On some questions concerning the differential geometry of curves in n-dimensional euclidean spaces

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1. Introduction

It is well-known that the classical Frenet-theory of curves in euclidean 3-space applies only to those curves which have non-vanishing curvature at each of their points. In this way, for example, the trivial case of a line segment is not covered by the usual treatment. Noticing this incompleteness K. NOMIZU set about to extend the classical theory to a larger class of curves, to the so-called Frenet-curves [1]. A rigorous treatment of Frenet-curves including such special ones as plane curves, spherical curves and helices was elaborated by YUNG-CHOW WONG and HON-FEI LAI [2] for the 3-dimensional case.

In the present paper some general methods will be developed in order to be able to extend a considerable part of the results in [2] for curves in higher dimensional euclidean spaces. Moreover some important results of R. BISHOP [3] will be also proved here for $n>3$.

Our methods are based mainly on a simple formula which relates two different families of moving frames being adapted to the same curve.

2. On moving frames of curves in $\mathbf{R}^n$

Let $\mathbf{R}^n$ be the n-dimensional euclidean space and let us consider such a curve in $\mathbf{R}^n$ which can be given in the following arc-length representation:

$$\Gamma: \underline{x} = \underline{x}(s), \quad s \in L = [0, \lambda],$$

where λ is the total length of the curve Γ, and the vectorfunction $\underline{x}(s)$ is supposed to be of class C^∞ on L. According to this, we will mean by a curve in this paper always an oriented C^∞ [and regular] curve having a finite length λ where, of course, by a C^∞-function on a closed interval L, we mean a function which may be extended to a C^∞-function on an open interval containing L. Γ is called regular, if $\underline{x}'(s) \neq \underline{0}$. Regularity will also be supposed.

Let $E(s)$, $s \in L$ be a matrix-function of class C^∞. It will be called a *moving frame* if

$$E(s) \in SO(n) \quad \text{holds for each} \quad s \in L,$$

or in other words if the vectors

$$\underline{e}_1(s), \quad \underline{e}_2(s), \dots, \underline{e}_n(s),$$

which are the consecutive columns of the matrix $E(s)$, form a positively oriented orthonormal basis of $\mathbf{R}^n$ for each $s \in L$.

We will say that a moving frame $E(s)$ is *adapted* to a curve $\Gamma: \underline{x} = \underline{x}(s)$, $s \in L$, if

$$\underline{x}'(s) = \underline{e}_1(s)$$

holds for each $s \in L$.

Let $E(s)$ and $\tilde{E}(s)$, $s \in L$ be two different moving frames. They are called *congruent* if there exists an orthogonal matrix $R \in SO(n)$ such that $\tilde{E}(s) = RE(s)$ holds on the whole interval L. It is easy to see that if $E(s)$ and $\tilde{E}(s)$, $s \in L$, are congruent moving frames being adapted to the curves $\Gamma: \underline{x} = \underline{x}(s)$ and $\tilde{\Gamma}: \underline{x} = \tilde{\underline{x}}(s)$, $s \in L$, respectively, then there exists an orientation preserving isometry of $\mathbf{R}^n$ which carries the curve Γ into the curve $\tilde{\Gamma}$, that is $\tilde{\underline{x}}(s) = R\underline{x}(s) + \underline{a}$ holds for each $s \in L$, where $\underline{a} \in \mathbf{R}^n$.

Let $E(s)$, $s \in L$ be an arbitrary moving frame. The usual derivational formulae for the frame-vectors can be conveniently expressed in the following matrix-equation:

$$E'(s) = -E(s)C(s), \quad s \in L \tag{1}$$

where $C(s)$ is the so-called *Cartan-matrix*.

It is uniquely defined by the equation:

$$C(s) = -E^*(s)E'(s), \quad s \in L$$

where asterisk denotes transposition.

It is evident that a Cartan-matrix is always skew-symmetric. This follows immediately from the fact that the matrix $(E^*(s)E(s))'$ is identically zero and thus

$$C^*(s) = -(E'(s))^*E(s) = E^*(s)E'(s) = -C(s)$$

holds for each $s \in L$.

Notice also that congruent moving frames must have the same Cartan-matrix. In fact, let $\tilde{E}(s) = RE(s)$, $s \in L$ and $R \in SO(n)$, then

$$\tilde{C}(s) = -\tilde{E}^*(s)\tilde{E}'(s) = -(E^*(s)R^*)(RE'(s)) = -E^*(s)E'(s) = C(s)$$

holds on L since R^*R is the unit matrix of $SO(n)$.

Theorem 1. *Let $C(s)$, $s \in L$ be an arbitrary skew-symmetric matrix-function of class C^∞. Then there always exists a moving frame $E(s)$, $s \in L$ whose Cartan-matrix is $C(s)$, and all the moving frames having the same Cartan-matrix $C(s)$ are congruent.*

PROOF. The matrix-equation (1) can be considered as a linear system of differential equations for the unknown entries of $E(s)$, $s \in L$. Let the initial condition be chosen so that, at a fixed value of parameter s_0 $E(s_0) \in SO(n)$ holds. If $E = \{e^i_j \mid i, j = 1, 2, \dots, n\} \in m_n(\mathbf{R})$, where $m_n(\mathbf{R})$ denotes the set of all matrices of degree n with coefficients in $\mathbf{R}$, then $E \cdot C(s)$ is a continuous function on the closed (n^2+1)-dimensional square domain given by $s \in L$ and $|e^i_j| \leqq 1$ for

$i, j=1, 2, \ldots, n$; consequently the existence of a solution satisfying the given initial condition is assured ([4], pp. 85—86).

On the other hand for the solution

$$(E(s)E^*(s))' = (-E(s)C(s))E^*(s) + E(s)(C(s)E^*(s)) = 0$$

holds on the whole interval L, and hence $E(s) \in SO(n)$ also holds for each $s \in L$ showing that $E(s)$ is a moving frame.

According to the usual proof of the uniqueness let $E(s)$ and $\tilde{E}(s)$, $s \in L$ be two solutions of the equation (1) for which $\underline{e}_j(s_0) = \tilde{\underline{e}}_j(s_0)$ holds for $j=1, 2, \ldots, n$. It is easy to see that the derivative of the scalar-function

$$f(s) = \sum_{j=1}^{n} \langle \underline{e}_j(s), \tilde{\underline{e}}_j(s) \rangle$$

is identically zero.

So, since $f(s)=f(s_0)=n$ and $|\langle \underline{e}_j(s), \underline{e}_j(s) \rangle| \leqq 1$ we have that $E(s)=\tilde{E}(s)$ holds for each $s \in L$.

At last, following of the uniqueness, all the moving frames which are solutions of the equation (1) must be congruent.

Corollary. *A curve in* $\mathbf{R}^n$ *can be given uniquely up to an orientation preserving isometry by a prescribed skew-symmetric matrixfunction of class* C^∞.

Theorem 2. *Let* $E(s)$ *and* $\tilde{E}(s)$, $s \in L$ *be two different moving frames and denote by* $C(s)$ *and* $\tilde{C}(s)$ *their Cartan-matrices, respectively. Then the following matrix-equation holds on the whole interval* L:

$$\tilde{C}(s) = A'(s)A^*(s) + A(s)C(s)A^*(s), \tag{2}$$

where the matrix-function $A(s)$ *expressing the unique* C^∞ *transfromation between the given moving frames is defined by*

$$A(s) = \tilde{E}^*(s)E(s), \quad s \in L. \tag{3}$$

Proof. It is enough to give a short verification of (2). From (3) we get that $\tilde{E}^*(s)=A(s)E^*(s)$ and $\tilde{E}(s)=E(s)A^*(s)$ hold for each $s \in L$.

On the other hand we will use the identity $(A(s)A^*(s))'=0$, $s \in L$, and also the very definitions of the Cartan-matrices $C(s)$ and $\tilde{C}(s)$. Thus

$$\tilde{C}(s) = -\tilde{E}^*(s)\tilde{E}'(s) = -A(s)E^*(s)(E'(s)A^*(s) + E(s)A^{*\prime}(s)) =$$

$$= A(s)C(s)A^*(s) + A'(s)A^*(s)$$

identically holds on L.

Remarks. Let $E(s)$, $s \in L$ be an arbitrary moving frame which is adapted to a curve Γ. It is easy to see that a matrix-function $\tilde{E}(s)$, $s \in L$ given in the form $\tilde{E}(s)=E(s)A^*(s)$, $s \in L$ will be also a moving frame of the same curve Γ if and only if the following conditions are satisfied for the C^∞ matrix-function $A(s)$, $s \in L$:

(i) $A(s) \in SO(n)$;

(ii) for entries in the first row and first column of $A(s)$ $a_{11}(s)=1$ and $a_{1j}(s)=$ $=a_{i1}(s)=0$ hold for $i, j \neq 1$ and for each $s \in L$.

We will call here a C^∞ matrix-function with the properties (i) and (ii) simply a *frame-transformator*.

Now some special types of moving frames will be defined for C^∞ and regular curves lying in $\mathbf{R}^n$:

(a) A moving frame $E(s)$, $s\in L$ will be called a *Frenet-frame* if for the frame vectors the following derivational formulae (the so-called Frenet-equations) hold for each $s\in L$:

$$e_i'(s) = k_1(s)\underline{e}_2(s),$$

$$e_1'(s) = -k_{i-1}(s)\underline{e}_{i-1}(s)+k_i(s)\underline{e}_{i+1}(s)$$

$$\text{for } i = 2, 3, \ldots, n-1 \quad \text{and}$$

$$e_n'(s) = -k_{n-1}(s)\underline{e}_{n-1}(s),$$

where the suitable coefficients $k_i(s)$, $i=1, 2, \ldots, n-1$, are called the *pseudo-curvatures* belonging to $E(s)$.

The definition shows that the entries in the corresponding Cartan-matrix $C(s)$, $s\in L$ are the following:

$$c_{ij}(s) = k_i(s) \quad \text{for} \quad i = 1, 2, \ldots, n-1 \quad \text{and} \quad j = i+1;$$

$$c_{ij}(s) = 0 \quad \text{for} \quad i = 1, 2, \ldots, n-2 \quad \text{and} \quad j > i+1;$$

and using the skew-symmetry

$$c_{ij}(s) = -c_{ji}(s) \quad \text{for} \quad j \leqq i.$$

(b) A moving frame $E(s)$, $s\in L$ will be called here a *Bishop-frame* if for the frame vectors the following derivational formulae (the so-called Bishop-equations) hold for each $s\in L$:

$$\underline{e}_1'(s) = \sum_{i=1}^{n-1} b_i(s)\underline{e}_{i+1}(s) \quad \text{and}$$

$$\underline{e}_{i+1}'(s) = -b_i(s)\underline{e}_1(s) \quad \text{for} \quad i = 1, 2, \ldots, n-1,$$

where the suitable coefficients $b_i(s)$, $i=1, 2, \ldots, n-1$, will be called the *Bishop-coefficients* belonging to $E(s)$.

The definition shows that the entries in the corresponding Cartan-matrix $C(s)$, $s\in L$ are the following

$$c_{ij}(s) = b_{j-1}(s) \quad \text{for} \quad i = 1 \quad \text{and} \quad j > 1;$$

$$c_{ij}(s) = 0 \quad \text{for} \quad i > 1 \quad \text{and} \quad j > i;$$

and using the skew-symmetry

$$c_{ij}(s) = -c_{ij}(s) \quad \text{for} \quad j \leqq i.$$

Notice that in the Cartan-matrices of both a Frenet-frame and a Bishop-frame $\binom{n-1}{2}$ out of the $\binom{n}{2}$ independent entries are identically zero.

3. On Frenet-curves

Following Nomizu, a curve will be called a *Frenet-curve* if there exists a Frenet-frame which is adapted to it. It should be noted that a Frenet-curve may have more than one Frenet-frames adapted to it. In other words, there may exist many different systems of pseudo-curvatures which determine the same Frenet-curve in $\mathbf{R}^n$.

The problem of finding a necessary and sufficient condition for a curve to be a Frenet-curve has been studied by several authors including K. NOMIZU [1] and A. WINTNER [8] but the problem in its entire generality has not been solved yet. In any case an example of Nomizu shows that there are C^∞ and regular curves in $\mathbf{R}^n$ which do not admit Frenet-frames.

The following simple but rather strong sufficient condition is well-known from the classical treatment: Let $\Gamma: x = \underline{x}(s)$, $s \in L$ be a C^∞ and regular curve in $\mathbf{R}^n$. If the consecutive derivative vectors $\underline{x}^{(i)}(s)$, for $i = 1, 2, \ldots, n-1$ are linearly independent at every $s \in L$ then Γ is a Frenet-curve.

The method for obtaining a Frenet-frame of such a curve Γ has been concisely presented by H. GLUCK [5] and [6], as follows: The Gram—Schmidt orthonormalization process applied to the vectors

$$\underline{x}'(s), \quad \underline{x}''(s), \ldots, \underline{x}^{(n-1)}(s)$$

gives the unit vectors

$$\underline{e}_1(s), \underline{e}_2(s), \ldots, \underline{e}_{n-1}(s)$$

uniquely at every $s \in L$ including the fact that

$$\langle \underline{x}^{(i)}(s), \underline{e}_i(s) \rangle > 0 \quad \text{for} \quad i = 1, 2, \ldots, n-1.$$

The choice of the last unit vector $\underline{e}_n(s)$, $s \in L$ is already independent of the derivative $\underline{x}^{(n)}(s)$, it is defined uniquely by the fixed orientation of $\mathbf{R}^n$. The pseudo-curvatures $k_i(s)$, $i = 1, 2, \ldots, n-1$, $s \in L$ belonging to the above given Frenet-frame will be positive for $i = 1, 2, \ldots, n-2$ and $\operatorname{sign} k_{n-1}(s) = \operatorname{sign} \langle \underline{x}^{(n)}(s), \underline{e}_n(s) \rangle$ holds on L. Moreover the absolute values of the pseudo-curvatures, called simply curvatures in this case, have concrete geometrical meaning. Namely, they are the velocities of the so-called osculating subspaces as it was pointed out first by E. EGERVÁRY [7].

In fact, the p-dimensional osculating subspaces $(p = 1, 2, \ldots, n-1)$ are spanned now just by the vectors $\underline{e}_1(s), \underline{e}_2(s), \ldots, \underline{e}_p(s)$, $s \in L$ instead of the derivative vectors

$$\underline{x}'(s), \underline{x}''(s), \ldots, \underline{x}^{(p)}(s), \quad s \in L.$$

Thus, as the method of H. Gluck shows, it is enough to consider the turning of the following unit p-vector:

$$n_p(s) = \underline{e}_1(s) \wedge \underline{e}_2(s) \wedge \ldots \wedge \underline{e}_p(s), \quad s \in L$$

where $p = 1, 2, \ldots, n-1$.

It can be noticed that the p-th curvature of Γ at a given parameter $s \in L$ is nothing else than the norm of the derivative p-vector $\underline{n}_p'(s)$, where the norm in the $\binom{n}{p}$-dimensional euclidean vector space $\Lambda^p(\mathbf{R}^n)$ is defined by the inner product induced by that of $\mathbf{R}^n$. Let us differentiate $n_p(s)$ with respect to the arc-length s.

Using the Frenet-equations and also some basic properties of the exterior product, we get the above mentioned result:

$$\|\underline{n}_p'(s)\| = |k_p(s)| \quad \textit{holds for} \quad p = 1, 2, \ldots, n-1 \quad \textit{and} \quad s \in L.$$

It is convenient to say that a curve $\Gamma: \underline{x} = \underline{x}(s)$, $s \in L$ has regularity of order p $(1 \leqq p \leqq n)$ if the consecutive derivative vectors $\underline{x}'(s), \underline{x}''(s), \ldots, \underline{x}^{(p)}(s)$ are linearly independent at every $s \in L$. Now, it is clear that the classical Frenet-theory of curves covers only those Frenet-curves which have regularity of order $(n-1)$.

For a further discussion of Frenet-curves we will prove here the following important theorem of R. BISHOP [3] for the general n-dimensional case:

Theorem 3. *Let $\Gamma: \underline{x} = \underline{x}(s)$, $s \in L$ be an arbitrary C^∞ and regular curve in $\mathbf{R}^n$. Then there always exists a Bishop-frame which is adapted to it.*

On account of this theorem we may call here all C^∞ and regular curves Bishop-curves, as well.

PROOF. Let s_0 be an arbitrarily fixed value in the parameter interval L and denote $\underline{e}_1^*, \underline{e}_2^*, \ldots, \underline{e}_n^*$ a positively oriented orthonormal basis of $\mathbf{R}^n$ where $\underline{e}_1^* = \underline{x}'(s_0)$ holds. As there always exists a closed interval $L(s_0) \subseteqq L$ containing s_0 where the vectors

$$\underline{x}'(s), \underline{e}_2^*, \ldots, \underline{e}_n^*, \quad s \in L(s_0)$$

remain linearly independent, the Gram—Schmidt orthonormalization process can be applied to these vectors at every $s \in L(s_0)$. So we have a local moving frame $E(s)$, $s \in L(s_0)$ being adapted to the corresponding arc of the curve Γ.

Assume now that there is a frame transformator $A(s)$, $s \in L(s_0)$ which carries the above chosen local moving frame $E(s)$ into a local Bishop-frame $E(s)$, $s \in L(s_0)$. Then on account of Theorem 2 the unknown entries in the matrix of the frame transformator $A(s)$, $s \in L(s_0)$ have to satisfy the following system of differential equations:

$$a_{ij}'(s) = -\sum_{l=1}^{n} a_{il}(s) c_{lj}(s) \quad \textit{for} \quad i, j = 2, 3, \ldots, n \quad \textit{and} \quad s \in L(s_0).$$

We can write these equations in the more convenient matrix-form:

$$A_0'(s) = -A_0(s) C_0(s), \quad s \in L(s_0)$$

where $A_0(s)$ and $C_0(s)$ denote the matrices obtained at every $\in L(s_0)$ by omitting the first rows and columns of the matrices $A(s)$ and $C(s)$, respectively. So, as it was shown in the proof of the Theorem 1, there exists a unique solution $A_0(s)$, $s \in L(s_0)$ satisfying a given initial condition $A_0(\bar{s}) \in SO(n-1)$, where $\bar{s} \in L(s_0)$, and

$$A_0(s) \in SC(n-1) \quad \textit{holds on the whole interval} \quad L(s_0).$$

The uniqueness of the local Bishop-frame $\tilde{E}(s)$, $s \in L(s_0)$ belonging to the above chosen initial condition

$$\tilde{E}(\bar{s}) = E(\bar{s}) A^*(\bar{S}), \quad \bar{s} \in L(s_0)$$

is merely a consequence of the fact that the frame-transformator $A(s)$ has been uniquely determined on the interval $L(s_0)$.

Only the global existence of a Bishop-frame adapted to the whole curve Γ remained to be shown. It is easy to see that there exists for L a finite system of open covering intervals on each of which the existence of a local Bishop-frame is assured. These local Bishop-frames can be patched together, and due to the above mentioned uniqueness they link together smoothly.

Remark. Let $E(s)$, $s \in L$ be a Bishop-frame adapted to a given curve Γ. The corresponding Bishop-coefficients will be denoted by $b_i(s)$ for $i = 1, 2, \dots, n-1$. Let further $A(s)$, $s \in L$ be an arbitrary frame-transformator.

Then, on account of Theorem 2. the C^∞ entries in the transformed Cartan-matrix $\tilde{C}(s)$, $s \in L$ are the following:

$$\tilde{c}_{ij}(s) = \sum_{l=2}^{n} b_{l-1}(s) a_{jl}(s) \quad \text{for} \quad i = 1 \quad \text{and} \quad j > i$$

$$\tilde{c}_{ij}(s) = \sum_{l=2}^{n} a'_{il}(s) a_{jl}(s) \quad \text{for} \quad i = 2, 3, \dots, n-1 \quad \text{and} \quad j > i$$

and using the skew-symmetry

$$\tilde{c}_{ij}(s) = -\tilde{c}_{ji}(s) \quad \text{for} \quad j \leqq i.$$

Corollary 1. *It is easy to see that* $\tilde{C}(s)$, $s \in L$ *will belong also to a Bishop-frame of the given curve* Γ *if and only if the frame-transformator* $A(s)$ *is constant on the parameter interval* L.

In fact, from $A'(s) = 0$ we get that $\tilde{c}_{ij}(s) = 0$ holds for $i, j \geqq 2$. Conversely, if $\tilde{C}(s)$ belongs to a Bishop-frame of Γ then $A'(s)A^*(s) = 0$ has to hold for each $s \in L$ implying that $A(s)$ is constant on the parameter interval L. This is in a complete accordance with the fact that if once the frame-vectors $\underline{e}_2^*, \underline{e}_3^*, \dots, \underline{e}_n^*$ orthogonal to the tangent vector $\underline{x}'(s_0)$ are chosen at an initial parameter $s_0 \in L$ then the Bishop-frame of Γ is already unique.

Corollary 2. *K. Nomizu's crucial problem of finding a necessary and sufficient condition for a* C^∞ *and regular curve to be a Frenet-curve can be answered now, as follows:*

The considered curve Γ *is a Frenet-curve if and only if there exists a suitable frame-transformator* $A(s)$, $s \in L$ *for which*

$$\sum_{l=2}^{n} b_{1-1}(s) a_{jl}(s) = 0 \quad \textit{for} \quad j > 2$$

and

$$\sum_{l=2}^{n} a'_{il}(s) a_{jl}(s) = 0 \quad \textit{for} \quad i = 2, 3, \dots, n-2 \quad j > i+1$$

hold at every $s \in L$.

Notice that the existence of a solution for $A(s)$ having $\binom{n-1}{2}$ unknown independent C^∞ entries does not depend on the actual choice of the Bishop-coefficients $b_i(s)$, $i = 1, 2, \dots, n-1$, characterizing the given curve Γ.

In the most interesting special case, where $n=3$, the above condition expressing which entries have to vanish in the Cartan-matrix of the transformed moving frame reduces to the following single equation:

$$b_1(s)\sin\varphi(s) = b_2\cos\varphi(s) \quad \text{at every} \quad s\in L,$$

where $\varphi(s)$, $s\in L$ is the only C^∞ scalar-function to be determined in the frame-transformator

$$A(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi(s) & \sin\varphi(s) \\ 0 & -\sin\varphi(s) & \cos\varphi(s) \end{pmatrix}, \quad s\in L.$$

Let us consider now only Frenet-curves in $\mathbf{R}^n$. The following theorems will show how far the pseudo-curvatures are determined by a given Frenet-curve.

Theorem 4. *Let $\Gamma\colon \underline{x}=\underline{x}(s)$, $s\in L$ be a Frenet-curve in $\mathbf{R}^n$ and p $(1\leqq p\leqq m)$ a fixed natural number. Let further $s_0\in L$ be an arbitrarily chosen value of parameter*

Then the following two conditions are equivalent:

(i) The consecutive derivative vectors

$$x'(s_0),\, x''(s_0),\, \dots,\, x^{(i)}(s_0) \quad \text{are}$$

linearly independent for $i\leqq p$, and linearly dependent for $i>p$,

(ii) $$k_i(s_0)\neq 0 \quad \text{for} \quad i<p \quad \text{and}$$
$$k_i(s_0)=0 \quad \text{for} \quad i=p$$

hold at the given parameter s_0, where $k_i(s)$ for $i=1,2,\dots,n-1$ and for $s\in L$ denote the pseudo-curvatures belonging to an arbitrary Frenet-frame adapted to the curve Γ.

PROOF. Let $E(s)$, $s\in L$ be one of the Frenet-frames adapted to the given curve Γ. Then the higher derivatives of $\underline{x}(s)$ can be obtained at each $s\in L$ as linear combinations of the frame vectors $\underline{e}_1(s)$, $\underline{e}_2(s)$, ..., $\underline{e}_n(s)$. Applying the Frenet-equations we can write:

$$\underline{x}'(s) = \underline{e}_1(s),$$
$$\underline{x}''(s) = k_1(s)\,\underline{e}_2(s),$$
$$\dots$$
$$\underline{x}^{(r)}(s) = \sum_{j=1}^{n}\lambda_{jr}(s)\,\underline{e}_j(s),$$

where the coefficient $\lambda_{jr}(s)$ is equal to zero for each $j>r$; moreover, it can be verified that $\lambda_{jr}(s)=\prod_{l=1}^{r-1}k_l(s)$ holds for $j=r$ and $2\leqq r\leqq n$. On account of basic properties of the exterior multiplication we can get the following expression:

$$\underline{x}'(s)\wedge\underline{x}''(s)\wedge\dots\wedge\underline{x}^{(r)}(s) = \left\{\prod_{l=1}^{r-1}k_l^{r-l}(s)\right\}e_1(s)\wedge e_2(s)\wedge\dots\wedge e_r(s)$$

for each $s\in L$ and $2\leqq r\leqq n$.

Using the fact that the vectors $\underline{x}'(s), \underline{x}''(s), \ldots, \underline{x}^{(r)}(s)$ are linearly independent in $\mathbf{R}^n$ if and only if the corresponding r-vector

$$\underline{x}'(s) \wedge \underline{x}''(s) \wedge \ldots \wedge \underline{x}^{(r)}(s)$$

is not zero, the proof of our theorem can easily be completed, since

$$\prod_{l=1}^{p-1} k_l(s_0) \neq 0 \quad \text{and} \quad \prod_{l=1}^{p} k_l(s_0) = 0$$

has to hold at $s_0 \in L$.

Remark. Let $E(s)$, $s \in L$ be a Frenet-frame adapted to a given curve Γ and $k_i(s)$ for $i=1, 2, \ldots, n-1$ and for $s \in L$ the corresponding pseudo-curvatures. Then the parameter interval L can be decomposed in the following form:

$$L = \bigcup_{i=1}^{n} L_i$$

where

$$L_1 = \{s\colon\ k_1(s) = 0\},$$

$$L_i = \left\{s\colon\ \prod_{l=1}^{i-1} k_l(s) \neq 0 \quad \textit{and} \quad k_i(s) = 0\right\} \quad \textit{for} \quad 2 \leqq i \leqq n-1$$

and

$$L_n = \left\{s\colon\ \prod_{l=1}^{n-1} k_l(s) \neq 0\right\}.$$

Theorem 5. *Let $E(s)$ and $\tilde{E}(s)$ be two different Frenet-frames adapted to a given Frenet-curve $\Gamma\colon \underline{x} = \underline{x}(s)$, $s \in L$. If $\underline{x}'(s), \underline{x}''(s), \ldots, \underline{x}^{(p)}(s)$ are linearly independent vectors at a given parameter $s_0 \in L$ then $\tilde{e}_i(s_0) = \varepsilon_i \underline{e}_i(s_0)$ holds for $i = 1, 2, \ldots, p$ and $s_0 \in L$, where ε_i is either $+1$ or -1.*

Proof. Let $A(s)$, $s \in L$ be the frame-transformator carrying $E(s)$ into $\tilde{E}(s)$. Then, on account of Theorem 2., the following conditions are satisfied for each $s \in L$ and $1 \leqq i < j \leqq n$:

(4)

$$\sum_{l=1}^{n} a'_{il}(s) a_{jl}(s) + \sum_{l=1}^{n-1} \{a_{i,l}(s) a_{j,l+1}(s) - a_{i,l+1}(s) a_{jl}(s)\} \cdot k_l(s) = \begin{cases} \tilde{k}_i(s) & \text{for} \quad j = i+1 \\ 0 & \text{for} \quad j > i+1, \end{cases}$$

where $k_i(s)$ and $\tilde{k}_i(s)$, for $i = 1, 2, \ldots, n-1$ and for $s \in L$ denote the pseudo-curvatures belonging to $E(s)$ and $\tilde{E}(s)$, respectively.

Let $I \subseteqq L$ denote an open neighbourhood of s_0 where the vectors $\underline{x}'(s), \underline{x}''(s), \ldots, \underline{x}^{(p)}(s)$, $s \in I$ are still linearly independent, and so we have that $k_i(s) \neq 0$ holds for each $s \in I$ and $i = 1, 2, \ldots, p-1$.

Now, we start proving our theorem step by step. First it is trivial that $\tilde{\underline{e}}_1(s) = \varepsilon_1 \underline{e}_1(s)$ holds on I, where $\varepsilon_1 = 1$. From (4) $\tilde{k}_1(s) = a_{11}(s) a_{22}(s) k_1(s)$, $s \in L$ can be obtained since for $i, j \neq 1$ $a_{i1}(s)$ and $a_{1j}(s)$ are identically zero on I. Let us compare now the Frenet-equations valid for $\tilde{\underline{e}}'_1(s)$ and $\underline{e}'(s)$. Using also the condition that $k_1(s) \neq 0$ for $s \in I$, we get that $a_{22} \tilde{\underline{e}}_2(s) = \underline{e}_2(s)$ holds for each $s \in I$.

Thus, we already know that $\underline{e}_2(s_0)=\varepsilon_2\underline{e}_2(s_0)$ holds at $s_0\in L$. Continuing this procedure analogously we will arrive at the last step. Then from (4)

$$\tilde{k}_{p-1}(s) = a_{p-1,p-1}(s)a_{pp}(s)k_{p-1}(s), \quad s\in I$$

can be obtained since we have already that $a_{i,p-1}(s)$ and $a_{p-1,j}(s)$ are identically zero on I for $i, j\neq p-1$. Let us compare now the Frenet-equations valid for $\tilde{\underline{e}}'_{p-1}(s)$ and $\underline{e}'_{p-1}(s)$. Using also the condition that $k_{p-1}(s)\neq 0$ for $s\in I$, we get that $a_{pp}(s)\tilde{\underline{e}}_p(s)=\underline{e}_p(s)$ holds for each $s\in I$. Thus, we know that $\tilde{\underline{e}}_p(s_0)=\varepsilon_p\underline{e}_p(s_0)$ holds at $s_0\in L$, as well.

Remark 1. Notice that the r-dimensional osculating subspaces of the given curve Γ surely exist at the parameter $s_0\in L$ for $r=1, 2, \ldots, p$. So, as it was shown earlier, their velocities also exist and are given by the absolute values of the corresponding pseudo-curvatures $k_r(s_0)$ for $r=1, 2, \ldots, p$. It is reasonable to ask whether these values do not depend on the choice of the Frenet-frame which is actually adapted to the curve Γ. In fact, we can see from the proof of the above theorem that for the different pseudo-curvatures $k_r(s)$ and $\tilde{k}_r(s)$ the following conditions hold at the given parameter $s_0\in L$: $|\tilde{k}_r(s_0)|=|k_r(s_0)|>0$ for $r=1, 2, \ldots, p-1$, and due to Theorem 4. $|\tilde{k}_p(s_0)|=|k_p(s_0)|$.

Finally, in order to cover also the case $p=n$, it seems convenient to accept for the n-th pseudo-curvature function $k_n(s)$ the following definition: $k_n(s)=0$ for each $s\in L$.

Remark 2. As the proof of the Theorem 5. shows the matrix of the frame-transformator $A(s)$, $s\in L$ must have the following form at $s_0\in L$:

$$A(s_0)=\begin{pmatrix} \varepsilon_1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & \varepsilon_p & 0 & \dots & 0 \\ 0 & \dots & 0 & a_{p+1,p+1} & \cdots & a_{p+1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & 0 & a_{n,p+1} & \dots & a_{nn} \end{pmatrix}$$

In particular, let $p=n-1$.
Then

$$A(s_0)=\begin{pmatrix} \varepsilon_1 & 0 & \dots & 0 \\ 0 & \varepsilon_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \varepsilon_n \end{pmatrix} \quad \text{holds, where}$$

$$\varepsilon_1 = 1 \quad \text{and} \quad \varepsilon_n = \prod_{l=1}^{n-1} \varepsilon_l.$$

Consequently there are only 2^{n-2} possibilities for the choice of different Frenet-frames adapted to a curve Γ having regularity of order $(n-1)$.

4. Characterization of curves lying in p-planes

Definition 1. Let $\Gamma: \underline{x}=\underline{x}(s)$, $s \in L$ be a C^∞ and regular curve in $\mathbf{R}^n$. It is said that Γ lies in a p-plane $(1 \leqq p < n)$ if there exists a p-dimensional linear submanifold of $\mathbf{R}^n$ containing Γ.

Definition 2. We say that Γ lies uniformly in a p-plane if it lies in a p-plane but it has no subarcs lying in an r-plane, where $r < p$.

Theorem 6. *Let $\Gamma: \underline{x}=\underline{x}(s)$, $s \in L$ be a Frenet-curve. If there exists a Frenet-frame $E(s)$, $s \in L$ adapted to the curve Γ where $k_p(s)=0$ identically holds on the whole interval L $(1 \leqq p < n)$, then Γ lies in a p-plane.*

PROOF. Let $s_0 \in L$ be an arbitrarily chosen parameter value. It is enough to show that $(\underline{x}(s)-\underline{x}(s_0)) \wedge \underline{n}_p(s_0)=0$ identically holds for each $s \in L$, where $\underline{n}_p(s)$ denotes the p-vector formed by the vectors $\underline{e}_j(s)$, for $j=1, 2, \dots, p$, of the considered Frenet-frame $E(s)$, $s \in L$. First, on account of the Frenet-formulas and some basic properties of the exterior multiplication, it is easy to see that $\frac{d}{ds}(n_p(s)) = 0$ holds for each $s \in L$. Thus we can get the following identity:

$$\underline{e}_1(s) \wedge \underline{n}_p(s_0) = 0, \quad s \in L.$$

An integration of this last equation already shows the desired result.

Remark. It should be noticed that the condition $k_p(s)=0$ for each $s \in L$, is only a sufficient but not a necessary condition for a Frenet-curve to lie in a p-plane.

Theorem 7. *If a Frenet-curve Γ lies in a p-plane then for each system of its pseudo-curvatures*

$$\prod_{l=1}^{p} k_l(s)=0 \quad \textit{holds for every } s \in L.$$

PROOF. Since Γ lies in a p-plane, there exists a suitable p-vector n_p, so that $(\underline{x}(s)-\underline{x}(s_0)) \wedge \underline{n}_p=0$ identically holds on L. Thus for the derivative vectors $\underline{x}^{(r)}(s)$ $\underline{x}^{(r)}(s) \wedge \underline{n}_p=0$ also holds for each natural number r and $s \in L$. It is easy to see that the consecutive derivative vectors $\underline{x}'(s), \underline{x}''(s), \dots, \underline{x}^{(r)}(s)$ cannot be linearly independent if $r > p$ and so $\underline{x}'(s) \wedge \underline{x}''(s) \wedge \dots \wedge \underline{x}^{(p+1)}(s)=0$ identically holds on L implying that

$$\prod_{l=1}^{p} k_l(s) = 0$$

also has to hold for each $s \in L$.

Remark: If a Frenet-curve Γ lies in a p-plane then the previously introduced decomposition of the parameter interval L is the following $L=\bigcup_{i=1}^{p} L_i$, in other words $L_i=0$ for $i=p+1, \dots, n$.

Definition: A curve $\Gamma: \underline{x}=\underline{x}(s)$, $s \in L$ will be called strictly p-regular curve if for the consecutive derivative-vectors $\underline{x}'(s), \underline{x}''(s), \dots, \underline{x}^{(r)}(s)$, $s \in L$ the following conditions hold:

(i) they are linearly independent for $r \leqq p$,
(ii) they are linearly dependent for $r > p$.

Theorem 8. *Every strictly p-regular curve Γ is a Frenet-curve lying uniformly in a p-plane.*

PROOF. First we show that Γ is a Frenet-curve. Applying the Gram—Schmidt orthonormalization process for the linearly independent derivative vectors $\underline{x}'(s)$, $\underline{x}''(s), \ldots, \underline{x}^{(p}(s)$, we get for each $s \in L$ the first p unit vectors of a Frenetframe $E(s)$. It is easy to see on account of the condition (ii) that

$$\frac{d}{ds}\left(\underline{e}_1(s) \wedge \underline{e}_2(s) \wedge \ldots \wedge \underline{e}_p(s)\right) = 0$$

has to hold for each $s \in L$. So, from the first p Frenet-equations we get for the pseudo-curvatures $k_i(s)$ the following results:

$$\prod_{l=1}^{p-1} k_l(s) \neq 0 \quad \text{and} \quad k_p(s) = 0$$

hold on the whole interval L.

Notice that the unit vectors

$$\underline{e}_{p+1}(s), \underline{e}_{p+2}(s), \ldots, \underline{e}_n(s), \quad s \in L,$$

of a suitable Frenet-frame $E(s)$ can be chosen from that constant $(n-p)$-dimensional subspace of $\mathbf{R}^n$ which is orthogonal to the p-dimensional subspace spanned by the linearly independent vectors $\underline{x}'(s), \underline{x}'(s), \ldots, \underline{x}^{(p)}(s)$. This choice, however, has much freedom and so the corresponding pseudo-curvatures

$$k_{p+1}(s). k_{p+2}(s), \ldots, k_{n-1}(s)$$

may be arbitrary C^∞ scalar-functions on L.

Let now $\tilde{E}(s)$, $s \in L$ denote a Frenet-frame adapted to the given curve Γ and $\tilde{k}_i(s)$, for $i = 1, 2, \ldots, n-1$ and for $s \in L$ be the corresponding pseudo-curvatures. Then by Theorem 4 $\tilde{k}_p(s) = 0$ holds on L and therefore we have that Γ lies in a p-plane. Moreover, again by Theorem 4

$$\prod_{l=1}^{p-1} k_l(s) \neq 0, \quad s \in L$$

holds and therefore on account of Theorem 7 we have that Γ has no subarcs lying in any lower dimensional r-plane.

Remark. For the parameter interval L of a strictly p-regular curve we have:

$$L = L_p = \left\{s\colon \prod_{l=1}^{p-1} k_l(s) \neq 0,\ k_p(s) = 0\right\}.$$

Lemma. *Let $\Gamma\colon x = x(s)$, $s \in L$ be an arbitrary Frenet-curve in $\mathbf{R}^n$. Then it has a dense subset which is the union of a countable number of strictly i-regular curves, where $i = 1, 2, \ldots, n$.*

PROOF. Let us consider the above mentioned decomposition of the parameter interval L

$$L = \bigcup_{i=1}^{n} L_i.$$

Then, on account of a simple topological lemma applied by [2] we can state that

$$L = \bigcup_{i=1}^{n} \bar{L}_i^0$$

also holds, where L_i^0 denotes the interior of L_i. Assuming that $\mathbf{R}$ has the usual topology, the open set L_i^0 can be given as union of a countable family of disjoint open intervals. It is evident that to each of such an open subinterval a strictly i-regular subarc of the given curve Γ will belong since

$$\prod_{l=1}^{i-1} k_l(s) \neq 0 \quad \text{and} \quad k_i(s) = 0$$

hold there identically.

Theorem 9. *Let $\Gamma: x = \underline{x}(s), s \in L$ be a Frenet-curve lying uniformly in a p-plane. Then for the p-th pseudo-curvature function of any Frenet-frame adapted to the given curve Γ $k_p(s) = 0$ holds on the whole interval L.*

PROOF. Since Γ lies in a p-plane $L = \bigcup_{i=1}^{p} L_i$ holds. On the other hand $L_i^0 = 0$ has to hold for $i = 1, 2, \ldots, p-1$ because Γ lies uniformly in a p-plane. Thus, for the parameter interval L is valid $L = \bar{L}_p^0$. Consider now a parameter value $s_0 \in L$ for which $s_0 \notin L_p^0$ holds. Then, due to the continuity of the function $k_p(s)$ $k_p(s_0) = 0$ holds, as well, and our theorem is proved.

Notice that a necessary and sufficient condition for a Frenet-curve to lie in a p-plane has not been found yet. The following theorem gives such a condition generally for C^∞ and regular curves in terms of the Bishop-coefficients.

Theorem 10. *Let $\Gamma: x = \underline{x}(s), s \in L$ be an arbitrary C^∞ and regular curve in $\mathbf{R}^n$. It lies in a p-plane if and only if there exists a Bishop-frame $E(s), s \in L$ adapted to Γ so that the corresponding Bishop-coefficients satisfy the following conditions: $b_p(s) = b_{p+1}(s) = \ldots = b_{n-1}(s) = 0$ for each $s \in L$.*

PROOF. First suppose that Γ lies in a p-plane, i.e. the vectorfunction $\underline{x}(s)$ satisfies the following equation for each $s \in L$:

$$(\underline{x}(s) - \underline{x}(s_0)) \wedge \underline{u}_1 \wedge \underline{u}_2 \wedge \ldots \wedge \underline{u}_p = 0$$

where $u_1, u_2, \ldots, \underline{u}_p$ are mutually orthogonal unit vectors spanning the p-plane which goes through a fixed point $\underline{x}(s_0)$, $s_0 \in L$. Let us complete now this system of vectors $u_1, u_2, \ldots, u_p$ to a positively oriented orthonormal basis $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_n$ of $\mathbf{R}^n$ and denote by $R \in SO(n)$ the matrix of that orthogonal transformation which carries the above chosen basis into the canonical basis of $\mathbf{R}^n$. Then, it is easy to see that the curve $\tilde{\Gamma}$ given by

$$\tilde{\Gamma}: \underline{x} = R(\underline{x}(s) - \underline{x}(s_0))$$

will lie in the p-dimensional linear subspace of $\mathbf{R}^n$ spanned by the first p vectors of its canonical basis and so it can be identified with a curve $(\tilde{\Gamma})_p$ lying in the space $\mathbf{R}^p$.

Consequently on account of Theorem 3 we can adapt to $\tilde{\Gamma}$ a Bishop-frame $\tilde{E}(s)$, $s\in L$ so that only the first p frame-vectors

$$\tilde{\underline{e}}_1(s), \tilde{\underline{e}}_2(s), \ldots, \tilde{\underline{e}}_p(s)$$

may change along the curve $\tilde{\Gamma}$. Let then

$$E(s) = R^*\tilde{E}(s), \quad s\in L$$

be the corresponding Bishop-frame adapted to the curve Γ. Since $E(s)$ and $\tilde{E}(s)$ are congruent moving frames, they have the same Cartan-matrix so that

$$b_p(s) = b_{p+1}(s) = \ldots = b_{n-1}(s) = 0$$

hold for each $s\in L$.

Conversely, let us suppose that the given curve Γ has a Bishop-frame $E(s), s\in L$ for the Cartan-matrix of which the considered conditions hold. Then, on account of the Bishop-equations the p-vector

$$\underline{n}_p(s) = \underline{e}_1(s)\wedge \underline{e}_2(s)\wedge\ldots\wedge \underline{e}_p(s), \quad s\in L$$

is constant on the parameter interval L. Thus, it is easy to see that

$$\big(\underline{x}(s)-\underline{x}(s_0)\big)\wedge\underline{n}_p(s_0) = 0$$

identically holds on L implying that Γ lies in a p-plane.

Remark 1. Let $\Gamma\colon \underline{x}=\underline{x}(s)$, $s\in L$ be a Frenet-curve in $\mathbf{R}^n$ lying in a p-plane. As it was shown in the above proof, there always exists an orientation preserving isometry carrying the given curve Γ into a curve $\tilde{\Gamma}$ which can be identified with a C^∞ and regular curve $(\tilde{\Gamma})_p$ lying in $\mathbf{R}^p$. The curve $(\tilde{\Gamma})_p$, however, may not be a Frenet-curve in $\mathbf{R}^p$. This is the reason of the fact that the proof of the Theorem 10 cannot be modified so as to yield an analogous theorem for Frenet-curves where the conditions would be given in terms of the pseudo-curvatures.

Remark 2. Let $\Gamma\colon x=x(s)$, $s\in T$, be a Frenet-curve in $\mathbf{R}^n$ and denote by $E(s)$, $s\in L$ a Frenet-frame which is adapted to it. The given curve Γ lies in a p-plane if and only if there exists a frame-transformator $A(s)$, $s\in L$ so that in the Cartan-matrix $\tilde{C}(s)$, $s\in L$ belonging to the transformed moving frame we have the following entries:

$$\tilde{c}_{1j}(s) = 0 \quad \text{for} \quad j = p+1,\ p+2, \ldots, n \quad \text{and} \quad \text{for} \quad s\in L,$$

$$\tilde{c}_{ij}(s) = 0 \quad \text{for} \quad i,\ j = 2, 3, \ldots, n \text{ and for } \quad s\in L.$$

Notice that this condition given now for a Frenet-curve is nothing else then the condition used in Theorem 10.

5. On Curves lying on p-spheres

Definition. Let $\Gamma: \underline{x} = \underline{x}(s)$, $s \in L$ be a C^∞ and regular curve in $\mathbf{R}^n$. It is said that Γ lies on a p-sphere having center-vector $\underline{c}$ and radius r if Γ lies in a $(p+1)$-plane which contains this p-sphere and

$$\langle \underline{x}(s) - \underline{c},\ \underline{x}(s) - \underline{c} \rangle = r^2$$

holds on the parameter interval L. $(1 \leqq p < n)$.

The following theorem gives a convenient characterization of the spherical curves in terms of the Bishop-coefficients:

Theorem 11. *Let $\Gamma: \underline{x} = \underline{x}(s)$, $s \in L$ be a C^∞ and regular curve in $\mathbf{R}^n$. It lies on a p-sphere (of radius r) if and only if there exists a Bishop-frame $E(s)$, $s \in L$ adapted to Γ so that the corresponding Bishop-coefficients satisfy the following conditions:*

$$b_p(s) = \frac{1}{r} \quad \text{and} \quad b_{p+1}(s) = b_{p+2}(s) = \ldots = b_{n-1}(s) = 0$$

hold for each $s \in L$, where $r > 0$ is the radius of the p-sphere.

Proof. First suppose that Γ lies on a p-sphere having center-vector $\underline{c}$ and radius r. Then by Theorem 10 there exists a Bishop-frame

$$E(s) = \{e_1(s),\ \underline{e}_2(s), \ldots, \underline{e}_n(s)\}, \quad s \in L$$

so that $e_i(s)$ is constant for $i > p+1$ and

$$(\underline{x}(s) - \underline{x}(s_0)) \wedge \underline{e}_1(s) \wedge \underline{e}_2(s) \wedge \ldots \wedge \underline{e}_{p+1}(s) = 0$$

identically holds on L.

Moreover by Theorem 3 the initial condition

$$\underline{e}_{p+1}(s_0) = \frac{1}{r}(\underline{c} - x(s_0)), \quad s_0 \in L$$

can be satisfied at the choice of this Bishop-frame $E(s)$. Using now the fact that the vectors $\underline{c} - \underline{x}(s)$ and $\underline{e}_1(s)$ are everywhere perpendicular to each other, we can write:

$$\underline{c} - \underline{x}(s) = A_1(s)\underline{e}_2(s) + A_2(s)\underline{e}_3(s) + \ldots + A_p(s)\underline{e}_{p+1}(s)$$

holds for each $s \in L$, where the coefficient-functions

$$A_i(s) = \langle c - \underline{x}(s),\ \underline{e}_{i+1}(s) \rangle \quad \text{for} \quad i = 1, 2, \ldots, p$$

are constant on L since

$$A_i'(s) = \langle -\underline{e}_1(s),\ \underline{e}_{i+1}(s) \rangle + \langle \underline{c} - \underline{x}(s),\ -b_i(s)\,\underline{e}_1(s) \rangle = 0$$

holds for each $s \in L$. Thus

$$\underline{c} - \underline{x}(s) = r\underline{e}_{p+1}(s)$$

identically holds on the whole interval L.

A derivation with respect to the arc-length s already shows that $b_p(s)=1/r$ holds for each $s\in L$, as well.

Conversely, let us suppose that the given curve Γ has a Bishop-frame $E(s)$, $s\in L$ for the Cartan-matrix of which the considered conditions hold. Then the vector-function

$$\underline{c}(s) = \underline{x}(s)+r\underline{e}_{p+1}(s)$$

is constant on L since its derivative is identically zero. So, using simply $\underline{c}$ instead of $\underline{c}(s)$, it is evident that

$$\langle \underline{x}(s)-\underline{c}, \underline{x}(s)-\underline{c}\rangle = r^2$$

holds for each $s\in L$, and consequently the curve Γ lies on a p-sphere having center-vector $\underline{c}$ and radius r.

Remark 1. Let Γ: $\underline{x}=\underline{x}(s)$, $s\in L$ be a C^∞ and regular curve in $\mathbf{R}^n$ and denote by $b_1(s), b_2(s), \dots, b_{n-1}(s)$ the Bishop-coefficients belonging to a Bishop-frame $E(s)$, $s\in L$ adapted to Γ. Then, according to R. Bishop [3], a curve defined in $\mathbf{R}^{n-1}$ by the vector-function $b(s)$, $s\in L$ whose coordinates are just the Bishop-coefficients $b_i(s)$, $i=1, 2, \dots, n-1$ and $s\in L$, is called a *normal development* of the given curve.

So, the above theorem, involving also Theorem 10 where $1/r=0$, can be given in the following form:

A C^∞ and regular curve Γ lies on a p-sphere of radius r $(1\leq p<n)$ if and only if it has in $\mathbf{R}^{n-1}$ a normal development $\underline{b}(s)$, $s\in L$ which lies in the $(p-1)$-plane spanned by the first $(p-1)$ element of the canonical basis of $\mathbf{R}^{n-1}$ and going through the point $P(0, 0, \dots, 0, 1/r, 0, \dots, 0)$ where only the p-th coordinate differs from zero for $r<\infty$.

On the other hand, if $\tilde{\underline{b}}(s)$, $s\in L$ is an arbitrary other normal development belonging to the given curve Γ, then on account of Theorem 2 and Theorem 3 we have that

$$\tilde{\underline{b}}(s) = A_0\underline{b}(s)$$

holds for each $s\in L$, where $A_0\in SO(n-1)$.

Thus the curve $\tilde{\underline{b}}(s)$, $s\in L$ lies also in a $(p-1)$-plane which has the distance $1/r$ from the origin of the vector-space $\mathbf{R}^{n-1}$.

So we have obtained the characterization of the spherical curves given in [3]. A proof of the corresponding theorem for the usual 3-dimensional case can be found there.

Remark 2. Let Γ: $x=\underline{x}(s)$, $s\in L$ be a Frenet-curve in $\mathbf{R}^n$ and denote by $E(s)$, $s\in L$ a Frenet-frame which is adapted to it. The given curve Γ lies on a p-sphere of radius r if and only if there exists a frame transformator $A(s)$, $s\in L$ so that in the Cartan-matrix $\tilde{C}(s)$, $s\in L$ belonging to the transformed moving frame we have the following entries:

$$\tilde{c}_{1j}(s) = \begin{cases} \dfrac{1}{r} & \text{for } j = p+1, \quad s\in L \\ 0 & \text{for } j = p+2, \dots, n, \quad s\in L; \end{cases}$$

$$\tilde{c}_{ij}(s) = 0 \quad \text{for } i, j = 2, 3, \dots, n, \quad s\in L.$$

Notice that this condition given now for a Frenet-curve is nothing else then the condition used in Theorem 11.

The above formulae are closely related to the ones given in [2] for the usual case where $n=3$ and $p=2$. In fact, let $k_1(s)$ and $k_2(s)$, $s \in L$ be pseudo-curvatures belonging to a Frenet-curve Γ in $\mathbf{R}^3$. Then, on account of Theorem 2, the matrix $\tilde{C}(s)$, $s \in L$ has the following independent entries:

$$\tilde{c}_{12}(s) = k_1(s) \cos \varphi(s)$$

$$\tilde{c}_{13}(s) = -k_1(s) \sin \varphi(s)$$

$$\tilde{c}_{23}(s) = k_2(s) + \varphi'(s),$$

where $\varphi(s)$, $s \in L$ is a C^∞ scalar-function to be determined in the frame-transformator $A(s)$, $s \in L$. Thus, the Frenet-curve Γ lies on a sphere of radius r if and only if there exists a C^∞ scalar-function $\varphi(s)$, $s \in L$ so that

$$-k_1(s) \sin \varphi(s) = \frac{1}{r} \quad \text{and} \quad k_2(s) + \varphi'(s) = 0 \quad \text{hold on } L.$$

Finally, we can easily obtain the following consequence by eliminating $\varphi(s)$ from the above two equations: If the Frenet-curve Γ lies on a sphere having radius $r < \infty$ then for the pseudo-curvatures $k_1(s)$ and $k_2(s)$, $s \in L$ we have that

$$(k_1'(s))^2 = (k_1(s) k_2(s))^2 ((r k_1(s))^2 - 1)$$

identically holds on the parameter interval L.

References

[1] K. Nomizu, On Frenet equations for curves of class C, *Tôhoku Math. J.* **11** (1959), 106—112.

[2] Yung-Chow Wong and Hon-Fei Lai, A critical examination of the theory of curves in thre-dimensional differential geometry, *Tôhoku Math. J.* **19** (1967), 1—31.

[3] R. Bishop, There is more than one way to frame a curve, *Amer. Math. Monthly* **82** (1975), 246—251.

[4] E. Kamke, Differentialgleichungen I. *Leipzig*, 1962.

[5] H. Gluck, Higher curvatures of curves in Euclidean spaces I., *Amer. Math. Monthly* **73** (1966), 699—704.

[6] H. Gluck, Higher curvatures of curves in Euclidean spaces II., *Amer. Math. Monthly* **74** (1967), 1049—1056.

[7] E. Egerváry, Über die Kurven des *n*-dimensionalen euklidischen Raumes, *Math. Term. Tud. Értesítő* **59** (1940), 796—797.

[8] A. Wintner, On Frenet's equations, *Amer. J. Math.* **78** (1956), 349—356.

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