On the convergence of iterated Pilgerschritt transformation in nilpotent Lie groups

By N. NETZER and H. REITBERGER (Innsbruck)

The solution of the translation equation in a real Lie group can be defined as the construction of a continuous homomorphism $h: \mathbf{R} \to G$ where $h(1)=f$ is a given element of G (c.f. R. LIEDL, [4]). In order to solve this problem requiring the restriction $h/[0, 1]$ to be homotopic to a given path $\varphi: [0,1] \to G$ from the unit element to the element f, R. LIEDL has proposed the following method called Pilgerschritt transformation.

First let G be a group of real $n \times n$-matrices and $\varphi: [0, 1] \to G$ a $\mathscr{C}^1$-path with $\varphi(0=E$ (E is the unit matrix). We consider the function $\tilde{\varphi}: [0, 1] \to G$ which is defined by $\tilde{\varphi}(\tau) = X(1)$, where X is the solution of the matrix differential equation $X'(t)X(t)^{-1} = \tau\varphi'(t)\varphi(t)^{-1}$ with initial condition $X(0)=E$. $\tilde{\varphi}$ is called the Pilgerschritt transform of φ. $\tilde{\varphi}$ is a $\mathscr{C}^\infty$-function and $\tilde{\varphi}(0)=\varphi(0)=E$ and $\tilde{\varphi}(1)=\varphi(1)$. This gives rise to the sequence $\varphi, \tilde{\varphi}, \ldots, \overset{n}{\tilde{\varphi}}, \ldots$ of iterated Pilgerschritt transforms. In this case the conjecture of R. LIEDL was that under weak conditions the sequence of iterated Pilgerschritt transforms converges to a function $\overset{\infty}{\tilde{\varphi}}: [0, 1] \to G$ such that $\overset{\infty}{\tilde{\varphi}}(t) = \exp tD$, where D is a logarithm of $\varphi(1)$ having the property that φ and $\overset{\infty}{\tilde{\varphi}}$ are homotopic. The first author proved that this is true, if $\max_{t \in [0,1]} \|\varphi'(t)\|$ is small enough (c.f. N. NETZER [7]). Integration shows that $\|\varphi(t)\|$ is small, too, so this is a local theorem. A corresponding global theorem cannot be expected because in general $\log(\varphi(1))$ does not exist in the Lie algebra of G. Therefore, it is necessary to specialize the group G, if we want to have global theorems. R. Liedl proved a global theorem for abelian groups (c.f. [4]). K. KUHNERT proved a corresponding theorem for groups of matrices $(\alpha_{ij})_{i,j=1,\ldots,n}$ with $\alpha_{ii}=1$ and $\alpha_{ij}=0$ for $j<i$ (c.f. [3]). Therefore, the second author proposed to investigate nilpotent Lie groups and the first author could give the proof of a global theorem for these groups. In this paper, we shall give this first proof in an abriged form which was proposed by the second author using the universal covering group.

If G is an arbitrary connected Lie group, we have to formulate the problem in a different way because, in general, such a group is only locally isomorphic with a matrix group. It is easy to show (c.f. R. LIEDL [4]) that in a matrix group the Pilgerschritt transform of φ can be defined as follows:

Let $\pi: 0=t_0<t_1<\ldots<t_{m-1}<t_m=1$ be a partition of $[0, 1]$ and $\tau \in [0, 1]$,

$\delta_k := t_{k+1} - t_k$, $t_k^* := t_k + \tau\delta_k$ $(k=0, \dots, m-1)$, $|\pi| := \max \{\delta_k | k=0, \dots, m-1\}$. Put

$$(*) \qquad \tilde{\varphi}(\tau) := \lim_{|\pi|\to 0} \varphi(t_{m-1}^*)\varphi(t_{m-1})^{-1} \dots \varphi(t_0^*)\varphi(t_0)^{-1}.$$

This product exists in an arbitrary Lie group too as can be shown by a simple compactness argument using the local isomorphism of a real Lie group with a suitable group of matrices. Further, $\tilde{\varphi}$ is a $\mathscr{C}^\infty$-path and $\tilde{\varphi}(1) = \varphi(1)$. In this paper we will consider a real connected nilpotent Lie group of dimension n. We will start with a $\mathscr{C}^1$-path $\psi: [0, 1] \to G$ such that $\psi(0) = e$ (e is the neutral element of G) and we use formula $(*)$ for calculating the Pilgerschritt transform (for other equivalent definitions of the Pilgerschritt transform see R. Liedl [4], [5]). We shall prove that $\overset{M}{\tilde{\psi}}(t) = \exp tD$, if M is greater or equal to the degree of nilpotence of the group G.

First we investigate the formal background of this method. Let $X = \{X_1, \dots, X_n\}$ be a finite set with $X_i \neq X_j$ if $i \neq j$. Let $A(X)$ denote the free (associative non-commutative) algebra over $\mathbf{R}$ generated by X and $\hat{A}(X)$ the algebra of formal power series with indeterminates $X_1, \dots, X_n$ and real coefficients. Further let $L(X)$ be the free Lie algebra generated by X, which we shall identify with its canonical image in $A(X)$, and $\hat{L}(X)$ the closure of $L(X)$ in $\hat{A}(X)$ (c.f. N. Bourbaki [1], chap. II, § 6, 2.). Therefore, an element of $\hat{L}(X)$ has the form $\sum_{k\geqq 1} \sum_{i=1}^{m_k} a_{kj} B_{kj}$ where the a_{kj} are reals and the B_{kj} are Lie brackets of order k. In particular $m_1 = n$ and $B_{1j} = X_j$ for $j = 1, \dots, n$. The Campbell—Baker—Hausdorff-formula $H(U, V) := U + V + \frac{1}{2}[U, V] + \dots$ $(U, V \in \hat{L}(X))$ gives rise to a group law $U \circ V := H(U, V)$ on $\hat{L}(X)$ (c.f. N. Bourbaki [1], chap. II; § 6, 4.). If $U = \sum_{k\geqq 1} \sum_{j=1}^{m_k} a_{kj} B_{kj}$ and $V = \sum_{k\geqq 1} \sum_{j=1}^{m_k} b_{kj} B_{kj}$ then $U \circ V = \sum_{k\geqq 1} \sum_{j=1}^{m_k} c_{kj} B_{kj}$, with

$$c_{kj} = a_{kj} + b_{kj} + P_{kj} \times$$
$$\times (a_{11}, \dots, a_{1m_1}, \dots, a_{k-1,1}, \dots, a_{k-1,m_{k-1}}, b_{11}, \dots, b_{1m_1}, \dots, b_{k-1,1}, \dots, a_{k-1,m_{k-1}})$$

where $P_{kj} \in \mathbf{R}[Y_1, \dots, Y_{m_1+\dots+m_{k-1}}, Z_1, \dots, Z_{m_1+\dots+m_{k-1}}]$, $P_{1j} = 0$ $(j = 1, \dots, n)$, $\deg P_{kj} \leqq k$ and the order of P_{kj} in Y and the order of P_{kj} in Z are greater or equal to 1. Further $P_{kj}(Y_1, \dots, Y_{m_1+\dots+m_{k-1}}, -Y_1, \dots, -Y_{m_1+\dots+m_{k-1}}) = 0$.

The following definitions are motivated by the fact that for a Lie group G and its Lie algebra $L(G)$ the exponential map is a local diffeomorphism and that $\exp(x) \circ \exp(y) = \exp H(x, y)$, if x and y are elements of a suitable neighbourhood of $0 \in L(G)$.

Let $\varphi: [0, 1] \to \hat{L}(X): t \mapsto \sum_{k\geqq 1} \sum_{j=1}^{m_k} a_{kj}(t) B_{kj}$ be a map such that the functions $a_{kj}: [0, 1] \to \mathbf{R}$ are continuous and $\varphi(0) = 0 \in \hat{L}(X)$. If π is a partition of $[0, 1]$ as above and $\tau \in [0, 1]$ and $t \in (t_i, t_{i+1}]$, we put $\tilde{\varphi}(t, \tau, \pi) := \sum_{k\geqq 1} \sum_{j=1}^{m_k} \hat{a}_{kj}(t, \tau, \pi) B_{kj} :=$ $:= \varphi(t_i^*) \circ \varphi(t_i)^{-1} \circ \dots \circ \varphi(t_0^*) \circ \varphi(t_0)^{-1}$. Further we put $\hat{a}_{kj}(0) := 0$. If $\lim_{|\pi|\to 0} \hat{a}_{kj}(t, \tau, \pi) =:$ $=: \hat{a}_{kj}(t, \tau)$ exists for each $t, \tau \in [0, 1]$ and for each $k \geqq 1$ and for each $j \in \{1, \dots, m_k\}$

and if the functions $\hat{a}_{kj}(t,\tau)$ are continuous, we put $\tilde{a}_{kj}(\tau):=\hat{a}_{kj}(1,\tau)$ and we call the map $\tilde{\varphi}\colon[0,1]\to\hat{L}(X)\colon t\mapsto\sum_{k\geqq 1}\sum_{j=1}^{m_k}\tilde{a}_{kj}(t)B_{kj}$ the Pilgerschritt transform of φ. If $\tilde{\varphi}$ is the Pilgerschritt transform of φ, we define analogously the functions $\tilde{\varphi}(t,\tau,\pi):=\sum_{k\geqq 1}\sum_{j=1}^{m_k}\hat{\tilde{a}}_{kj}(t,\tau,\pi)B_{kj}$ resp. $\hat{\tilde{a}}_{kj}(t,\tau):=\lim_{|\pi|\to 0}\tilde{a}_{kj}(t,\tau,\pi)$ and we call $\tilde{\tilde{\varphi}}(t)=\sum_{k\geqq 1}\sum_{j=1}^{m_k}\tilde{\tilde{a}}_{kj}B_{kj}$ the twice iterated Pilgerschritt transform of φ. Analogously we use the notation $\overset{M}{\tilde{\varphi}}$ and $\overset{M}{\tilde{a}}_{kj}\,(M\in\mathbf{N})$ to denote the M-times iterated Pilgerschritt transform.

A sufficient condition for the existence of the Pilgerschritt transform is given by the following

Lemma. *If all a_{kj} are $\mathscr{C}^1$-functions, then the Pilgerschritt transform $\tilde{\varphi}$ of φ exists and the $\tilde{a}_{kj}$ are polynomial functions.*

PROOF. For $s,t\in[0,1]$ we have

$$\varphi(s)\circ\varphi(t)^{-1}=\Big(\sum_k\sum_j a_{kj}(s)B_{kj}\Big)\circ\Big(\sum_k\sum_j -a_{kj}(t)B_{kj}\Big)=$$
$$=\sum_k\sum_j\big(a_{kj}(s)-a_{kj}(t)+g_{kj}(s,t)\big)B_{kj},$$

where $g_{kj}(s,t)=P_{kj}(a_{11}(s),\dots,-a_{11}(t),\dots)$. The g_{kj} are $\mathscr{C}^1$-functions and $g_{kj}(t,t)=0$. Let π be a partition of $[0,1]$ as above and $0\leqq i\leqq m-1$.

Then

$$\varphi(t_i^*)\circ\varphi(t_i)^{-1}=\sum_k\sum_j\big(a_{kj}(t_i^*)-a_{kj}(t_i)+g_{kj}(t_i^*,t_i)\big)B_{kj}=$$
$$=\sum_k\sum_j\tau\delta_i\left(a'_{kj}(\xi_{kj}^{(i)})+\frac{\partial g_{kj}}{\partial s}(\eta_{kj}^{(i)},t_i)\right)B_{kj}=:$$
$$=:\sum_k\sum_j\tau\delta_i\alpha_{kj}^{(i)}B_{kj}\quad\text{where}\quad t_i\leqq\xi_{kj}^{(i)},\ \eta_{kj}^{(i)}\leqq t_i^*.$$

If $t\in(t_i,t_{i+1}]$, the $\hat{a}_{kj}(t,\tau,\pi)$ are given inductively by $\hat{\varphi}(0)=0=\sum_k\sum_j\hat{a}_{kj}(0,\tau,\pi)B_{kj}$

$$\hat{\varphi}(t,\tau,\pi)=\varphi(t_i^*)\circ\varphi(t_i)^{-1}\circ\hat{\varphi}(t_i,\tau,\pi)=$$
$$=\Big(\sum_k\sum_j\tau\delta_i\alpha_{kj}^{(i)}B_{kj}\Big)\circ\Big(\sum_k\sum_j\hat{a}_{kj}(t_i,\tau,\pi)B_{kj}\Big)=$$
$$=\sum_k\sum_j\big(\hat{a}_{kj}(t_i,\tau,\pi)+\tau\delta_i\alpha_{kj}^{(i)}+P_{kj}(\tau\delta_i\alpha_{11}^{(i)},\dots,\hat{a}_{11}(t_i,\tau,\pi),\dots)\big)B_{kj}=$$
$$=:\sum_k\sum_j\hat{a}_{kj}(t,\tau,\pi)B_{kj}.$$

We interprete the P_{kj} as polynomials in the indeterminates $Z_1,\dots,Z_{m_1+\dots+m_{k-1}}$ with coefficients in $\mathbf{R}[Y_1,\dots,Y_{m_1+\dots+m_{k-1}}]$ and denote the r-th homogeneous part of P_{kj} by $P_{kj}^{(r)}$. Then $\hat{\varphi}(t,\tau,\pi)=\sum_k\sum_j\big(\hat{a}_{kj}(t_i,\tau,\pi)+\tau\delta_i\big(\alpha_{kj}^{(i)}+\sum_{r=1}^{k-1}(\tau\delta_i)^{r-1}P_{kj}^{(r)}\big(\alpha_{11}^{(i)},\dots,$ $\dots,\hat{a}_{11}(t_i,\tau,\pi),\dots\big)\big)B_{kj}=\sum_k\sum_j\big(\hat{a}_{kj}(t_i,\tau,\pi)+\tau\delta_i\big(\alpha_{kj}^{(i)}+P_{kj}^{(1)}(\alpha_{11}^{(i)},\dots,\hat{a}_{11}(t_i,\tau,\pi),\dots)+$

$+o(\delta_i))B_{kj}=\sum_k\sum_j \hat{a}_{kj}(t,\tau,\pi)B_{kj}$, where $\lim_{h\to 0}\frac{o(h)}{h}=0$. Let k_0 be a natural number. We show that by

$$\hat{a}_{kj}(0,\tau,\pi)=0,$$

$$\hat{a}_{kj}(t_{i+1},\tau,\pi)=$$

$$=\hat{a}_{kj}(t_i,\tau,\pi)+\tau\delta_i\left(\alpha_{kj}^{(i)}+\sum_{r=1}^{k-1}(\tau\delta_i)^{r-1}P_{kj}^{(i)}\left(\alpha_{11}^{(i)},\ldots,\hat{a}_{11}(t_i,\tau,\pi),\ldots\right)\right)=:$$

$$=:\hat{a}_{kj}(t_i,\tau,\pi)+\tau\delta_i f_{kj}^{(i)}(t_i,\hat{a}_{11},\ldots,\hat{a}_{k-1,m_{k-1}}),$$

$$(1\leqq k\leqq k_0,\ 1\leqq j\leqq m_k,\ 0\leqq i\leqq m-1)$$

there is given a one-step-method (c.f. R. GRIGORIEFF [2]), which solves the initial value problem:

$$v_{kj}'(t)=\tau\left(a_{kj}'(t)+\frac{\partial g_{kj}}{\partial s}(t,t)+P_{kj}^{(1)}\left(a_{11}'(t)+\frac{\partial g_{11}}{\partial s}(t,t),\ldots,v_{11}(t),\ldots\right)\right)=:$$

$(**)$

$$=:\left(a_{kj}'(t)+f_{kj}(t,v_{11},\ldots,v_{k-1,m_{k-1}})\right)$$

$$v_{kj}(0)=0.$$

Consistency follows from the fact that

$$\max_{i\in\{0,\ldots,m-1\}}\tau\left|f_{kj}^{(i)}(t_i,v_{11}(t_i),\ldots)-f_{kj}(t_i,v_{11}(t_i),\ldots)\right|=$$

$$=\max_{i\in\{0,\ldots,m-1\}}\tau\Bigg|\alpha_{kj}^{(i)}+\sum_{r=1}^{k-1}(\tau\delta_i)^{r-1}P_{kj}^{(r)}\left(\alpha_{11}^{(i)},\ldots,v_{11}(t_i),\ldots\right)-$$

$$-\left(a_{kj}'(t_i)+\frac{\partial g_{kj}}{\partial s}(t_i,t_i)+P_{kj}^{(1)}\left(a_{11}'(t_i)+\frac{\partial g_{11}}{\partial s}(t_i,t_i),\ldots,v_{11}(t_i),\ldots\right)\right)\Bigg|=$$

$$=\max_{i\in\{0,\ldots,m-1\}}\tau\Bigg|\alpha_{kj}'(\xi_{kj}^{(i)})-a_{kj}'(t_i)+\frac{\partial g_{kj}}{\partial s}(\eta_{kj}^{(i)},t_i)-\frac{\partial g_{kj}}{\partial s}(t_i,t_i)+$$

$$+P_{kj}^{(1)}\left(a_{11}'(\xi_{kj}^{(i)})+\frac{\partial g_{11}}{\partial s}(\eta_{11}^{(i)},t_i),\ldots,v_{11}(t_i),\ldots\right)-$$

$$-P_{kj}^{(1)}\left(a_{11}'(t_i)+\frac{\partial g_{11}}{\partial s}(t_i,t_i),\ldots,v_{11}(t_i),\ldots\right)+o(\delta_i)\Bigg|$$

converges to 0, if $|\pi|\to 0$ (c.f. R. D. GRIGORIEFF [2], 1.1. (11)). Convergence will be proved in the following way (c.f. [2], 2.2.): The $P_{kj}^{(r)}$ are polynomials and therefore there exist an $\varepsilon>0$ and a Lipschitz constant $L>0$ such that $|\pi|<\varepsilon$ implies

$$|f_{kj}^{(i)}(t_i,y_{11},\ldots,y_{k-1,m_{k-1}})-f_{kj}^{(i)}(t_i,z_{11},\ldots,z_{k-1,m_{k-1}})|=$$

$$=\tau\left|\alpha_{kj}^{(i)}+\sum_{r=1}^{k-1}(\tau\delta_i)^{r-1}P_{kj}^{(r)}(\alpha_{11}^{(i)},\ldots,y_{11},\ldots)-\alpha_{kj}^{(i)}-\sum_{r=1}^{k-1}(\tau\delta_i)^{r-1}P_{kj}^{(r)}(\alpha_{11}^{(i)},\ldots,z_{11},\ldots)\right|\leqq$$

$$\leqq L\|(y_{11}-z_{11}),\ldots,(y_{k-1,m_{k-1}}-z_{k-1,m_{k-1}})\|.$$

If $k \leqq k_0$, the solution does not depend on the choice of k_0. This proves the existence of $\tilde{\varphi}$.

The initial value problem $(**)$ has the solution

$$v_{kj}(t) = \tau \left(\int_0^t a'_{kj}(\xi)\, d\xi + \int_0^t \frac{\partial g_{kj}}{\partial s}(\xi, \xi)\, d\xi + \right.$$

$$\left. + \int_0^t P_{kj}^{(1)} \left(a'_{11}(\xi) + \frac{\partial g_{11}}{\partial s}(\xi, \xi), \ldots, v_{11}(\xi), \ldots \right) d\xi \right)$$

and therefore

$$\tilde{a}_{kj}(\tau) = \tau \left(a_{kj}(1) + \int_0^1 \frac{\partial g_{kj}}{\partial s}(\xi, \xi)\, d\xi + \right.$$

$$\left. + \int_0^1 P_{kj}^{(1)} \left(a'_{11}(\xi) + \frac{\partial g_{11}}{\partial s}(\xi, \xi), \ldots, \tilde{a}_{11}(\xi), \ldots \right) d\xi \right).$$

Induction on k shows that the $\tilde{a}_{kj}$ are polynomials. ∎

To prove that the iterated Pilgerschritt transformation is a method which solves the translation equation in a nilpotent Lie group, we need the following

Proposition. *If* $M \in \mathbf{N}$ *and* $k \leqq M$, *then* $\overset{M}{\tilde{a}}_{kj}(t) = t a_{kj}(1)$ *for each* $j \in \{1, \ldots, m_k\}$.

PROOF. The proof is by induction on M. Since $P_{1j} = 0$ for $1 \leqq j \leqq n$, we have $\frac{\partial g_{1j}}{\partial s} = 0$ and therefore $\tilde{a}_{1j}(t) = t a_{1j}(1)$. We assume that

$$\overset{M-1}{\tilde{\varphi}}(t) = tA + \sum_{k \geqq M} \sum_{j=1}^{m_k} \overset{M-1}{\tilde{a}_k}(t) B_{kj}, \quad \text{where} \quad A = \sum_{k=1}^{M-1} \sum_{j=1}^{m_k} a_{kj}(1) B_{kj}.$$

If π is a partition of $[0, 1]$ as above, we have:

$$\overset{M-1}{\tilde{\varphi}}(t_i^*) \circ \overset{M-1}{\tilde{\varphi}}(t_i)^{-1} = t_i^* A - t_i A + \sum_{j=1}^{m_M} \left(\overset{M-1}{\tilde{a}_{Mj}}(t_i^*) - \overset{M-1}{\tilde{a}_{mj}}(t_i) \right) B_{Mj} +$$

$$+ \text{terms with Lie brackets of order} \geqq M+1 = \tau \delta_i A + \sum_{j=1}^{m_M} \tau \delta_i \left(\overset{M-1}{\tilde{a}_{Mj}} \right)(\xi_{Mj}^{(i)}) B_{Mj} +$$

$+$ terms with Lie brackets of order $\geqq M+1$ $(t_{i-1} \leqq \xi_{Mj}^{(i)} \leqq t_i)$.

This implies

$$\overset{\widehat{M-1}}{\tilde{\varphi}}(1, \tau, \pi) = \tau A + \sum_{j=1}^{m_M} \sum_{i=1}^{m} \tau \delta_i \left(\overset{M-1}{\tilde{a}_{mj}} \right)'(\xi_{Mj}^{(i)}) B_{mj} +$$

$$+ \text{terms with Lie brackets of order} \geqq M+1$$

and

$$\overset{M}{\tilde{a}}_{Mj}(\tau) = \lim_{|\pi|\to 0} \sum_{i=1}^{m} \tau\delta_i (\overset{M-1}{\tilde{a}}_{Mj})'(\xi_{Mj}^{(i)}) = \tau \int_0^1 (\overset{M-1}{\tilde{a}}_{Mj})'(\xi)\, d\xi = \tau \overset{M-1}{\tilde{a}}_{Mj}(1).$$

Since $\hat{\varphi}(1, \pi, 1) = \varphi(1)$ we have $\overset{r}{\tilde{a}}_{kj}(1) = a_{kj}(1)$ for each $j, k, r \in \mathbf{N}$.

Now we can prove the announced global theorem for nilpotent Lie groups.

Theorem. *Let G be a real connected Lie group of dimension n and order of nilpotence N. Let $\psi: [0, 1] \to G$ be a $\mathscr{C}^1$-path with $\psi(0) = e$. Then there exists an element D of the Lie algebra $L(G)$ of G such that the iterated Pilgerschritt transform $\overset{M}{\tilde{\psi}}(t) = \exp(tD)/[0, 1]$ whenever $M \geqq N$. Further, $\exp(tD)/[0, 1]$ is homotopic to ψ.*

Proof. Let $(A_1, \ldots, A_n)$ be a basis of $L(G)$. By Φ we denote the homomorphism $\Phi: L(X) \to L(G): X_i \mapsto A_i$. A group law on $L(G)$ is given by $\Phi(u) \circ \Phi(v) = \Phi(H(u, v))$ $(u, v \in L(X))$. The group $(L(G), \circ)$ is the universal covering group of G with the covering projection $\exp: L(G) \to G$ which is a group homomorphism (c.f. J. Tits [9], 2.4.). Let $\psi^*: [0, 1] \to L(G): t \mapsto \sum_{j=1}^{n} a_j(t) A_j$ be the lifting of ψ with $\psi^*(0) = 0 \in L(G)$. Since exp is a local diffeomorphism, ψ^* is a $\mathscr{C}^1$-path. We put $\varphi: [0, 1] \to \hat{L}(X): t \mapsto \sum_{j=1}^{n} a_j(t) X_j$. Then $\overset{M}{\tilde{\psi}}{}^*(t) = \sum_{k=1}^{N} \sum_{j=1}^{m_k} \overset{M}{\tilde{a}}_j(t) \Phi(B_{kj})$ and therefore $\overset{M}{\tilde{\psi}}{}^*(t) = \sum_{j=1}^{n} t a_j(1) A_j$ if $M \geqq N$. Since $\exp: L(G) \to G$ is a continuous group homomorphism we have $\tilde{\psi} = \exp \tilde{\psi}^*$ and therefore $\overset{M}{\tilde{\psi}}{}^*$ is the lifting of $\overset{M}{\tilde{\psi}}$. This implies $\overset{M}{\tilde{\psi}}(t) = \exp\left(\sum_{j=1}^{n} t a_j(1) A_j\right)$ for $M \geqq N$. For the homotopy see Liedl [4].

References

[1] N. Bourbaki, Groupes et algèbres de Lie, Hermann, *Paris* 1971/72.

[2] R. D. Grigorieff, Numerik gewöhnlicher Differentialgleichungen I, Teubner Studienbücher, *Stuttgart* 1972.

[3] K. Kuhnert, Die Konvergenz des Pilgerschrittverfahrens für unipotente und auflösbare lineare Gruppen, *Berichte der math.-stat. Sektion im Forschungszentrum Graz, Ber. Nr.* **87** (1978).

[4] R. Liedl, Über eine Methode zur Lösung der Translationsgleichung, *Berichte der math.-stat. Sektion Forschungszentrum Graz, Ber. Nr.* **84** (1978).

[5] R. Liedl, Non-commutative calculus and Pilgerschritt transformation, *Zeszyty Nauk. Uniw. Jagiello. Prace Mat. (to appear).*

[6] N. Netzer, Differentialgleichungen im Zusammenhang mit der Pilgerschritt-Transformation, *Berichte der math.-stat. Sektion im Forschungszentrum Graz, Ber. Nr.* **86** (1978).

[7] N. Netzer, On the convergence of iterated Pilgerschritt-transform, *Zeszyty Nauk. Uniw. Jagiello. Prace Mat. (to appear).*

[8] H. Reitberger, Pilgerschritt-Transformation und verallgemeinerte Liegruppen. *Berichte der math.-stat. Sektion im Forschungszentrum Graz, Ber. Nr.* **85** (1978).

[9] J. Tits, Liesche Gruppen und Algebren, *Math, Inst. Bonn* 1965.

INSTITUT FÜR MATHEMATIK
UNIVERSITÄT INNSBRUCK
INNRAIN 52
A-6020 INNSBRUCK/AUSTRIA

(Received November 29, 1979)