Commutativity of n-torsion-free rings

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Abstract

Let m, n be fixed positive integers, not necessarily distinct, and let R be an mn-torsion-free ring with identity. Suppose that for all x, y in R, (i) $x^ny^n=y^nx^n$ and (ii) $(xy)^m-(yx)^m$ is in the center. Then R is commutative. As a corollary, it is shown that if $m=n$ above and if instead of (i) and (ii), R is assumed to satisfy the identity $(xy)^n=(yx)^n$, then R is commutative, and this recovers a recent result of Bell.

Recently, AWTAR [1] showed that if R is an $n!$-torsion-free ring with identity 1 such that for all x, y in $R, x^ny^n=y^nx^n$, then R is commutative. Subsequently BELL [2] gave an example of an n-torsion-free ring with 1 which satisfies the identity $x^ny^n=y^nx^n$ but which is *not* commutative. BELL [2] also observed that in an n-torsion-free ring with 1, the identity $(xy)^n=(yx)^n$ necessarily implies the identity $x^ny^n=y^nx^n$, and this led him to prove that an n-torsion-free ring with 1 which satisfies the *stronger* identity $(xy)^n=(yx)^n$ is commutative. Our objective is to generalize Bell's result by showing that an mn-torsion-free ring R with 1 which satisfies the *weaker* identity $x^ny^n=y^nx^n$ and for which $(xy)^m-(yx)^m$ is always in the center of R is necessarily commutative. We also give examples which show that none of the hypotheses of this theorem can be dropped.

In preparation for the proof of our main theorem, we first consider the following lemmas. We use the usual notation $[x, y]=xy-yx$. Our first lemma is well known [4; p. 221].

Lemma 1. *If* $[x, y]$ *commutes with* x, *then* $[x^k, y]=kx^{k-1}[x, y]$ *for all positive integers* k.

Lemma 2. *Let* R *be a ring with identity* 1 *and let* $x, y\in R$. *Suppose that for some positive integer* $n, x^ny=0=(x+1)^ny$. *Then necessarily* $y=0$.

PROOF. The proof of this lemma has already been given in [5], but for convenience we reproduce it. Multiply the equation $(x+1)^ny=0$ by x^{n-1} on the left and expand the binomial to get

$$0=\sum_{k=0}^{n}\binom{n}{k}x^{k+n-1}y=x^{n-1}y,$$

since $x^ny=0$. Similarly, write $x^ny=0$ in the form $(-1+(x+1))^ny=0$. Expand this and multiply by $(x+1)^{n-1}$ to get

$$0 = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(x+1)^{k+n-1}y = (-1)^n(x+1)^{n-1}y.$$

Hence we have shown $x^{n-1}y=0=(x+1)^{n-1}y$. Continue this process to obtain, eventually, $xy=0=(x+1)y$, and hence $y=0$.

We are now in a position to prove our main theorem. We state this theorem in a somewhat more general form in that we merely assume that *commutators* are mn-torsion-free (instead of R being mn-torsion-free).

Main theorem. *Let m, n be fixed positive integers, not necessarily distinct, and let R be a ring with identity* 1. *Suppose that for all x, y in R,* (i) $x^ny^n=y^nx^n$, (ii) $(xy)^m-(yx)^m$ *is in the center Z of R,* (iii) $mn[x, y]=0$ *implies* $[x, y]=0$. *Then R is commutative (and conversely).*

PROOF. Let $x\in R$ and suppose that u is a unit in R. Then, by hypothesis (ii),

$$[(xu^{-1})u]^m-[u(xu^{-1})]^m\in Z[=\text{center of } R],$$

and hence $x^m-ux^mu^{-1}$ commutes with u. Thus,

$$(x^m-ux^mu^{-1})u = u(x^m-ux^mu^{-1}),$$

and hence $x^mu-ux^m=ux^m-u^2x^mu^{-1}$. Therefore,

$$(x^mu-ux^m)u = (ux^m-u^2x^mu^{-1})u = u(x^mu-ux^m).$$

We have thus shown that

(1) $\qquad [x^m, u]$ commutes with u, $(u=\text{unit in } R,\ x\in R)$.

By hypothesis (i), $[x^{mn}, u^{mn}]=0$, and hence by (1) and Lemma 1, $mnu^{mn-1}[x^{mn}, u]=0$. Since u is a unit in R and in view of hypothesis (iii), we conclude that $[x^{mn}, u]=0$. Thus,

(2) $\qquad [x^{mn}, u]=0$ for all units u in R and all x in R.

Let a be a nilpotent element of R. Then $u=1+a$ is a unit in R, and hence by (2),

(3) $\qquad [x^{mn}, a]=0$ for all nilpotents a and all $x\in R$.

Let N denote the set of nilpotent elements of R. Suppose that u is a unit in R, and suppose $a\in N$. Then there exists a *minimal* positive integer p such that

(4) $\qquad [a^k, u]=0$ for all integers $k\geqq p$, p *minimal*.

Suppose $p>1$. Since u is a unit in R, we have by (2),

(5) $\qquad [(1+a^{p-1})^{mn}, u] = 0,$

which by (4) reduces to $mn[a^{p-1}, u]=0$. Therefore, by hypothesis (iii), $[a^{p-1}, u]=0$, which contradicts the minimality of p [see (4)]. This contradiction shows that

$p=1$, and hence by (4), $[a, u]=0$. We have thus shown that

(6) $[a, u]=0$ for all nilpotents a and all units u in R.

Next, suppose that u is a unit in R and $x \in R$. By hypothesis (ii),

$$[(x^{mn-1}u)x]^m - [x(x^{mn-1}u)]^m \in Z [= \text{center of } R],$$

and hence

(7) $$[x^{mn-1}ux^{mn}ux^{mn}ux^{mn} \dots ux^{mn}ux - (x^{mn}u)^m] \in Z.$$

Therefore, by (2) and (7), we have

$$[x^{m^2n-1}u^m x - x^{m^2n}u^m] \in Z,$$

and hence

$$x^{m^2n-1}[u^m, x] \in Z.$$

Thus, in particular,

$$x(x^{m^2n-1}[u^m, x]) = (x^{m^2n-1}[u^m, x])x,$$

and hence

(8) $$x^{m^2n-1}(x[u^m, x]-[u^m, x]x) = 0.$$

Replacing x by $x+1$ in the above argument, we see that (8) becomes

(9) $$(x+1)^{m^2n-1}(x[u^m, x]-[u^m, x]x) = 0.$$

Now, by (8), (9), and Lemma 2, we conclude that $x[u^m, x]-[u^m, x]x=0$, and hence

(10) $[u^m, x]$ commutes with x, (u=unit in R, $x \in R$).

But, by hypothesis (i), $[u^{mn}, x^{mn}]=0$, and hence by (10) and Lemma 1, $mnx^{mn-1} \cdot [u^{mn}, x]=0$. Therefore,

(11) $$x^{mn-1}[mnu^{mn}, x] = 0, \quad (u = \text{unit in } R, \ x \in R).$$

Replacing x by $x+1$ in (11), we obtain

(12) $$(x+1)^{mn-1}[mnu^{mn}, x] = 0.$$

Hence, by (11), (12), and Lemma 2, we conclude that $[mnu^{mn}, x]=0$, and hence $mn[u^{mn}, x]=0$. Therefore, by hypothesis (iii), $[u^{mn}, x]=0$. Thus,

(13) $[u^{mn}, x]=0$ for all units u in R and all $x \in R$.

Now, by a theorem of Herstein [3], hypothesis (i) implies that the commutator ideal of R is nil, and hence the *nilpotent* element $[u, x]$ commutes with u [see (6)]. Combining this fact with Lemma 1, we see that (13) yields $mnu^{mn-1}[u, x]=0$, and hence $mn[u, x]=0$ (since u is a unit in R). Thus, by hypothesis (iii), $[u, x]=0$. Therefore,

(14) $[u, x] = 0$ for all units u in R and all $x \in R$.

Let a be any nilpotent element of R. Then $u=1+a$ is a unit in R, and hence by (14),

(15) $$[a, x] = 0 \quad \text{for all nilpotents} \quad a\in R \quad \text{and all} \quad x\in R.$$

To complete the proof of our theorem, let $x\in R, y\in R$. By hypothesis (i), $[x^n, y^n]=0$. Moreover, as we have remarked above, $[x, y]$ is nilpotent (see [3]) and hence $[x, y]$ is in the center of R, by (15). Hence, by Lemma 1, $[x^n, y^n]=0$ now yields $nx^{n-1}[x, y^n]=0$ and hence $x^{n-1}[x, ny^n]=0$. Replacing x by $x+1$, we see that $(x+1)^{n-1}[x, ny^n]=0$, and hence by Lemma 2, $[x, ny^n]=0$. Thus, $n[x, y^n]=0$, and hence by hypothesis (iii), $[x, y^n]=0$. Repeating the above argument to $[x, y^n]=0$, we see that $y^{n-1}[x, y]=0=(y+1)^{n-1}[x, y]$, and hence by Lemma 2, $[x, y]=0$. This completes the proof.

It was shown in [2] that in an n-torsion-free ring with identity, the identity $(xy)^n=(yx)^n$ necessarily implies the identity $x^ny^n=y^nx^n$. Combining this fact with our main theorem (with $m=n$), we obtain the following result of BELL [2]:

Corollary. *Let R be an n-torsion-free ring with identity such that, for all x, y in R, $(xy)^n=(yx)^n$. Then R is commutative.*

We conclude with the following

Remark. Our main theorem need not be true if any of the hypotheses is deleted. To see this, first let

$$R = \left\{\begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c\in GF(4)\right\}.$$

Note that the ring R is *not* commutative. Also, R with $m=n=3$ satisfies all the hypotheses of our main theorem *except* hypothesis (ii).

Next, let

$$R_0 = \left\{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| a, b, c, d\in GF(3)\right\}.$$

Note that the ring R_0 is *not* commutative. Also,

(a) The ring R_0 with $m=n=1$ satisfies all the hypotheses of our main theorem *except* hypothesis (i).

(b) The ring R_0 with $m=n=3$ satisfies all the hypotheses of our main theorem *except* hypothesis (iii).

(c) Let N_0 be the subring of R_0 consisting of the nilpotent elements. Note that N_0 with $m=n=2$ satisfies all the hypotheses of our main theorem *except* the hypothesis that the ring has an identity.

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(Received February 20, 1980; revised form: July 29, 1980)