x<b-2\varepsilon)(\varepsilon>0),$$

where $q\in C_\nu[a, b-2\varepsilon]$, $(\varepsilon>0)$, moreover it $q(x)$ is increasing on (a, b), then also $q_\varepsilon(x)$ is increasing for $(a<x<b-2\varepsilon)$ and $\lim_{\varepsilon\to 0} q_\varepsilon(x)=q(x)$. The condition $q\in C_\nu[a, b]$ is equivalent to the simpler condition

(2) $$(\nu-1)q'^2+qq'\geqq 0 \quad (a<x<b)$$

We introduce

$$u=\int_a^x\sqrt{q}\,dx \quad (a\leqq x<b).$$

Considering u as independent variable we have (with $x=x(u)$)

$$dx/du = q(x)^{-1/2}.$$

In the sequel we shall use the notation $\cdot = d/du$. Introducing

$$s(u) = 1/2 \cdot q' \cdot q^{-3/2} \quad (0 < u < u(b)) \tag{2a}$$

we have

$$\dot{s}(u) = 1/2 \cdot (-3/2q'^2 + q \cdot q'') \cdot q^{-3}.$$

For $q \in C_\nu[a, b]$ $(\nu > 0)$ (2) yields

$$\dot{s}(u) \geqq -(1+2\nu) \cdot s^2 \quad (0 < u < u(b)). \tag{3}$$

Assuming $q \in C_\nu[a, b]$, $\nu < -1/2$ and integrating we get

$$s(u) \geqq (1-2\cdot\mu)\cdot\mu^{-1} \quad (0 < u < u(b)) \tag{4}$$

where

$$\mu = \nu/(1+2\nu).$$

In the sequel it will be more convenient to use the following simpler form of the function $s(u)$:

$$s(u) = 1/2 \cdot \dot{q} \cdot q^{-1}. \tag{5}$$

We shall need the following lemmas.

Lemma 1. I. *The differential equation* (1) *has a solution* $y(x)$ *satisfying the initial conditions* $y(x_0)=0$, $y'(x_0)=1$, *with domain of definition* $[x_0, b)$ *if and only if*

$$\int_{x_0}^{x} g(v)(v-x_0)\,dv < \infty \quad (x_0 \leqq x < b). \tag{6}$$

II. *The differential equation* (1) *has a solution* $y(x)$ *satisfying the initial conditians* $y(x_0')=1$, $y(x_0')=0$, *with domain of definition* $[x_0', b)$, *if and only if*

$$\int_{x_0'}^{x} q(v)\,dv < \infty \quad (x_0 \leqq x < b). \tag{7}$$

Proof. I. Assuming (6), we shall show that the differential equation (1) has a solution $y(x)$ satisfying $y(x_0)=0$, $y'(x_0)=1$.

We define

$$I_0(x) = x - x_0$$

and, for $j \geqq 1$,

$$I_j(x) = -\int_{x_0}^{x}\int_{x_0}^{v} q(u) I_{j-1}(u)\,du\,dv \quad (x_0 \leqq x < b).$$

For $j=1$ the definition is meaningful by (6).

It is easy to prove by induction, that

$$(8)\qquad |I_j(x)| \leqq \frac{(x-x_0)\cdot\Big(\int\limits_{x_0}^{x} q(u)(u-x_0)\,du\Big)^j}{j!} \quad \begin{matrix}(x_0 \leqq x < b)\\ (j=0,1,2,\ldots).\end{matrix}$$

The required solution $y(x)$ of the differential equation (1) has the form

$$(9)\qquad y(x) = \sum_{j=0}^{\infty} I_j(x) \quad (x_0 \leqq x < b).$$

Indeed, the series together with its derivative converges absolutely on the interval $[x_0, b]$ by (8). It is easy to see, that $y(x)$ satisfies (1) and the initial conditions $y(x_0)=0,\ y'(x_0)=1$.

Suppose conversely that the differential equation (1) has a solution $y(x)$ satisfying the initial conditions $y(x_0)=0,\ y'(x_0)=1$. We shall show, that condition (6) holds.

Integrating (1), we get

$$(9a)\qquad \int\limits_{\xi}^{x} q(v)\cdot y(v)\cdot dv = y'(\xi)-y'(x) \quad (x_0 < \xi < x < b)$$

$y(x)$ is nonincreasing and concave on the interval $[x_0, x_0']$, hence

$$(9b)\qquad y(x) \geqq \alpha(x-x_0) \quad (x_0 \leqq x \leqq x_0')$$

where $\alpha=y(x_0')/(x_0'-x_0)$. By (9a) and (9b)

$$\lim_{\xi\to x_0}\int\limits_{\xi}^{x} q(v)(v-x_0)\,dv \leqq 1-y'(x) \leqq 1 \quad (x_0 \leqq x \leqq x_0')$$

whence the integral on the left hand side of (6) is bounded.

II. Assume, that condition (7) holds. Define

$$L_0(x) = 1$$

and for $j \geqq 1$

$$L_j(x) = -\int\limits_{x_0'}^{x}\int\limits_{x_0'}^{v} q(u)L_{j-1}(u)\,du\,dv \quad (x_0' \leqq x < b).$$

Using (7) it can be shown by induction that

$$(10)\qquad |L_j(x)| \leqq \frac{\Big((x-x_0')\int\limits_{x_0'}^{x} q(u)\,du\Big)^j}{j!} \quad \begin{matrix}(x_0' \leqq x < b),\\ (j=0,1,2,\ldots).\end{matrix}$$

Defining $y(x)=\sum\limits_{j=0}^{\infty} L_j x$, (10) yield the absolute convergence of the series of $y(x)$ and $y'(x)$ on $[x_0', b]$. It can be shown easily that the function $y(x)$ satisfies (1) as well as the initial conditions $y(x_0')=1,\ y'(x_0')=0$.

On the same way as in the proof of case I, it can be proved, that condition (7) is also necessary.

Lemma 2. *For* $q \in C_\nu[a, b]$ $(\nu < -1/2)$

$$\int_{x_0}^{x_0'} \sqrt{q}\, dx \leqq j_{\mu-1}$$

holds, where $j_{\mu-1}$ *denotes the first positive root of the Bessel function* $J_{\mu-1}$ *of first kind and of order* $\mu-1$, $\mu=\nu/(1+2\nu)$.

Proof. Define

$$u(x) = \int_{x_0}^{x} \sqrt{q}\, dx \quad (x_0 \leqq x < b).$$

Denote by $y(x)$ the solution of the differential equation (1) satisfying $y(x_0)=0$, $y'(x_0)=1$ and let

$$Y(u) = y(x(u)).$$

With the independent variable u, differential equation (1) has the form

$$\ddot{Y} + s\dot{Y} + Y = 0. \tag{11}$$

Let $\xi = \min(j_{\mu-1}, u(x_0'))$. We shall show that

$$\xi = u(x_0'). \tag{12}$$

We introduce the function $W(u)$ by

$$W(u) = u^{1-\mu}(J_{\mu-1}Y - J_\mu \dot{Y}).$$

It is easy to see that $W(0)=0$. Differentiating and using the known recursion formulas

$$\dot{J}_\mu = J_{\mu-1} - \frac{\mu}{u} \cdot J_\mu,$$

$$\dot{J}_{\mu-1} = \frac{\mu-1}{u} \cdot J_{\mu-1} - J_\mu$$

for Bessel-functions (see for example [4]) we get

$$\dot{W}(u) = -J_\mu u^{1-\mu}\big(Y + (1-2\mu)u^{-1}\dot{Y} + Y\big) \quad (0 < u \leqq \xi).$$

Using (11) and (4)

$$\dot{W}(u) = Y \cdot J_\mu u^{1-\mu}\big(s - (1-2\mu)u^{-1}\big) \geqq 0 \quad (0 < u \leqq \xi)$$

whence it follows, that

$$W(u) \geqq 0 \quad (0 < u \leqq \xi). \tag{13}$$

Assume, that (12) would not hold, i.e. $u(x_0') > \xi$. By definition $\xi = j_{\mu-1}$, and since $j_{\mu-1} < j_\mu$, $W(\xi) = -\xi^{1-\mu} J_\mu(\xi)\dot{Y}(\xi) < 0$, what contradicts (13). Hence (12) holds.

Lemma 3. *According to* $q' \gtreqless 0$ *we have*

$$\int_{x_0}^{x_0'} \sqrt{q}\, dx \gtreqless \pi/2.$$

PROOF. The proof is similar to that of Lemma 2. Introduce $\xi = \min\left(\pi/2, u(x_0')\right)$. The function $W(u)$ should be defined by

$$W(u) = \sin u \cdot \dot{Y} - \cos u \cdot Y.$$

Differentiating and using (10) we get $\dot{W}(u) = -s.\ \dot{Y} \sin u \gtreqless 0 (0 < u \leqq \xi)$, whence $W(u) \gtreqless 0$. Now, it can be shown easily that $u(x_0') \gtreqless \xi$.

Lemma 4. *If* $q \in C_\nu[a, b]$, *then*

$$\int_{x_0'}^{x_1} \sqrt{q}\, dx \begin{cases} \geqq j_{-\mu} & \text{for} \quad \nu < -1 \\ > 0 & \text{for} \quad -1 \leqq \nu < -1/2. \end{cases}$$

In the latter case the inequality cannot be improved.

PROOF. Consider first the case $\nu < -1$. The proof is similar to that of Lemma 2. Let

$$u = \int_{x_0'}^{x} \sqrt{q}\, dx,$$

and denote by $y(x)$ the solution of the differential equation (1) satisfying the initial conditions $y(x_0') = 1,\ y'(x_0') = 0$. Moreover, let $Z(u) = y'(x(u))$ $(u \geqq 0)$. It follows from (1) that

$$\ddot{Z} - s\dot{Z} + Z = 0. \tag{14}$$

Let $\xi = \min\left(j_{\mu_1 - 1}, u(x_1)\right)$. Define the function $W(u)$ by

$$W(u) = u^{1-\mu_1}(J_{\mu_1 - 1} \cdot Z - J_{\mu_1} \cdot \dot{Z}).$$

Where $\mu_1 = 1 - \mu$. From (14) and (4) it follows, that $W(u) \leqq 0$ $(0 < u \leqq \xi)$, whence $\xi = j_{\mu_1 - 1} = j_{-\mu}$. Consider the case $-1 \leqq \nu < -1/2$. Denote by k_ν the infimum of the functional $\int_{x_0'}^{x_1} \sqrt{q}\, dx$ for $q \in C_\nu[a, b]$, $(a \leqq x_0',\ x_1 < b)$. For $\nu_1 < \nu_2$ we have $C_{\nu_1} \subset C_{\nu_2}$, whence

$$k_{\nu_1} \geqq k_{\nu_2}. \tag{15}$$

Since for $\nu < -1$ we already proved $k_\nu = j_{-\mu}$ and $j_{-\mu}$ depends continuously on μ for $\mu > -1$, we have

$$\lim_{\nu \to -1} k_\nu = j_{-1} = 0.$$

It follows by (15) that $k_\nu = 0$ for all $\nu \geqq -1$.

Lemma 5. *It* $q \in C_\nu[a, b]$ $(\nu \leqq -1/4)$, *then*

$$\int_{x_n}^{x_0} \sqrt{q}\, dx \geqq n\pi \quad (a \leqq x_0 < x_n < b).$$

PROOF. Consider the differential equation (11). Introducing the notation

$$U(u) = Y(u) e^{\frac{1}{2}\int_1^u s(v)\,dv}$$

(see E. MAKAI [2]) the differential equation (11) can be written in the following form:

$$\ddot{U} + PU = 0 \tag{16}$$

where

$$P(u) = 1 - \frac{1}{2}\left(\dot{s} + \frac{1}{2}s^2\right).$$

By (3)

$$P(u) \leqq 1 + \left(\frac{1}{4} + \nu\right) s^2 \quad (0 < u < u(b))$$

whence

$$P(u) \leqq 1 \quad (0 < u < u(b)). \tag{17}$$

Applying Sturm's comparison theorem for the differential equation (16), we have by (17), that the n-th root of the solution U of the differential equation (16) lies to the left from $n\pi$, what proves the lemma.

Lemma 6. *If* $q \in C_\nu[a, b]$ $(\nu \leqq -3/4)$ *then*

$$\int_{x_0'}^{x_n'} \sqrt{q}\, dx \leqq n\pi \quad (a \leqq x_0' < x_n' < b).$$

PROOF. Consider the solution of the differential equation (14) satisfying the initial condition $Z(0)=0$, provided it exists. Introducing the notation

$$V(u) = Z(u) \cdot e^{-\frac{1}{2}\int_1^u s(v)\,dv}$$

(see COHN [3]). Equation (14) can be written in the form

$$\ddot{V} + QV = 0 \tag{18}$$

where

$$Q(u) = 1 + \frac{1}{2}\left(\dot{s} - \frac{1}{2}s^2\right).$$

By (3)

$$Q(u) \geqq 1 - \left(\frac{3}{4} + \nu\right) s^2 \quad (0 < u < u(b)) \tag{19}$$

whence

$$Q(u) \geqq 1 \quad (0 < u < u(b)).$$

Applying Sturm's comparison theorem it can be shown that the n-th root of the solution $V(u)$ is smaller than $n\pi$.

Lemma 7. *If* $q\ C_\nu[a, b]\ (-3/4 < \nu < -1/2)$, *then*

$$\int_{x_0'}^{x_n'} \sqrt{q}\,dx \begin{cases} < j_{\mu-1,n} & \text{for } q \text{ monotonic} \\ < j_{\mu-1,\left[\frac{n}{2}\right]} + j_{\mu-1,n-\left[\frac{n}{2}\right]} & \text{otherwise.} \end{cases}$$

PROOF. Consider first the case of monotonic q. Assume $q'(x) \leqq 0\ (a < x < b)$. (The proof can be easily modified for q nondecreasing.) Since $q' \leqq 0$, we have by (4)

$$s^2 \leqq (1-2\mu)^2 u^{-2}$$

and, by (18)

$$Q(u) \geqq 1 + (1/4-(1-\mu)^2)u^{-2} \quad (0 < u < u(b)). \tag{20}$$

Consider a solution of the differential equation

$$\ddot{V}^* + (1/4-(1-\mu)^2 u^{-2})V^* = 0 \tag{21}$$

satisfying the initial condition $V^*(0)=0$. It can be seen easily that

$$V(u) = Cu^{1/2} \cdot J_{\mu-1}(u)$$

(where $C<0$ denote, an arbitrary constant). Using Sturm's comparison theorem and (20) it can be shown, that the n-th root of the solution $V(u)$ of the differential equation (18) is smaller than the n-th root of the solution $V^*(u)$ of the differential equation (21).

Note that in (20) equality takes place only for $q = Cx^{1/\nu}$ $(C>0)$, $(a=0, b>0, x_0'=a)$. At the same time, the differential equation $y''+x^{1/\nu}y=0$ $(-1<\nu<0)$ has no solution satisfying $y(0)=1, y'(0)=0$, since it has the independent pair of solutions

$$y_1(x) = C\sqrt{x}J_\mu(2\mu x^{1/2\mu}), \quad y_2(x) = C\sqrt{x}J_{-\mu}(2\mu x^{1/2\mu}).$$

Therefore only the estimation

$$\int_{x_0'}^{x_n'} \sqrt{q}\,dx < j_{\mu-1,n}$$

holds.

Consider now the case of non monotonic q. Suppose that

$$q'(x) \leqq 0 \quad \text{for} \quad x \leqq x^*$$

and

$$q'(x) \geqq 0 \quad \text{for} \quad x \geqq x^*. \qquad (a \leqq x^* < b)$$

We may assume

$$x_i \leqq x^* \leqq x_i' \quad (0 \leqq i \leqq n).$$

Define

$$q^*(x) = \begin{cases} q(x) & \text{for} \quad x_0 \leqq x \leqq x^* \\ q(x^*) & \text{for} \quad x^* \leqq x \leqq x_i'. \end{cases}$$

Denote by $y^*(x)$ the solution of the differential equation $y^{*''}+qy^*=0$ satisfying the initial conditions $y^*(x_0')=y(x_0')$, $y^*(x_0')=y'(x_0')$. Let ξ be the i-th root of the equation $y^{*'}(x)=0$ $(x>x_0')$. It can be shown, that

$$\int_{x_0'}^{x_i'} \sqrt{q}\,dx \leqq \int_{x_0'}^{\xi} \sqrt{q^*}\,dx. \tag{22}$$

Since q and q^* are monotonic on the intervals $[x_0', x_i']$, $[x_i', x_n']$ respectively, (22) implies

$$\int_{x_0}^{x_i'} \sqrt{q}\,dx \leqq j_{\mu-1,i} \quad \text{and} \quad \int_{x_i'}^{x_n'} \sqrt{q}\,dx \leqq j_{\mu-1,n-i}.$$

It is easy to see that

$$j_{\mu-1,i}+j_{\mu-1,n-i} \leqq j_{\mu-1,\left[\frac{n}{2}\right]}+j_{\mu-1,n-\left[\frac{n}{2}\right]}.$$

Lemma 8. *If* $q\in C_\nu[a,b]$ $(-3/4<\nu<-1/2)$ *then*

$$\int_{x_0'}^{x_n'} \sqrt{q}\,dx \begin{cases} \leqq j_{\mu-1,n+1} & \textit{for } q \textit{ monotonic} \\ < j_{\mu-1,\left[\frac{n+1}{2}\right]}+j_{\mu-1,n+1-\left[\frac{n+1}{2}\right]} & \textit{otherwise.} \end{cases}$$

Proof. Let $y_\xi(x)$ be the solution of the differential equation (1) satisfying the initial conditions $y_\xi(\xi)=1$, $y_\xi'(\xi)=0$ $(x_0>\xi>x_n')$, and denote by $(\xi=)x_{\xi,0}, x_{\xi,1}', \ldots, \ldots, x_{\xi,n+1}'$ the roots of the equation $y_\xi'(x)=0$ $(x>\xi)$.

Assume $x_{\xi,i+1}'\leqq b$ $(i=1,\ldots,n+1)$. It can be shown, that $x_i'\leqq x_{\xi,i+1}'$ $(i=1,2,\ldots)$, whence

$$\int_{x_0}^{x_n'} \sqrt{q}\,dx \leqq \int_{x_0}^{\xi} \sqrt{q}\,dx+\int_{\xi}^{x_{\xi,n+1}'} \sqrt{q}\,dx. \tag{23}$$

Consider the right hand side of (23). The first term can be arbitrarilis small. Applying Lemma 7 for the second term, we get the assertion.

Lemma 9. *If* $q\in C_\nu[a,b]$ $(-3/4<\nu<-1/2)$ *then*

$$\int_{x_0}^{x_n} \sqrt{q}\,dx \begin{cases} \leqq j_{\mu-1,n}\pi/2 & \textit{for } q \textit{ monotonic,} \\ \leqq j_{\mu-1,\left[\frac{n+1}{2}\right]}+j_{\mu-1,n+1-\left[\frac{n+1}{4}\right]} & \textit{otherwise.} \end{cases}$$

Proof. Consider first the case of monotonic q. Decampose the interval of integration into the subintervals $[x_0, x_{n-1}']$, $[x_{n-1}', x_n]$. Thus the assertion follows at once from Lemma 8 and Lemma 3.

If q is not assumed to be monotonic, the proof is similar to that of Lemma 8.

Theorem 1.

If $q'(x)\leqq 0$ $(a<x<b)$ *and* $q\in C_\nu[a,b]$, *then*

a., $$n\pi \leqq \int_{x_0}^{x_n} \sqrt{q}\,dx \begin{cases} \leqq j_{\mu-1,n}+\pi/2 & \text{for} \quad -3/4 \leqq \nu < -1/2 \\ \leqq j_{\mu-1}+(n-1/2)\pi & \text{for} \quad \nu < -3/4, \end{cases}$$

b., $$(n+1/2)\pi \leqq \int_{x_0}^{x_n'} \sqrt{q}\,dx \begin{cases} \leqq j_{\mu-1,n+1} & \text{for} \quad -3/4 \leqq \nu < -1/2 \\ \leqq j_{\mu-1}+n\pi & \text{for} \quad \nu < -3/4, \end{cases}$$

c., $$\left.\begin{cases} (n-1)\pi < \\ (n-1)\pi < \\ j_\mu+(n-1)\pi \leqq \end{cases}\right\} \int_{x_0'}^{x_n} \sqrt{q}\,dx \begin{cases} < j_{\mu-1,n-1}\pi/2 & \text{for} \quad -3/4 < \nu < -1/2 \\ \leqq (n-1/2)\pi & \text{for} \quad -1 \leqq \nu \leqq -3/4 \\ \leqq (n-1/2)\pi & \text{for} \quad \nu < -1, \end{cases}$$

d., $$\left.\begin{cases} (n-1/2)\pi < \\ (n-1/2)\pi < \\ j_{-\mu}+(n-1/2)\pi \leqq \end{cases}\right\} \int_{x_0'}^{x_n'} \sqrt{q}\,dx \begin{cases} < j_{\mu-1,n} & \text{for} \quad -3/4 < \nu < -1/2 \\ \leqq n\pi & \text{for} \quad -1 \leqq \nu \leqq -3/4 \\ \leqq n\pi & \text{for} \quad \nu < -1. \end{cases}$$

PROOF. a., The left hand side coincides with that in Lemma 5. Consider the right hand side. For $-3/4\leqq\nu<-1/2$, Lemma 9 applies. For $\nu\leqq -3/4$ the interval $[x_0, x_n]$ should be decomposed as follows: $[x_0, x_n]=[x_0, x_0']\cup[x_0, x_{n-1}']\cup x_{n-1}', x_n]$. Thus Lemmas 2,6 and 3 can be applied.

b., For the left hand side inequality the decomposition $[x_0, x_n']=[x_0, x_0']\cup[x_0', x_n']$ and Lemmas 3. and 6. can be applied. The right hand side inequality in the case $-3/4\leqq\nu<-1/2$ follows from Lemma 8. For $\nu\leqq -3/4$ Lemmas 2. and 6. should be applied.

c., The left hand side inequalities follows from Lemma 4. and 5. $[x_0', x_n]=$ $=[x_0, x_1]\cup[x_1, x_n]$. For the right hand side inequalities Lemmas 6., 7. and 3. should be applied $([x_0', x_n]=[x_0', x_{n-1}']\cup[x_{n-1}', x_n])$.

d., The left hand side inequality follows from Lemma 4., 5. and 3. $([x_0', x_n']=$ $=[x_0', x_1]\cup[x_1, x_{n-1}]\cup[x_{n-1}, x_n'])$. The right hand side inequalities coincides with the assertions of Lemma 6. and Lemma 7., respectively.

Theorem 2.

If $q\in C_\nu[a,b]$, *then*

a., $$n\pi \leqq \int_{x_0}^{x_n} \sqrt{q}\,dx \begin{cases} \leqq j_{\mu-1,\left[\frac{n}{2}\right]}+j_{\mu-1,n-\left[\frac{n}{2}\right]} & \text{for} \quad -3/4 \leqq \nu < -1/2 \\ \leqq j_{\mu-1}+(n-1)\pi & \text{for} \quad \nu < -3/4, \end{cases}$$

b., $$\left.\begin{cases} n\cdot\pi < \\ n\cdot\pi < \\ j_{-\mu}+n\cdot\pi \leqq \end{cases}\right\} \int_{x_0'}^{x_n'} \sqrt{q}\,dx \begin{cases} \leqq j_{\mu-1,\left[\frac{n}{2}\right]}+j_{\mu-1,n-\left[\frac{n}{2}\right]} & \text{for} \quad -3/4 \leqq \nu < -1/2 \\ \leqq j_{\mu-1}+n\cdot\pi & \text{for} \quad -1 \leqq \nu \leqq -3/4 \\ \leqq j_{\mu-1}+n\cdot\pi & \text{for} \quad \nu < -1, \end{cases}$$

c., $$\left.\begin{cases} (n-1)\cdot\pi < \\ (n-1)\cdot\pi < \\ 2\cdot j_{-\mu}+(n-1)\cdot\pi \leqq \end{cases}\right\} \int_{x_0'}^{x_n'} \sqrt{q}\,dx \begin{cases} < j_{\mu-1,\left[\frac{n}{2}\right]}+j_{\mu-1,n-\left[\frac{n}{2}\right]} & \text{for} \quad -3/4 \leqq \nu < -1/2 \\ \leqq n\cdot\pi & \text{for} \quad -1 \leqq \nu \leqq -3/4 \\ \leqq n\cdot\pi & \text{for} \quad \nu < -1. \end{cases}$$

PROOF. Similar to the proof of Theorem 1., based on the Lemmas 2.—9.

Remark. By our method also the case $\nu > 0$ can be treated. If $\nu > 0$, inequalities (2), (3), and (4) turn to their opposite. Hence the statements and proofs of Lemmas 2.—6. remain valid with opposite relation signes. Thus we get the above mentioned results of Á. ELBERT [1].

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(Received February 21, 1980)