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Pointwise convergence and a new inversion theorem for Hankel transforms

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Abstract. In this note we obtain necessary and sufficient conditions for a measurable function f on $(0,\infty)$ to satisfy

$$\lim_{T\to\infty} S_T(f,\mu;x) = f(x),$$

for almost every $x \in (0,\infty)$ and for every $\mu \in (-1/2, 1/2)$, where

$$S_T(f,\mu;x) = \int_0^T y^{2\mu+1}(xy)^{-\mu} J_\mu(xy) h_\mu(f)(y)dy,\ x \in (0,\infty) \text{ and } T \in (0,\infty),$$

and

$$h_\mu(f)(y) = \int_0^\infty z^{2\mu+1}(yz)^{-\mu} J_\mu(yz) f(z)dz,\ y \in (0,\infty).$$

Here J_μ denotes the Bessel function of the first kind and order μ.

Finally we prove a new inversion theorem for the Hankel transformation h_μ for $\mu \geq -1/2$.

1. Introduction

The Hankel transformation appears in different forms ([16], [18] and [21], amongst others). One of them is that defined by

$$(1) \qquad h_\mu(f)(y) = \int_0^\infty x^{2\mu+1}(xy)^{-\mu} J_\mu(xy) f(x)dx,\ y \in (0,\infty).$$

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As usual J_μ denotes the Bessel function of the first kind and order μ. Usually μ represents a real number greater or equal to $-1/2$. The h_μ-transformation has been extensively studied in the last years (see, for example, [1], [2], [5], [7], [10] and [13]). A. L. Schwartz [16] established an inversion formula for the transformation (1). More specifically he proved that if $x^{2\mu+1}f(x)$ is absolutely integrable on $(0,\infty)$, $x^{\mu+1/2}f(x)$ is absolutely integrable on $(0,1)$ and f is of bounded variation in a neighborhood of $x_0 \in (0,\infty)$ then

$$\lim_{T\to\infty} S_T(f,\mu;x_0) = \frac{f(x_0+0)+f(x_0-0)}{2}$$

where $S_T(f,\mu;x) = \int_0^T y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)h_\mu(f)(y)dy$, $T \in (0,\infty)$ and $x \in (0,\infty)$.

According to Corollary 1 [19] one obtains (Lemma 2.1 in the next section) that

$$\lim_{T\to\infty} \sigma_T(f,\mu;x) = f(x), \text{ for almost every } x \in (0,\infty)$$

provided that $x^{2\mu+1}f$ is absolutely integrable in $(0,\infty)$ and $\mu\in(-1/2,1/2)$, where

$$\sigma_T(f,\mu;x) = \int_0^T y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)\Big(1-\Big(\frac{y}{T}\Big)^2\Big)h_\mu(f)(y)dy,$$

for every $x \in (0,\infty)$ and $T \in (0,\infty)$.

Other important results are due to D. T. Haimo [7] and I. I. Hirschman [8].

The first part of this note is inspired by the paper of D. S. Lubinsky and F. Moricz [11] where the pointwise convergence of the Fourier transforms is investigated. Here we establish relations between S_T and σ_T. Those relations allow us to obtain new necessary and sufficient conditions for a measurable function f on $(0,\infty)$ to satisfy $\lim\limits_{T\to\infty} S_T(f,\mu;x) = f(x)$, for almost every $x \in (0,\infty)$, when $\mu \in (-1/2,1/2)$.

Finally, motivated by the paper of T. G. Genchev [6], we will prove a new inversion theorem for the Hankel transformation h_μ, when $\mu \geq -1/2$.

Throughout this paper we will denote by L_μ the space constituted by all measurable functions f on $(0,\infty)$ such that $x^{2\mu+1}f$ is absolutely integrable on $(0,\infty)$.

2. Pointwise convergence of Hankel transforms

We obtain in this section necessary and sufficient conditions in order that $\lim_{T\to\infty} S_T(f,\mu;x) = f(x)$, for almost every $x \in (0,\infty)$. We previously establish some useful results.

An immediate consequence of Corollary 1 [19] is the following

Lemma 2.1. *Let $f \in L_\mu$ and $\mu \in (-1/2, 1/2)$. Then*

$$\lim_{T\to\infty} \sigma_T(f,\mu;x) = f(x), \text{ for almost every } x \in (0,\infty).$$

Throughout this section we will assume that $\mu \in (-1/2, 1/2)$.

Lemma 2.2. *Let $f \in L_\mu$. If for every $T \in (0,\infty)$, $x \in (0,\infty)$ and $\lambda \in (1,\infty)$ we define $I_T(f,\mu,\lambda;x) = \frac{\lambda^2}{\lambda^2-1}\int_T^{\lambda T} y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)\times \Big(1-\big(\frac{y}{\lambda T}\big)^2\Big)h_\mu(f)(y)dy$, then*

$$S_T(f,\mu;x)-\sigma_T(f,\mu;x) = \frac{\lambda^2}{\lambda^2-1}\Big[\sigma_{\lambda T}(f,\mu;x)-\sigma_T(f,\mu;x)\Big]-I_T(f,\mu,\lambda;x)$$

for each $x \in (0,\infty)$, $\lambda \in (1,\infty)$ and $T \in (0,\infty)$.

Proof. Let $x \in (0,\infty)$, $\lambda \in (1,\infty)$ and $T \in (0,\infty)$. It is not hard to see that

$$\begin{aligned}
&\sigma_{\lambda T}(f,\mu;x) - \sigma_T(f,\mu;x) \\
&\quad= \int_T^{\lambda T} y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)\Big(1-\Big(\frac{y}{\lambda T}\Big)^2\Big)h_\mu(f)(y)dy \\
&\qquad+ \frac{\lambda^2-1}{\lambda^2}\int_0^{T} y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)\Big(\frac{y}{T}\Big)^2 h_\mu(f)(y)dy \\
&\quad= \int_T^{\lambda T} y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)\Big(1-\Big(\frac{y}{\lambda T}\Big)^2\Big)h_\mu(f)(y)dy \\
&\qquad+ \frac{\lambda^2-1}{\lambda^2}\Big(S_T(f,\mu;x)-\sigma_T(f,\mu;x)\Big)
\end{aligned}$$

and thus the proof is finished. □

Lemma 2.3. *Let f be in L_μ and such that $x^{2k}f \in L_\mu$ where $k \in \mathbb{N}\cup\{0\}$. Then $h_\mu(f)$ is k-times differentiable on I and*

$$\Big(\frac{1}{x}D\Big)^l h_\mu(f)(x) = (-1)^l h_{\mu+l}(f)(x),\ x \in (0,\infty), \tag{2}$$

for every $l \in \mathbb{N} \cup \{0\}$ *satisfying* $0 \leq l \leq k$. *Moreover if we also assume that* $x^{-\mu-1/2-k} f \in L_{\mu+k}$ *and* $x^{-\mu-1/2} f \in L_\mu$ *then*

$$x^{\mu+1/2+l}\left(\frac{1}{x}D\right)^l h_\mu(f)(x) \longrightarrow 0, \text{ as } x \to \infty, \quad \text{for every } l \in \mathbb{N} \cup \{0\},\ 0 \leq l \leq k. \tag{3}$$

Proof. According to Lemma 5.4-1 [21], in order to see (2) it is sufficient to differentiate under the integral sign. On the other hand (3) is an immediate consequence of (2) and of the Riemann Lebesgue Lemma for the Hankel transformation ([20], p. 457). □

We remark here that [15] and [17] established results similar to the one presented in Lemma 2.3.

In the sequel, for each $f \in L_\mu$ we will denote by A_f the set of all those $x \in (0, \infty)$ for which $\lim_{T\to\infty} \sigma_T(f, \mu; x) = f(x)$. According to Lemma 2.1 the Lebesgue measure of $(0, \infty)\backslash A_f$ is zero.

Now we obtain a necessary and sufficient condition for the validity of

$$\lim_{T\to\infty} S_T(f, \mu; x) = f(x), \text{ for every } x \in A_f.$$

Theorem 2.1. *Let* $f \in L_\mu$ *and* $x \in A_f$. *Then* $\lim_{T\to\infty} S_T(f, \mu; x) = f(x)$ *if and only if*

$$\lim_{\lambda\to 1^+} \limsup_{T\to\infty} |I_T(f, \mu, \lambda; x)| = 0,$$

where $I_T(f, \mu, \lambda; x)$ *is defined as in Lemma 2.2.*

Proof. According to Lemma 2.2 we have

$$\begin{aligned} &\Big| |S_T(f, \mu; x) - \sigma_T(f, \mu; x)| - |I_T(f, \mu, \lambda; x)| \Big| \\ &\leq \frac{\lambda^2}{\lambda^2 - 1} \Big| \sigma_{\lambda T}(f, \mu; x) - \sigma_T(f, \mu; x) \Big| \end{aligned} \tag{4}$$

for every $T \in (0, \infty)$ and $\lambda \in (1, \infty)$.

Our result can now be deduced from (4) and Lemma 2.1. □

As a consequence of Theorem 2.1 we obtain new sufficient conditions to ensure that

$$\lim_{T\to\infty} S_T(f, \mu; x) = f(x),$$

for every $x \in A_f$ and f being a function in L_μ.

Theorem 2.2. *Let f be in L_μ such that $x^{-\mu-1/2}f \in L_\mu$, $x^{2k}f \in L_\mu$ and $x^{-\mu+k-3/2}f \in L_\mu$, for some $k \in \mathbb{N}\cup\{0\}$. If*

$$\lim_{\lambda\to 1^+}\limsup_{T\to\infty}\int_T^{\lambda T} y^{2(\mu+k)+1}|h_{\mu+k}(f)(y)|dy = 0 \tag{5}$$

then $\lim_{T\to\infty} S_T(f,\mu;x) = f(x)$, *for every* $x \in A_f$.

Proof. Assume firstly $k=0$. Since $z^{-\nu}J_\nu(z)$ is a bounded function on $(0,\infty)$, in case $\nu \geq -1/2$ we have

$$\begin{aligned}
&\Big|I_T(f,\mu,\lambda;x)\Big| \\
&\leq \frac{\lambda^2}{\lambda^2-1}\int_T^{\lambda T} y^{2\mu+1}|(xy)^{-\mu}J_\mu(xy)|\Big(1-\Big(\frac{y}{\lambda T}\Big)^2\Big)|h_\mu(f)(y)|dy \\
&\leq C\int_T^{\lambda T} y^{2\mu+1}|h_\mu(f)(y)|dy,\ \text{for every } x, T\in(0,\infty) \text{ and } \lambda\in(1,\infty).
\end{aligned}$$

Here C denotes a suitable positive constant.

Hence (5) implies that

$$\lim_{\lambda\to 1^+}\limsup_{T\to\infty}|I_T(f,\mu,\lambda;x)| = 0,\ \text{for every } x\in(0,\infty).$$

Then according to Theorem 2.1 $\lim_{T\to\infty} S_T(f,\mu;x) = f(x)$, for every $x \in A_f$.

Assume now $k \in \mathbb{N}$. By virtue of 5.1(6) [21] partial integration leads to

$$\begin{aligned}
&\int_T^{\lambda T} y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)\Big(1-\Big(\frac{y}{\lambda T}\Big)^2\Big)h_\mu(f)(y)dy \\
&= x^{-2\mu-1}\int_T^{\lambda T}(xy)^{\mu+1}J_\mu(xy)\Big(1-\Big(\frac{y}{\lambda T}\Big)^2\Big)h_\mu(f)(y)dy \\
&= x^{-2\mu-2}\int_T^{\lambda T}\frac{d}{dy}\Big[(xy)^{\mu+1}J_{\mu+1}(xy)\Big]\Big(1-\Big(\frac{y}{\lambda T}\Big)^2\Big)h_\mu(f)(y)dy \\
&= x^{-2\mu-2}\Big\{\Big[(xy)^{\mu+1}J_{\mu+1}(xy)\Big(1-\Big(\frac{y}{\lambda T}\Big)^2\Big)h_\mu(f)(y)\Big]_{y=T}^{y=\lambda T}
\end{aligned}$$

$$-\int_T^{\lambda T}(xy)^{\mu+1}J_{\mu+1}(xy)\Big[\Big(1-\Big(\frac{y}{\lambda T}\Big)^2\Big)\frac{d}{dy}(h_\mu(f)(y))-h_\mu(f)(y)\frac{2y}{(\lambda T)^2}\Big]dy\Big\}$$

$$= x^{-2\mu-2}\Big\{-(xT)^{\mu+1}J_{\mu+1}(xT)\frac{\lambda^2-1}{\lambda^2}h_\mu(f)(T)$$

$$-\int_T^{\lambda T}(xy)^{\mu+1}J_{\mu+1}(xy)\Big(1-\Big(\frac{y}{\lambda T}\Big)^2\Big)y\Big(\frac{1}{y}\frac{d}{dy}\Big)\Big(h_\mu(f)(y)\Big)dy$$

$$+\frac{2}{(\lambda T)^2}\int_T^{\lambda T}(xy)^{\mu+1}J_{\mu+1}(xy)yh_\mu(f)(y)dy\Big\}$$

$$= -T^{2\mu+2}(xT)^{-\mu-1}J_{\mu+1}(xT)\frac{\lambda^2-1}{\lambda^2}h_\mu(f)(T)$$

$$-\int_T^{\lambda T}y^{2(\mu+1)+1}(xy)^{-\mu-1}J_{\mu+1}(xy)\Big(1-\Big(\frac{y}{\lambda T}\Big)^2\Big)\Big(\frac{1}{y}\frac{d}{dy}\Big)(h_\mu(f)(y))dy$$

$$+\frac{2}{(\lambda T)^2}\int_T^{\lambda T}y^{2(\mu+1)+1}(xy)^{-\mu-1}J_{\mu+1}(xy)h_\mu(f)(y)dy,$$

for every $x, T \in (0,\infty)$ and $\lambda \in (1,\infty)$.

Now we can write

$$\begin{aligned}I_T(f,\mu,\lambda;x) = &-T^{2\mu+2}(xT)^{-\mu-1}J_{\mu+1}(xT)h_\mu(f)(T)\\ &+\frac{2}{(\lambda^2-1)T^2}\int_T^{\lambda T}y^{2(\mu+1)+1}(xy)^{-\mu-1}J_{\mu+1}(xy)h_\mu(f)(y)dy\\ &+I_T(f,\mu+1,\lambda;x),\end{aligned}$$

for every $x \in (0,\infty)$, $\lambda \in (1,\infty)$ and $T \in (0,\infty)$.

By repeating the argument one obtains

$$I_T(f,\mu,\lambda;x) = -T^{2\mu+2}\sum_{j=1}^{k}(xT)^{-\mu-j}J_{\mu+j}(xT)h_{\mu+j-1}(f)(T)T^{2(j-1)}$$

$$+\frac{2}{(\lambda^2-1)T^2}\sum_{j=1}^{k}\int_T^{\lambda T}y^{2(\mu+j)+1}(xy)^{-\mu-j}J_{\mu+j}(xy)h_{\mu+j-1}(f)(y)dy$$

$$+I_T(f,\mu+k,\lambda;x),$$

for every $x, T \in (0,\infty)$ and $\lambda \in (1,\infty)$.

We now analyze each of the three terms of the above sum. Firstly we note that by proceeding as in the case $k = 0$ (5) implies that

$$\lim_{\lambda\to 1^+} \limsup_{T\to\infty} |I_T(f, \mu + k, \lambda; x)| = 0, \text{ for every } x \in (0, \infty). \tag{6}$$

Moreover, since for every $\nu \geq -1/2$, $\sqrt{z}J_\nu(z)$ is a bounded function on $(0, \infty)$, there exists $C > 0$ such that

$$\begin{aligned}\left|T^{2\mu+2}\sum_{j=1}^{k}(xT)^{-\mu-j}J_{\mu+j}(xT)h_{\mu+j-1}(f)(T)T^{2(j-1)}\right|& \\ \leq Cx^{-\mu-1/2}\sum_{j=1}^{k}x^{-j}T^{j+\mu-1/2}|h_{\mu+j-1}(f)(T)|,& \\ \text{for every } T, x \in (0, \infty).&\end{aligned}$$

Hence, since $x^{-(\mu+3/2-j)}f \in L_\mu$ for every $j = 1, 2, \ldots, k$, by virtue of the Riemann Lebesgue Lemma (Lemma 2.3) for the Hankel transformation we conclude that

$$\begin{aligned}\lim_{T\to\infty} T^{2\mu+2}\sum_{j=1}^{k}(xT)^{-\mu-j}J_{\mu+j}(xT)h_{\mu+j-1}(f)(T)T^{2(j-1)} = 0,& \\ \text{for every } x \in (0, \infty).&\end{aligned} \tag{7}$$

Finally, again taking into account that $\sqrt{z}J_\nu(z)$ is a bounded function on $(0, \infty)$ for every $\nu \geq -1/2$ there exists $C > 0$ such that

$$\begin{aligned}\frac{2}{(\lambda^2-1)T^2}&\left|\sum_{j=1}^{k}\int_T^{\lambda T} y^{2(\mu+j)+1}(xy)^{-\mu-j}J_{\mu+j}(xy)h_{\mu+j-1}(f)(y)dy\right| \\ &\leq \frac{Cx^{-\mu-1/2}\lambda}{(\lambda^2-1)T}\sum_{j=1}^{k}x^{-j}\int_T^{\lambda T}|h_{\mu+j-1}(f)(y)|y^{\mu+j-1/2}dy \\ &\leq \frac{Cx^{-\mu-1/2}\lambda}{\lambda+1}\sum_{j=1}^{k}x^{-j}\sup_{y>T}\left|h_{\mu+j-1}(f)(y)y^{\mu+j-1/2}\right|.\end{aligned}$$

Hence by invoking Lemma 2.3 we infer

$$\lim_{T\to\infty}\frac{2}{(\lambda^2-1)T^2} \tag{8}$$
$$\times\sum_{j=1}^{k}\int_T^{\lambda T} y^{2(\mu+j)+1}(xy)^{-\mu-j}J_{\mu+j}(xy)h_{\mu+j-1}(f)(y)dy=0$$

for every $\lambda\in(1,\infty)$ and $x\in(0,\infty)$.

By combining (6), (7) and (8) we conclude that

$$\lim_{\lambda\to 1^+}\limsup_{T\to\infty}|I_T(f,\mu,\lambda;x)|=0,\ \text{for every } x\in(0,\infty).$$

Now the result follows from Theorem 2.1. □

Theorem 2.3. *Let $f\in L_\mu$ be such that $x^{-\mu-1/2}f\in L_\mu$, $x^{2k}f\in L_\mu$ and $x^{-\mu+k-3/2}f\in L_\mu$, for some $k\in\mathbb{N}\cup\{0\}$. Then*

$$\lim_{T\to\infty}S_T(f,\mu;x)=f(x),\ \text{for every } x\in A_f,$$

provided that $\lim_{y\to\infty}y^{2(\mu+k+1)}|h_{\mu+k}(f)(y)|<\infty$.

Proof. It is clear that there exist C and $y_0\in(0,\infty)$ such that

$$y^{2(\mu+k)+1}|h_{\mu+k}(f)(y)|\leq\frac{C}{y},\ \text{for every } y\in(y_0,\infty).$$

Hence for every $T\geq y_0$ and $\lambda\in(1,\infty)$ one has

$$\int_T^{\lambda T}y^{2(\mu+k)+1}|h_{\mu+k}(f)(y)|dy\leq C\int_T^{\lambda T}\frac{dy}{y}=C\log\lambda.$$

Then

$$\lim_{\lambda\to 1^+}\limsup_{T\to\infty}\int_T^{\lambda T}y^{2(\mu+k)+1}|h_{\mu+k}(f)(y)|dy=0,$$

and by invoking Theorem 2.2 we conclude the proof. □

Theorem 2.4. *Let f be in L_μ such that $x^{-\mu-1/2}f\in L_\mu$, $x^{2k}f\in L_\mu$ and $x^{-\mu+k-3/2}f\in L_\mu$, for some $k\in\mathbb{N}\cup\{0\}$. If*

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x y^{2(\mu+k+1)}|h_{\mu+k}(f)(y)|dy \tag{9}$$

is finite, then $\lim_{T\to\infty}S_T(f,\mu;x)=f(x)$ for every $x\in A_f$.

Proof. For every $T \in (0,\infty)$ and $\lambda \in (1,\infty)$, we have

$$\int_T^{\lambda T} y^{2(\mu+k)+1}|h_{\mu+k}(f)(y)|dy$$
$$\leq \frac{1}{T}\left(\int_0^{\lambda T} y^{2(\mu+k+1)}|h_{\mu+k}(f)(y)|dy - \int_0^{T} y^{2(\mu+k+1)}|h_{\mu+k}(f)(y)|dy\right).$$

If $L = \lim_{x\to\infty} \frac{1}{x}\int_0^x y^{2(\mu+k+1)}|h_{\mu+k}(f)(y)|dy$, then for every $\epsilon > 0$

$$\int_T^{\lambda T} y^{2(\mu+k)+1}|h_{\mu+k}(f)(y)|dy \leq \lambda(L+\epsilon) - (L-\epsilon)$$

for T sufficiently large and $\lambda \in (1,\infty)$.

Hence, the arbitrariness of ϵ allows us to write

$$\limsup_{T\to\infty} \int_T^{\lambda T} y^{2(\mu+k)+1}|h_{\mu+k}(f)(y)|dy \leq (\lambda-1)L \text{ for every } \lambda \in (1,\infty).$$

Thus we conclude

$$\lim_{\lambda\to 1^+} \limsup_{T\to\infty} \int_T^{\lambda T} y^{2(\mu+k)+1}|h_{\mu+k}(f)(y)|dy = 0$$

and our result is deduced from Theorem 2.2. □

Note that (9) is equivalent, under the imposed requirements for the functions f, to the following condition

$$\lim_{x\to\infty} \frac{1}{x-x_0}\int_{x_0}^x y^{2(\mu+k+1)}|h_{\mu+k}(f)(y)|dy < \infty \tag{10}$$

for some $x_0 \in (0,\infty)$. (10) is analogous to the condition that appears in Theorem 4 [11].

Remark 1. Results established in Theorems 2.1–2.4 complete well-known results about convergence of $S_T(f,\mu;x)$ (see, for example, Corollary 2 [9] and Corollary 3.2 [4]).

Remark 2. The results in this Section hold when $-1/2 < \mu < 1/2$. This condition allows to establish Lemma 2.1. It is well-known ([19]) that for every $f \in L_\mu$ $\lim_{T\to\infty} \sigma_T^\beta(f,\mu;x) = f(x)$, a.e. $x \in (0,\infty)$, provided that $-1/2 < \mu < \beta - 1/2$, where

$$\sigma_T^\beta(f,\mu;x) = \int_0^T y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)\Big(1-(\frac{y}{T})^2\Big)^\beta h_\mu(f)(y)dy,$$
$$x \in (0,\infty).$$

However our technique is not suitable when β is not equal to 1. It is an open problem to obtain results similar to the ones established in Theorems 2.1–2.4. Thus we could obtain conditions to reconstruct a radial function f from the Fourier transform of f because the n-dimensional transform of radial functions reduces to Hankel transform.

3. A new inversion theorem for the Hankel transformation

Different inversion theorems for Hankel transformations have been established (see, for example, [7], [8], [14], [16] and [20]). We now prove, inspired by the work of T.G. Genchev [6] on the Fourier transform, a new inversion theorem for the Hankel transformation. We note that our inversion theorem holds for those measurable functions f on $(0,\infty)$ such that $x^{-\mu-1/2}f$ and $x^{-\mu-1/2}h_\mu(f)$ are in L_μ. However the inversion formulas for h_μ established earlier by other authors apply to other functions. Specifically Theorem 1 [14] (adapted to h_μ by making a single change of variable) holds provided that $x^{-\mu-1/2}f \in L_\mu$ and f is continuous on $(0,\infty)$. A.L. Schwartz [16] established that if $f \in L_\mu$, $x^{-\mu-1/2}f$ is absolutely integrable on $(0,1)$ and f is of bounded variation in a neighborhood of x then

$$\lim_{y\to\infty}\int_0^y (zx)^{-\mu}J_\mu(zx)z^{2\mu+1}h_\mu(f)(z)dz = \frac{f(x+0)+f(x-0)}{2}. \tag{11}$$

In Theorem 5.1-1 [21] it is showed that (11) holds provided that $x^{-\mu-1/2}f \in L_\mu$ and f is of bounded variation in a neighborhood of x. Inversion theorems proved in Corollary 2.10 [7] and Corollary 2.e [8] hold when $f \in L_\mu$ and $h_\mu(f) \in L_\mu$.

Theorem 3.1. *Let $\mu \geq -1/2$ and let f be a measurable function on $(0,\infty)$ such that $x^{-\mu-1/2}f \in L_\mu$ and $x^{-\mu-1/2}h_\mu(f) \in L_\mu$. Then $h_\mu(h_\mu f)(x) = f(x)$ for almost every $x \in (0,\infty)$.*

Proof. Let $0 < a < b < \infty$ and define

$$\varphi_{a,b}(x) = \begin{cases} x^{-\mu-1/2}, & \text{if } x \in (a,b) \\ 0, & \text{otherwise.} \end{cases}$$

By invoking [16] we can obtain

$$\lim_{T\to\infty}\int_0^T x^{2\mu+1}(xy)^{-\mu}J_\mu(xy)h_\mu(\varphi_{a,b})(x)dx = \varphi_{a,b}(y) \tag{12}$$

$$\text{for almost every } y \in (0,\infty).$$

By invoking Fubini's Theorem it is easy to see that

$$\int_0^\infty h_\mu(h_\mu f)(x)\varphi_{a,b}(x)x^{2\mu+1}dx = \int_0^\infty h_\mu(f)(x)h_\mu(\varphi_{a,b})(x)x^{2\mu+1}dx. \tag{13}$$

Also, for every $m \in \mathbb{N}$ one has

$$\int_0^m h_\mu(f)(x)h_\mu(\varphi_{a,b})(x)x^{2\mu+1}dx$$
$$= \int_0^\infty f(x)x^{2\mu+1}\int_0^m y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)h_\mu(\varphi_{a,b})(y)dydx.$$

Let $m \in \mathbb{N}$. According to 5.11 (8) [20] we have

$$x^{\mu+1/2}\int_0^m y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)h_\mu(\varphi_{a,b})(y)dy$$
$$= \int_a^b (xz)^{1/2}\int_0^m yJ_\mu(zy)J_\mu(xy)dydz = \int_a^b \frac{(xz)^{1/2}}{x^2-z^2}$$
$$\times\Big(xmJ_{\mu+1}(xm)J_\mu(zm) - zmJ_\mu(xm)J_{\mu+1}(zm)\Big)dz,\ x \in (0,\infty).$$

Hence by virtue of Lemma 7 [12] there exists $C > 0$ such that

$$\left|x^{\mu+1/2}\int_0^m y^{2\mu+1}(xy)^{-\mu}J_\mu(xy)h_\mu(\varphi_{a,b})(y)dy\right| \le C,$$
$$x \in (0,\infty) \text{ and } m \in \mathbb{N}.$$

Then, since $x^{-\mu-1/2}f \in L_\mu$, by (12) and (13) the dominated convergence theorem leads to

$$\int_0^\infty h_\mu(h_\mu f)(x)\varphi_{a,b}(x)x^{2\mu+1}dx = \int_0^\infty f(x)\varphi_{a,b}(x)x^{2\mu+1}dx.$$

We conclude that if φ is a step function with compact support on $(0,\infty)$ then

$$\int_0^\infty x^{\mu+1/2}\Big[h_\mu(h_\mu f)(x) - f(x)\Big]\varphi(x)dx = 0.$$

Hence $h_\mu(h_\mu f)(x) = f(x)$ for almost every $x \in (0,\infty)$ and the proof is finished. $\square$

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