Alternative loop rings

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§ 1. Introduction

Throughout this paper, R will denote a commutative (associative) ring with identity süch that $2x=0$ implies that $x\neq 0$ and L will be a *loop*; that is, a set on which there is defined a closed binary operation $(g, h)\mapsto gh$ relative to which there is a two-sided identity element and such that the left and right translation maps $R(x)\colon g\mapsto gx$ and $L(x)\colon g\mapsto xg$ are one-to-one maps of L onto L. In particular, both left and right cancellation laws hold in L. The *loop ring* RL is the free (left) R-module with the elements of L as a basis and distributive multiplication induced by that of L. Thus if $x=\sum_{g\in L}\alpha_g g$ and $y=\sum_g \beta_g g$, $\alpha_g, \beta_g\in R$ are elements of RL we have

$$x=y \quad \text{if and only if} \quad \alpha_g=\beta_g \quad \text{for all} \quad g\in L$$

$$x+y=\sum_g (\alpha_g+\beta_g)g \quad \text{and}$$

$$xy=\sum_g \Big(\sum_{hk=g} \alpha_h\beta_k\Big)g$$

The group ring is an object about which much is known and of great interest from many different points of view (see for example the books by PASSMAN [9] or SEHGAL [12]). On the other hand, the literature on non-associative loop rings is sparce. There is a semi-simplicity result in a 1944 paper by R. H. BRUCK [1] and a brief mention of loop rings in another of Bruck's papers about loops two years later [2], but we are unaware of any loop ring research since 1955 when a most interesting article by LOWELL—PAIGE [8] appeared. Here it is proven that if a commutative loop algebra over a field of characteristic different from 2 satisfies even the mild power associative identity $x^2\cdot x^2=x^3\cdot x$ then it is necessarily an associative group algebra. The discovery that the loop of units in the Cayley numbers has an alternative (and so power associative) loop ring which is not associative showed that the commutativity assumption by Paige is a vital ingredient in his strong theorem and encouraged us to consider further the situation in which alternative loop rings arise. In the first section of this paper, we exhibit a number of loops whose loop rings are alternative and completely classify those which are Moufang of "$M(G, 2)$ type". Then in Section 2 we find some properties necessarily shared by loops which have alternative loop rings and discuss which of these are also sufficient.

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§2. Examples

An *alternative ring* is one in which $yx \cdot x = yx^2$ and $x \cdot xy = x^2 y$ are identities. We refer the reader to the book by Schafer [11] for many of the properties of alternative rings. Among these is the fact that they satisfy the *Moufang identities*

$$xy \cdot zx = (x \cdot yz)x = x(yz \cdot x) \tag{1}$$

$$(xy \cdot x)z = x(y \cdot xz) \tag{2}$$

$$x(y \cdot zy) = (xy \cdot z)y \tag{3}$$

Since a loop is naturally embedded in any of its loop rings, if RL is alternative, L will also satisfy these Moufang identities. In fact, these identities are known to be equivalent in a loop and define the class of *Moufang loops*. One important feature of Moufang loops is their *diassociativity* (the subloop generated by any pair of elements is a group) so that, for example, the second equality in equation (1) above is unnecessary. Many Moufang loops are formed from groups in the following way (see Chein [4]):

(i) The set L is the disjoint union of a non-abelian group G and the set Gu, where u is an indeterminate.
(ii) G has an involution $g \mapsto g^*$.
(iii) Multiplication in L is given by the rules

$$g \cdot hu = hg \cdot u$$

$$gu \cdot h = gh^* \cdot u$$

$$gu \cdot hu = g_0 h^* g \quad \text{for all} \quad h, g, \in G,$$

where g_0 is an element in the centre of G which is fixed by the involution.

If G is any non-abelian group, $g^* = g^{-1}$, and g_0 is the identity of G, then the corresponding Moufang loop L is denoted $M(G, 2)$. That L is not a group is assured by the requirement that G be non-abelian. The loop of units in the Cayley numbers can be constructed by the above process taking G to be the quaternion group, the involution to be the inverse map again, and g_0 to be the generator of the centre of G. In addition, any of the loops obtained by Chein's Theorem 2′ construction with $k=2$ [4, p. 24] arise by the process described. All loops which can be constructed as above we will say are of "$M(G, 2)$ type". There is a simple test for determining which of these loops have alternative loop rings.

Theorem 1. *Let L be a loop of $M(G, 2)$ type and R be a commutative ring* such that $2x = 0 \Rightarrow x = 0$. *Then RL is alternative if and only if $g + g^*$ is in the centre of the group ring RG for all $g \in G$.*

PROOF. First of all, the involution on G extends to a map on RG, $x=\sum\alpha_g g\mapsto x^*:=\sum\alpha_g g^*$, which is in fact a ring involution. Then observe that an element of RL has a unique expression as $x+yu$ with $x, y\in RG$ and that multiplication in RL is given by

$$(x+yu)(a+bu) = (xa+g_0 b^* y)+(bx+ya^*)u.$$

For $X=x+yu$ and $A=a+bu$, it is straightforward to check that

$$XA\cdot A-XA^2 = g_0([b^*b, x]+[b^*y, a+a^*])+(b[x, a+a^*]+g_0[b, b^*]y+g_0[b^*b, y])u$$

and that

$$A\cdot AX-A^2X = g_0([x, b^*b]+[a+a^*, y^*b])+(b[x^*, a+a^*]+g_0[bb^*, y]+g_0 y[b, b^*u])$$

where $[u, v]=uv-vu$ is the ring commutator of u and v. It follows readily that the left and right alternative laws are equivalent for this kind of loop ring and that they will hold precisely when $x+x^*$ and xx^* are central in RG and $xx^*=x^*x$, for all $x\in RG$. These conditions clearly imply that $g+g^*$ is in the centre of RG for all $g\in G$. Conversely, if this condition is satisfied then certainly $gh^*+hg^*=(gh^*)+(gh^*)^*$ is central for all $g, h\in G$. Since $2x=0$ implies $x=0$ in R, we see also that gg^* must be central and hence that xx^* will be central. Also x and x^* must commute since xx^*-x^*x is a linear combination of the expressions gg^*-g^*g and $gh^*+hg^*-g^*h-h^*g$, for $h\neq g\in G$. The former is 0 because $gg^*=g(g+g^*)-g^2=(g+g^*)g-g^2=g^*g$ and the latter is $g(h+h^*)-gh+hg^*-(g+g^*)h+gh-h^*g=(h+h^*)g+hg^*-h(g+g^*)-h^*g$ which is also equal to 0.

Corollary 1. *Suppose $L=M(G,2)$ for some (non-abelian) group G and R is commutative without 2-torsion. Then RL is alternative if and only if G is the direct product of the group of quaternions with an abelian group of exponent 2.*

PROOF. In this situation $g^*=g^{-1}$ and we must determine the groups for which $g+g^{-1}$ is always in the centre of the group ring. It is well known (see for example [9, p. 113]) that the centre of RG is spanned by the class sums, a class sum being the sum of the elements in a conjugacy class of G. Thus $g+g^{-1}$ is in the centre of RG for all g if and only if G has the property that $h^{-1}gh\in\{g, g^{-1}\}$ for all g and h in G. Obviously this forces G to be Hamiltonian and hence the direct product of the quaternions, an abelian group of exponent 2, and an abelian group in which every element has odd order. In our situation, this third factor cannot occur. Conversely, it is easy to see that $g+g^{-1}$ is central for any quaternion g since the conjugacy classes in the group of quaternions are either singletons or sets of the form $\{g, g^{-1}\}$.

Corollary 2. *Any loop ring of the Cayley loop is alternative.*

PROOF. The realization of the Cayley loop as a loop of $M(G,2)$ type has already been described and we have just noted that the quaternion group has the desired property with respect to the involution $g\mapsto g^{-1}$.

This corollary actually implies that any loop ring of a Hamiltonian Moufang loop is alternative for this kind of loop is just a product of the Cayley loop with an abelian group [3, p. 87—88] and hence has an alternative loop ring because of

Proposition 2. *The loop rings of a loop L are alternative if and only if L satisfies two conditions:*

A. *If $g, h, k \in L$ associate in some order then they associate in all orders.*

B. *If $g, h, k \in L$ do not associate, then $gh \cdot k = g \cdot kh = h \cdot gk$.*

In particular a direct product of loops will yield loop rings which are alternative but not associative if and only if precisely one of the loops has properties A and B while the remaining loops are abelian groups.

PROOF. We have already noted that if any loop ring of a loop L is alternative, then L is Moufang. Property A is known to hold for Moufang loops because actually, the subloop generated by three elements which associate in some order is a group [3, p. 117]. To obtain B, let g, h and k be any elements in L which do not associate and let x and y be the loop ring elements $h+k$ and $1+g$ respectively. Since $yx \cdot x = yx^2$ we see that $gh \cdot k + gk \cdot h = g \cdot hk + g \cdot kh$. By the linear independence of the loop elements in a loop ring it follows that $gh \cdot k = g \cdot kh$. Considering $x \cdot xy = x^2 y$ with $x = g+h$ and $y = 1+k$, we obtain also $gh \cdot k = h \cdot gk$. Conversely, assuming that L is a loop with properties A and B, note first that if h and k commute (for instance if $h=k$) then g, h and k must associate in all orders; otherwise, $g \cdot kh = g \cdot kh = gh \cdot k$. Now let R be any commutative ring and let $x = \sum \alpha_g g$, $y = \sum \beta_g g$ be elements of RL. Consider

$$yx.x - yx^2 = \sum_g \beta_g \big(\sum_{h,k} \alpha_h \alpha_k (gh \cdot k - g \cdot hk) \big)$$

where we can assume that the inner sum is taken over just those h and k for which $gh \cdot k - g \cdot hk \neq 0$, implying in particular that $h \neq k$ by our preliminary remark. But for a fixed g and a pair of distinct elements h and k, the inner sum will contain the expression $gh \cdot k - g \cdot hk + gk \cdot h - g \cdot kh$. Assuming $gh \cdot k - g \cdot hk \neq 0$, then also $gk \cdot h - g \cdot kh \neq 0$ by A and so the expression is 0 by B. Thus $yx \cdot x = yx^2$. The left alternative law in RL follows in a similar fashion.

We prove the last statement of the proposition for direct products of two loops, the general case being an easy induction. It is straightforward to check that the direct product of an abelian group and a loop satisfying A and B is a loop with the same two properties. On the other hand, if $L_1 \times L_2$ satisfies A and B then certainly both L_1 and L_2 do also. Since $L_1 \times L_2$ is not associative we may assume that L_1 is not associative and show why L_2 must be an abelian group. In fact we have only to show that L_2 is abelian because of our earlier observation that if two elements commute they necessarily associate with any third. For this, we simply let g, h and k be three elements of L_1 which do not associate, a and b any two elements of L_2 and notice that since $(g, 1)$, (h, a) and (k, b) do not associate in $L_1 \times L_2$, it must be that $(g, 1)(h, a) \cdot (k, b) = (g, 1) \cdot (k, b)(h, a)$.

In closing this section we remark that no Hamiltonian Moufang loop which is not a group is $M(G, 2)$ for any group G since in $M(G, 2)$ every minimal set of generators contains an element of order 2. Thus there is no overlap in our examples.

§3. What Loops Arise?

If g, h and k are elements of a loop L then the *associator* (g, h, k) and *commutator* (g, h) are defined by

$$gh \cdot k = (g \cdot hk)(g, h, k)$$

and

$$gh = (hg)(g, h).$$

The *associator* and *commutator subloops* are those subloops generated by all associators and all commutators respectively. The set $\{(g, h, k)|k \in L\}$ is denoted by (g, h, L) and the meaning of (g, L) is clear too. The *nucleus* $N(L)$ and *centre* $Z(L)$ of L are the subgroups of L

$$N(L) = \{g \in L|(g, h, k) = (h, g, k) = (h, k, g) = 1 \text{ for all } h, k \in L\}$$

$$Z(L) = \{g \in N(L)|(g, h) = 1 \text{ for all } h \in L\}.$$

In any ring all of the above definitions have obvious analogues. We use $[x, y, z]$ to denote the *(ring) associator* $xy \cdot z - x \cdot yz$ and $[x, y] = xy - yx$ for the *(ring) commutator*.

Theorem 3. *Suppose L is a loop (but not a group) which has an alternative loop ring RL, R a commutative ring without 2-torsion. Then*

(i) $g^2 \in N(L)$ *for all* $g \in L$.
(ii) $N(L) = Z(L)$.
(iii) For $g, h \in L$, $(g, h) = 1$ *if and only if* $(g, h, L) = 1$.
(iv) If $g, h, k \in L$ *and* $(g, h, k) \neq 1$, *then* $(g, h, k) = (g, h) = (h, k) = (g, k)$ *is a central element of order* 2.
(v) *The commutator and associator subloops are equal subgroups of order* 2 *contained in* $Z(L)$.

Proof. The reader is reminded that L is necessarily Moufang so that if three elements in L associate in any order, then they generate a subgroup. We use this fact, Proposition 2, and the linear independence of the loop elements in a loop ring implicitly and freely in what follows.

(i) The alternative ring RL satisfies the linearization of the Moufang identity $xy \cdot zx = (x \cdot yz)x$; namely,

$$xy \cdot zw + wy \cdot zx = (x \cdot yz)w + (w \cdot yz)x.$$

Setting $x = y = g$, $z = h$, $w = k$ with $g, h, k \in L$, we obtain $g^2 \cdot hk + kg \cdot hg = g^2h \cdot k + (k \cdot gh)g$. Assume g^2, h and k do not associate. Then $g^2h \cdot k = kg \cdot hg$. Also, neither triple g, h, k nor g, h, kg can associate, the latter because $gh \cdot kg = g(h \cdot kg)$ would imply $g(hk \cdot g) = g(h \cdot kg)$ by (1) and hence $hk \cdot g = h \cdot kg$ upon cancellation. Hence $kh \cdot g^2 = (kh \cdot g)g = (h \cdot kg)g = kg \cdot hg = g^2h \cdot k = g^2 \cdot kh = kg^2 \cdot h = k \cdot hg^2$; i.e. k, h and g^2 do associate. (Moufang loops in which all squares are in the nucleaus have been identified by Chein and Robinson [5] as precisely the *extra loops* of F. Fenyves [6]; that is, those loops which satisfy any one of the three equivalent identities $(xy \cdot z)x = x(y \cdot zx)$, $yx \cdot zx = (y \cdot xz)x$ and $xy \cdot xz = x(yx \cdot z)$. We also refer the reader to the paper by D. A. Robinson in these *Publicationes* [10] in which a holomorphy theory for extra loops is developed.)

(ii) First we note that the nucleus of a Moufang loop is normal [3, p. 114], that L/N is then a dissociative loop of exponent 2, hence commutative, and hence a group [6]. Also it is clear that the nucleus of L is contained in the nucleus of RL. Now let $n\in N(L)$ and let g, h, k be any three elements of L which do not associate. Then $gh\cdot k=n_1(g\cdot hk)$, $1\neq n_1\in N(L)$ (since L/N is a group) and $gn=(ng)n_2$ for some $n_2\in N(L)$ (since L/N is commutative) and so the (ring) associator $[g, h, k]=$ $=(n_1-1)g\cdot hk$ and the (ring) commutator $[g, n]=ng(n_2-1)$. Now Kleinfeld has shown that in any alternative ring, $[x, n][x, y, z]=0$, where n is in the nucleus but x, y, z are arbitrary [7, p. 132]. Making the obvious substitutions, we obtain here that $(ng(n_2-1))((n_1-1)g\cdot hk)=0$. Since n_2-1 and n_1-1 are in the nucleus of RL, we have $ng((n_2-1)(n_1-1))g\cdot hk=0$ and then $(n_2-1)(n_1-1)=0$ since ng and $g\cdot hk$ are invertible in RL (being elements of the loop L) and for any y and invertible x in an alternative ring, it is true that $yx\cdot x^{-1}=x^{-1}\cdot xy=y$. Thus $n_1+n_2=1+n_1n_2$ and since $n_1\neq 1$, $n_2=1$ and $gn=ng$. So n commutes with all non-nuclear elements and is therefore central because the complement of any proper subloop of a diassociative loop generates the loop.

(iii) We noted in the proof of Proposition 2 that $(g, h)=1$ implies $(g, h, L)=1$. For the converse we suppose that $(g, h, L)=1$, equivalently that $[g, h, RL]=0$ and conclude as does Kleinfeld [7, p. 133] that $[g, gh]$ is in the nucleus of RL. But writing $gh\cdot g=ng\cdot gh=ng^2h$, $n\in N(L)$, we have $[g, gh]=(1-n)g^2h$ and so $[g^2h, x, y]=[ng^2h, x, y]$ for all $x, y\in RL$. Since n and g^2 are in $N(L)=Z(L)$, it follows that $[h, x, y]=n[h, x, y]$ for all $x, y\in RL$. If $h\in N(L)=Z(L)$, obviously $(g, h)=1$; otherwise, choose $x, y\in L$ so that $[h, x, y]\neq 0$ and then as before obtain $[h, x, y]=(1-n')hx\cdot y$ with $1\neq n'\in N(L)$. Then we see that $(1-n')hx\cdot y=$ $=n(1-n')hx\cdot y$ and thus $1-n'=n(1-n')$; i.e., $1+nn'=n+n'$ and so $n=1=(g,h)$.

(iv) Suppose g, h and k do not associate. Then $(g, h, k)=n\in N(L)=Z(L)$ so we can write $gh\cdot k=n(g\cdot hk)$. But $gh\cdot k=g\cdot kh=kg\cdot h$ and $g\cdot hk=gk\cdot h$ and so $kg\cdot h=n(gk\cdot h)=ngk\cdot h$ and $kg=ngk$. Also, because all squares are in $N(L)=Z(L)$, $g^2k=kg^2=ngk\cdot g=ng\cdot kg=ng\cdot ngk=(ng)^2k=n^2g^2k$ and $n^2=1$. Since $kg=ngk$, it now follows that $gk=nkg=kgn$ so that $n=(g, k)$. Now $hg\cdot k=g\cdot hk$ and $h\cdot gk=gh\cdot k$ and so $(h, g, k)=(g, h, k)$. The above argument now yields $(g, h, k)=(h, k)$. Similarly $(g, h)=(g, k, h)=(g, h, k)$.

(v) Part (iii) shows that the associator and commutator subloops are equal and part (iv) shows that this subloop is a central subgroup of exponent 2. Therefore to establish (v), it is enough to show that if two associators are not 1, then they are equal. We will rely heavily now on two consequences of (iv); firstly, that the associator of three elements is independent of the order in which those elements appear and secondly, that if two associators, neither equal to 1, have a pair of elements in common, then they are equal (to the commutator of the common pair). Suppose two associators (g, h, k) and (g, b, c), neither 1, have just the one element g in common. It is known that in any alternative ring, the function $f(x, y, z, w)=$ $=[xy, z, w]-y[x, z, w]-[y, z, w]x$ is skew-symmetric [7]. Consider $f(b, c, g, h)=$ $=[bc, g, h]-c[b, g, h]-[c, g, h]b$ in the alternative ring RL. Each of the associators (b, g, h) and (c, g, h) has two elements in common with each of (g, h, k) and (g, b, c) so that if either of the former associators is not 1, we can easily establish

$(g, h, k)=(g, b, c)$. Also if $(bc, g, h)\neq 1$ we have $(bc, g, h)=(g, h, k)$ on the one hand, and $(bc, g, h)=(bc, g)$ on the other, but $bc\cdot g=b\cdot gc=gb\cdot c=(g\cdot bc)(g,b,c)$ says that $(bc, g)=(g, b, c)$. So again we would have $(g, b, c)=(bc, g, h)=(g, h, k)$. Thus it is possible now to assume that $[bc, g, h]=[b, g, h]=[c, g, h]=0$ and so also that $f(b, c, g, h)=0$. By skew symmetry, $0=f(b, h, g, c)=[bh, g, c]-h[b,g, c]-[h, g, c]b=[bh, g, c]-h[b, g, c]$ since we are assuming that $[h, g, c]=0$. Hence $[bh, g, c]=h[b, g, c]$. Since $[b, g, c]\neq 0$ and h is invertible in RL, we see that $[bh, g, c]\neq 0$ and so $(bh, g, c)\neq 1$. Applying the above argument to $f(h, k, g, b)$ allows us to assume also that $(bh, g, k)\neq 1$ and so that $(bh, g, k)\neq 1$ because $(b, h)\in Z(L)$. But (bh, g, c) and (bh, g, k) have two elements in common. Hence $(g, h, k)=(bh, g, k)=(bh, g, c)=(g, b, c)$. We conclude that if two associators, neither 1, have just a single element in common, then they are equal. Finally suppose that (g, h, k) and (a, b, c) are two arbitrary associators, neither equal to 1. Considering $f(g, h, k, a)=[gh, k, a]-h[g, k, a]-[h, k, a]g$ we see that each associator on the right has an element in common with (g, h, k) and (a, b, c) and so if any is not 0 we can easily obtain $(g, h, k)=(a, b, c)$. Thus we may assume that each of the associators $[gh, k, a]$, $[g, k, a]$ and $[h, k, a]$ and so $f(g, h, k, a)$ as well are zero. By skew symmetry, $0=f(g, a, h, k)=[ga, h, k]-a[g, h, k]-[a, h, k]g=[ga, h, k]-a[g, h, k]$. As before we see that $[ga, h, k]\neq 0$ and similarly $[ga, b,c]\neq 0$. Thus $(g, h, k)=(ga, h, k)=(ga, b, c)=(a, b, c)$, concluding the proof.

Any list of identities or properties which would characterize precisely those loops whose loop rings are alternative (other than the two conditions of Proposition 2) seems to require the inclusion of some form of property (iv) of this last theorem. We offer

Theorem 4. *The following are equivalent:*

(1) *L is a loop with an alternative loop ring*

(2) *L is a loop with the property that if three elements associate in some order then they associate in all orders and if g, h and k are elements of L which do not associate then $gh\cdot k=g\cdot kh=h\cdot gk$.*

(3) *L is an extra loop which satisfies the identity $((x, y, z), x)=1$ and is such that if g, h and k are elements of L which do not associate then $(g, h, k)=(g, h)=(h,k)$.*

Proof. Before commencing we suggest it interesting to observe that at present we know of no entirely loop theoretical way of establishing the equivalence of (2) and (3); that is, the proof that a loop described by the loop theoretical properties in (2) is the same as one described by those in (3) (and, for example, has a centre equal to its nucleus) relies heavily on the fact that such a loop has an alternative loop ring. The equivalence of (1) and (2) is of course part of Proposition 2 and the fact that (2) implies (3) is part of Theorem 3. To see why (3) implies (2), we use Lemma 5.5 of Bruck [3, p. 125] to note that in a Moufang loop satisfying $((x, y, z), x)=1$, the associator (x, y, z) is in the centre of the subloop generated by x, y and z and has n^{th} power equal to (x^n, y, z). If in addition then the loop is extra, all associators must have square 1. Now if g, h and k associate in some order, they associate in all orders because L is Moufang, and if they do not associate, then $(g, h, k)=n\in N(L)$ because L/N is a group when L is extra. Thus $gh\cdot k=(g\cdot hk)n=n(g\cdot hk)=ng\cdot hk=(ng)(khn)=(gn)(nkh)=g(n^2kh)=g\cdot kh$. Also if $(h, g, k)=n'\in N(L)$, then $n'=(h, g)=(g, h)^{-1}=n$ since $n^{-1}=n$. So we have also $gh\cdot k=nhg\cdot k=n(h\cdot gk)(h, g, k)=n^2h\cdot gk=h\cdot gk$.

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