A characterization of two classes of Dedekind domains by properties of their modules

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Let R be a ring with identity and let M be a unital left R-module. The investigation of projective and quasi-projective covers led a number of authors to the consideration of two module-theoretical properties which we shall call "supplemented" and "amply supplemented" respectively [2, 3, 7, 8, 9]. In this note we give characterizations of two classes of Dedekind domains in terms of their supplemented and apply supplemented modules. The Dedekind domains over which there exists non-torsion supplemented modules are exactly the local ones. Among the local Dedekind domains, the complete discrete valuation rings are characterized by the property that every supplemented module is amply supplemented.

The module M is called supplemented if, given any submodule N of M, there exists a submodule S of M which is minimal with respect to the property that $M=N+S$; such S, though not unique (not even up to isomorphism), is called a minimal supplement of N. If, given any pair of submodules N and U such that $M=N+U$, there exists a minimal supplement of N which is contained in U, then M is called amply supplemented. Our notation and terminology is that of [1]. We let R-Mod denote the category of left R-modules and write ${}_RM$ if we want to emphasize that $M \in R$-Mod.

We will establish the following two theorems.

Theorem 1. *For R a Dedekind domain, the following statements are equivalent.*

(i) *R is a local ring.*

(ii) *${}_RR$ is a supplemented module.*

(iii) *R-Mod is generated by supplemented modules.*

(iv) *There exists a non-zero torsion-free supplemented R-module.*

This characterization of the local rings among the Dedekind domains can be refined: among the local domains, the complete discrete valuation rings are distinguished by closure properties of their class of supplemented modules.

Theorem 2. *For a local Dedekind domain with maximal ideal P, the following conditions are equivalent.*

(i) *R is complete in its P-adic topology.*

(ii) *Submodules of supplemented R-modules are supplemented.*

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(iii) *Extensions of supplemented R-modules by supplemented R-modules are supplemented.*

(iv) *If* $0 \to L \to M \to N \to 0$ *is an exact sequence of R-modules, then* M *is supplemented if and only if both* L *and* N *are supplemented.*

(v) *Supplemented R-modules are amply supplemented.*

If R is a ring with identity, we let $\underline{P}$ denote the set of maximal ideals of R and, for $P \in \underline{P}$, we let M_P be the set of all elements x in M such that $P^n x = 0$ for some integer $n \geqq 0$ (by convention, $P^0 = R$). The R-module B is bounded if $IB = 0$ for some non-zero ideal I of R. We will require the following structure theorem [4, Theorem 4.14].

Theorem 3. *Let* R *be a Dedekind domain which is not a field and let* M *be an R-module. Then* M *is supplemented if and only if* $M = Y \oplus \bigoplus_{P \in \underline{P}} M_P$ *where, for each* $P \in \underline{P}$, M_P *is a direct sum of a bounded and an artinian submodule, and* $Y \neq 0$ *implies* a) R *is local, and* b) Y *is a direct sum of finitely many torsion-free modules of rank one.*

This theorem readily implies our Theorem 1. For the proof of Theorem 2 we require some auxiliary lemmas. The proof of the first is straightforward and will be omitted.

Lemma 4. *If every submodule of* M *is supplemented, then* M *is amply supplemented.*

Lemma 5. *Let* R *be a local Dedekind domain with maximal ideal* P *which is complete in its P-adic topology, and let*

$$0 \to L \to M \to N \to 0$$

be an exact sequence of R-modules. Then M *is supplemented if and only if both* L *and* N *are supplemented.*

PROOF. By [6; p. 48, Theorem 20], every torsion-free R-module of finite rank is a direct sum of submodules of rank one. An application of Theorem 3 completes the proof.

The R-module M is called divisible if $rM = M$ for every non-zero element $r \in R$. One easily verifies the following result.

Lemma 6. *Let* T *be a submodule of* M *such that* M/T *is divisible. If* S *is a minimal supplement of* T, *then* S *is divisible.*

If R is a Dedekind domain, then every divisible submodule of M is a direct summand [5; p. 334, Theorem 6 (b)]. A module M is said to be reduced if the only divisible submodule is the zero-module. We are now ready to prove our second theorem.

PROOF OF THEOREM 2. In view of Lemmas 4 and 5, it suffices to show that (i) follows from each of (ii), (iii), and (v). Let K denote the quotient field of R and let $M = {}_RK \oplus {}_RK$. By Theorem 3, M is supplemented. Assume R is not complete in its P-adic topology. Then, by a theorem of Kaplansky's [6, p. 46, Theorem 19], M contains a submodule N of rank two which is indecomposable. Theorem 3 then implies N is not supplemented. Let A be a submodule of N such that N/A is torsion-free of rank one. Again, by Theorem 3, both A and N/A are supplemented. We have shown that each of (ii) and (iii) imply (i). For the remaining implication observe that N/A cannot be free so that, as R-modules, $N/A \simeq K$, by [6; p. 45]. Thus, there exists a submodule B of N such that $N/B \simeq K/R$ is divisible and, hence, a direct summand of M/B. If T is a submodule of M containing B such that T/B is a complement of N/B, then $T \cap N = B$ and

$$(*) \qquad M = N+T, \quad T \neq M.$$

Assume M were amply supplemented. Then there exists a submodule S of N which is a minimal supplement of T. But M being divisible implies M/T divisible so that, by Lemma 6, S is divisible. Since N is indecomposable we must have $S=0$. By $(*)$, this is impossible and we have derived (i) from (v), which completes the proof.

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