Topogenous g-mappings

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0. Introduction

This paper presents a generalization of the notion of an *E-mapping*, which was introduced by M. HACQUE in [3] and [4].

Considering a set E, an E-mapping is a mapping $\mathfrak{z}$ of the power set 2^E into the set of all systems of subsets of E with the following properties for $A, B, X, Y \subset E$:

(E0) $\mathfrak{z}(A) \neq \emptyset$, *and* $Y \supset X \in \mathfrak{z}(A)$ *implies* $Y \in \mathfrak{z}(A)$.

(E1) $\mathfrak{z}(\emptyset) = 2^E$.

(E2) $X \in \mathfrak{z}(A)$ *implies* $A \subset X$.

(E3) $A \subset B$ *implies* $\mathfrak{z}(A) \supset \mathfrak{z}(B)$.

In [2] S. GACSÁLYI proved that the correspondence

$$(0.1) \qquad < \to \mathfrak{z}_< : \mathfrak{z}_<(A) = \{X : A < X\}$$

defined between semi-topogeneous orders on E and E-mappings is one-to-one, and its inverse can be produced as follows:

$$(0.2) \qquad \mathfrak{z} \to <_\mathfrak{z} : A <_\mathfrak{z} B \quad \textit{iff} \quad B \in \mathfrak{z}(A).$$

These concepts can be improved by a simple idea. Let g be a (single-valued) mapping of a set E onto a dense subset of the syntopogenous space $[E', \mathcal{S}']$; we frequently meet this situation in theory of extensions of syntopogenous spaces. If $<' \in \mathcal{S}'$, and as a value of an arbitrary $A' \subset E'$ we prescribe the system of those subsets X of E, which include the inverse image $g^{-1}(X')$ of a set $X' \subset E'$ such that $A' <' X'$, we get a mapping of $2^{E'}$ into the set of all systems of subsets of E with properties similar to (E0)—(E3).

Starting out from the observation mentioned above, in chapter 1 of the paper the notion of a *g-mapping* will be defined for a fixed mapping $g: E \to E'$. In part 2 a correspondence will be produced between semi-topogeneous orders on E' and g-mappings like in (0.1), but in general this will be not one-to-one. The main result of this paragraph is a generalization of Gacsályi's perception, namely the correspondence in question is an injection (a surjection), iff the mapping g is a surjection (an injection), consequently it is one-to-one, iff g is of this kind, too.

The special families of topogeneous g-mappings (so called *syntopogenous g-families*) and their application in the theory of extensions of syntopogenous spaces will be found in another notice of the author [5].

1. The notion of a g-mapping

Let us consider two sets E, E' and a single valued mapping $g: E \to E'$ (E, E' and g will be fixed throughout the whole paper). By a *g-mapping* we shall mean a mapping $\mathfrak{z}$ defined on the power set $2^{E'}$ into the set of all systems of subsets of E, which satisfies the following conditions for $X, Y \subset E$ and $A', B' \subset E'$:

(M0) $\mathfrak{z}(A') \neq \emptyset$, *and* $Y \supset X \in \mathfrak{z}(A')$ *implies* $Y \in \mathfrak{z}(A')$.

(M1) $\emptyset \in \mathfrak{z}(A')$ *iff* $A' = \emptyset$.

(M2) $X \in \mathfrak{z}(A')$ *implies* $g^{-1}(A') \subset X$.

(M3) $A' \subset B'$ *implies* $\mathfrak{z}(A') \supset \mathfrak{z}(B')$.

(1.1) **Proposition.** *If $E = E'$ and g is the identity of E, then the g-mappings are identical with the E-mappings.*

Proof. Under the conditions of the proposition the equivalences (M0)$\Leftrightarrow$(E0), (M2)$\Leftrightarrow$(E2), (M3)$\Leftrightarrow$(E3) and the implication (M0)$\cap$(M1)$\Rightarrow$(E1) are obvious. For the verification of (E1)$\cap$(E2)$\Rightarrow$(M1) let us observe that by (E2) $\emptyset \in \mathfrak{z}(A)$ implies $A = g^{-1}(A) \subset \emptyset$, thus $A = \emptyset$. ∎

(1.2) *Remark.* In (M1) we had to postulate that $\emptyset \in \mathfrak{z}(A')$ implies $A' = \emptyset$, because in general for a set $A' \subset E' - g(E)$ the assumption $\emptyset \in \mathfrak{z}(A')$ and (M2) does not imply $A' = \emptyset$. Leaving this out of consideration we should have got a generalization of the notion of an E-mapping infiitted for the applications in extension theory.

A g-mapping $\mathfrak{z}$ will be said to be *topogenous*, if for any $X, Y \subset E$ and $A', B' \subset E'$

(MQ′) $X \in \mathfrak{z}(A'), Y \in \mathfrak{z}(B')$ *implies* $X \cap Y \in \mathfrak{z}(A' \cap B')$

and

(MQ″) $X \in \mathfrak{z}(A'), Y \in \mathfrak{z}(B')$ *implies* $X \cup Y \in \mathfrak{z}(A' \cup B')$

(cf. ch. 3 of [1]). The topogenous g-mappings have a simple lattice theoretical characterization, in which, as in general topology it is usual, by a (filter) ideal in E we mean a (proper filter) ideal of the lattice of all subsets of E.

(1.3) **Lemma.** *A g-mapping $\mathfrak{z}$ is topogenous iff*

(1.3.1) *$\mathfrak{z}(A')$ is a filter in E for any $\emptyset \neq A' \subset E'$,*

and

(1.3.2) *the system $\{A' \subset E' : X \in \mathfrak{z}(A')\}$ is an ideal in E' for every $X \subset E$.*

PROOF. Let $\mathfrak{z}$ be topogenous. By (M1) $\emptyset \notin \mathfrak{z}(A')$, and because of (M0) $\mathfrak{z}(A')$ is increasing in E. If $X, Y \in \mathfrak{z}(A')$, then $X \cap Y \in \mathfrak{z}(A' \cap A') = \mathfrak{z}(A')$. Similarly, by (M3) the system in question is decreasing in E' for every $X \subset E$. If $X \in \mathfrak{z}(A')$ and $X \in \mathfrak{z}(B')$, then $X = X \cup X \in \mathfrak{z}(A' \cup B')$.

Conversely, if (1.3.1) is satisfied by $\mathfrak{z}$, then $X \in \mathfrak{z}(A')$, $Y \in \mathfrak{z}(B')$ together with (M3) imply $X, Y \in \mathfrak{z}(A' \cap B')$, and from this $X \cap Y \in \mathfrak{z}(A' \cap B')$ follows. If (1.3.2) holds, then because of (M0) $X \in \mathfrak{z}(A')$, $Y \in \mathfrak{z}(B')$ gives $X \cup Y \in \mathfrak{z}(A') \cap \mathfrak{z}(B')$, thus from our condition $X \cup Y \in \mathfrak{z}(A' \cup B')$. ∎

Let $\mathfrak{z}_1, \mathfrak{z}_2$ be two g-mappings. We shall say that $\mathfrak{z}_1$ is *coarser* than $\mathfrak{z}_2$, or equivalently $\mathfrak{z}_2$ is *finer* than $\mathfrak{z}_1$, iff for every $A' \subset E'$ the inequality $\mathfrak{z}_1(A') \subset \mathfrak{z}_2(A')$ holds. This will be denoted by $\mathfrak{z}_1 \sqsubset \mathfrak{z}_2$.

(1.4) *Remarks*. Let $\mathfrak{z}$ be a g-mapping. If there exists a topogenous g-mapping finer than $\mathfrak{z}$, then $\mathfrak{z}(A')$ is centred for any $A' \neq \emptyset$. Using the simple notation $\mathfrak{z}(x')$ instead of $\mathfrak{z}(\{x'\})$ for points x' of E', this condition is satisfied by $\mathfrak{z}$ iff $\mathfrak{z}(x')$ is centred for every $x' \in E'$. In this case there exists a topogenous g-mapping $\mathfrak{z}^q$, which is the coarsest of all topogenous g-mappings finer than $\mathfrak{z}$. $\mathfrak{z}^q$ can be defined as follows: (1.4.1) $X \in \mathfrak{z}^q(A')$ *iff there are natural numbers* m *and* n *such that* $A' = \bigcup_{i=1}^{m} A'_i$, $X = \bigcap_{j=1}^{n} X_j$ *and* $X_j \in \mathfrak{z}(A'_i)$ *for any* $1 \leqq i \leqq m$, $1 \leqq j \leqq n$.

(Cf. [1], (3.7).) These facts have no importance from the point of view of our further studies, therefore their proof will be omitted.

A g-mapping $\mathfrak{z}$ will be called *perfect*, if for an arbitrary set I of indices

$$\text{(MP)} \qquad X_i \in \mathfrak{z}(A'_i) \quad (i \in I) \quad \textit{implies} \quad \bigcup_{i \in I} X_i \in \mathfrak{z}\Big(\bigcup_{i \in I} A'_i\Big)$$

(cf. ch. 4 of [1]). We shall prove that the structure of a perfect g-mapping is uniquely determined by its values taken on the subsets of E' having one element.

(1.5) **Theorem.** *A g-mapping $\mathfrak{z}$ is perfect iff*

$$\text{(1.5.1)} \qquad \mathfrak{z}(A') = \bigcap \{\mathfrak{z}(x') : x' \in A'\}$$

holds for any $A' \neq \emptyset$. If $\mathfrak{z}$ is an arbitrary g-mapping, then there exists a perfect g-mapping $\mathfrak{z}^p$, which is the coarsest of all perfect g-mappings finer than $\mathfrak{z}$. For $A' \neq \emptyset$, $\mathfrak{z}^p(A')$ can be defined by

$$\text{(1.5.2)} \qquad \mathfrak{z}^p(A') = \bigcap \{\mathfrak{z}(x') : x' \in A'\}$$

and $\mathfrak{z}^p(\emptyset) = 2^E$.

PROOF. For an arbitrary g-mapping $\mathfrak{z}$ the inequality

$$\mathfrak{z}(A') \subset \bigcap \{\mathfrak{z}(x') : x' \in A'\}$$

is obvious for $\emptyset \neq A' \subset E'$ (see (M3)). Let $\mathfrak{z}$ be perfect, and $X \in \mathfrak{z}(x')$ for any $x' \in A'$. In this case because of $A' = \bigcup_{x' \in A'} \{x'\}$ we have $X \in \mathfrak{z}(A')$. Conversely, suppose (1.5.1) and $X_i \in \mathfrak{z}(A'_i)$ $(i \in I)$, $X = \bigcup_{i \in I} X_i$, $A' = \bigcup_{i \in I} A'_i$. Then $x' \in A'$ implies $x' \in A'_i$

for some $i \in I$, form this $X \supset X_i \in \mathfrak{z}(x')$. Consequently $X \in \mathfrak{z}(x')$ for every $x' \in A'$, thus by (1.5.1) $X \in \mathfrak{z}(A')$.

It is easy to prove that $\mathfrak{z}^p$ is a g-mapping for any g-mapping $\mathfrak{z}$, and it is perfect, since $\mathfrak{z}^p(x') = \mathfrak{z}(x')$ for any $x' \in E'$, hence $\mathfrak{z}^p$ satisfies (1.5.1). Finally if $\mathfrak{z}_1$ is a perfect g-mapping finer than $\mathfrak{z}$, then, for any $x' \in E'$, $\mathfrak{z}(x') \subset \mathfrak{z}_1(x')$, consequently by (1.5.1) $\mathfrak{z}^p(A') \subset \mathfrak{z}_1(A')$ for every $\emptyset \neq A' \subset E'$. ∎

(1.6) **Theorem.** *A mapping* $\mathfrak{z}$ *of* $2^{E'}$ *into the set of all systems of subsets of* E *is a perfect topogenous g-mapping iff for every* $x' \in E'$ *there exists a filter* $\mathfrak{f}(x')$ *in* E *such that*

(1.6.1) $x \in E$, $X \in \mathfrak{f}(\mathfrak{g}(x))$ *implies* $x \in X$,

further

(1.6.2) $\mathfrak{z}(\emptyset) = 2^E$ *and* $\mathfrak{z}(A') = \cap \{\mathfrak{f}(x') : x' \in A'\}$ *for any* $\emptyset \neq A' \subset E'$.

Proof. If $\mathfrak{z}$ is a perfect topogenous g-mapping, then $\mathfrak{f}(x') = \mathfrak{z}(x')$ satisfies these conditions for any $x' \in E'$ (see (1.3) and (1.5)). Conversely, let $\mathfrak{f}(x')$ be a filter in E for $x' \in E'$ with conditions (1.6.1)—(1.6.2). $\emptyset \neq \mathfrak{z}(A')$ for any $A' \subset E'$, since $E \in \mathfrak{z}(A')$. Every $\mathfrak{f}(x')$ is increasing in E, therefore $Y \supset X \in \mathfrak{z}(A')$ implies $Y \in \mathfrak{z}(A')$. The fulfilment of (M1) follows from $\emptyset \notin \mathfrak{f}(x')$ $(x' \in E')$, and from (1.6.2). $X \in \mathfrak{z}(A')$, $x \in g^{-1}(A')$ implies $X \in \mathfrak{f}(\mathfrak{g}(x))$, thus $x \in X$. Finally if $A' \subset B'$, then

$$\cap \{\mathfrak{f}(x') : x' \in B'\} \subset \cap \{\mathfrak{f}(x') : x' \in A'\}$$

is trivial. We got that $\mathfrak{z}$ is a g-mapping. $\mathfrak{z}(x') = \mathfrak{f}(x')$ and (1.5.1) give that $\mathfrak{z}$ is perfect, consequently it satisfies axiom (MQ″), too. Let $X \in \mathfrak{z}(A')$ and $Y \in \mathfrak{z}(B')$. If $x' \in A' \cap B'$, then $X, Y \in \mathfrak{f}(x')$, and since $\mathfrak{f}(x')$ is a filter, we get $X \cap Y \in \mathfrak{f}(x')$, accordingly $X \cap Y \in \mathfrak{z}(A' \cap B')$. ∎

(1.7) **Corollary.** *If* $\mathfrak{z}$ *is a topogenous g-mapping, then* $\mathfrak{z}^p$ *is perfect and topogenous.* ∎

2. g-mappings and semi-topogenous orders

Let $<$ be a semi-topogenous order on the set X [1]. A subset X_0 of X is called *$<$-dense*, if $x < V \subset X$ implies $X_0 \cap V \neq \emptyset$ for any $x \in X$ (or equivalently, if $x \in X$, then $x < X - X_0$ does not hold).

We shall prove that a g-mapping $\mathfrak{z}_{<'}$ can be assigned to any semi-topogenous order $<'$ on E', provided $g(E)$ is $<'$-dense.

(2.1) **Proposition.** *If* $<'$ *is a semi-topogenous order on* E' *such that* $g(E)$ *is* $<'$*-dense, then the definition*

(2.1.1) $\mathfrak{z}_{<'}(A') = \{X \subset E : A' <' E' - \mathfrak{g}(E - X)\}$ $(A' \subset E')$ *yields a g-mapping* $\mathfrak{z}_{<'}$, *which will be called the g-mapping deduced from* $<'$.

Before the verification of (2.1) we give a lemma explaining (2.1.1):

(2.2) **Lemma.** *Under the conditions of* (2.1) $X \in \mathfrak{z}_{<'}(A')$ *iff there exists a set* $X' \subset E'$ *such that* $A' <' X'$ *and* $g^{-1}(X') \subset X$.

PROOF. In fact, if there is such a set X', then $E-X\subset E-g^{-1}(X')$, thus $g(E-X)\subset g(E-g^{-1}(X'))\subset E'-X'$, therefore $A'<'X'\subset E'-g(E-X)$. Conversely, if $X\in \mathfrak{z}_{<'}(A')$, then putting $X'=E'-g(E-X)$, we have $A'<'X'$ and $g^{-1}(X')= =E-g^{-1}(g(E-X))\subset X$. ∎

PROOF OF (2.1). Let us consider the system of axioms defining a semi-topogenous order (see [1], $(0_1)-(0_3)$), and prove the validity of (M0) – (M3).

(M0): $A'<'E'-g(E-E)$, thus $E\in \mathfrak{z}_{<'}(A')$. If $X\subset Y$ and $X\in \mathfrak{z}_{<'}(A')$, then by (2.2) $Y\in \mathfrak{z}_{<'}(A')$.

(M1): $\emptyset<'\emptyset$ implies $\emptyset\in \mathfrak{z}_{<'}(\emptyset)$. If $\emptyset\in \mathfrak{z}_{<'}(A')$ and $x'\in A'$, then $x'<'E'-g(E)$ (see (0_3)), but this contradicts the density of $g(E)$.

(M2) If $X\in \mathfrak{z}_{<'}(A')$ and X' is a set required by (2.2), then because of (0_2) we have $g^{-1}(A')\subset g^{-1}(X')\subset X$.

(M3): If $A'\subset B'$ and $X\in \mathfrak{z}_{<'}(B')$, then $B'<'E'-g(E-X)$, thus (0_3) and (2.1.1) imply $X\in \mathfrak{z}_{<'}(A')$. ∎

We show that the notion of a topogenous or perfect g-mapping introduced in part 1 very exactly corresponds to the concept of a topogenous or perfect semi-topogenous order [1].

(2.3) **Proposition.** *Let $<'$ be a semi-topogenous order on E', and $g(E)$ be $<'$-dense.*

(2.3.1) *If $<'$ is topogenous, then $\mathfrak{z}_{<'}$ is topogenous, too.*

(2.3.2) *$\mathfrak{z}^p_{<'}=\mathfrak{z}_{<'^p}$ consequently if $<'$ is perfect, then $\mathfrak{z}_{<'}$ is also of this kind.*

PROOF. (2.3.1): If $X\in \mathfrak{z}_{<'}(A')$ and $Y\in \mathfrak{z}_{<'}(B')$, then $A'<'E'-g(E-X)$ and $B'<'E'-g(E-Y)$, consequently $A'\cap B'<'(E'-g(E-X))\cap(E'-g(E-Y))=E'- -g(E-(X\cap Y))$, and $A'\cup B'<'(E'-g(E-X))\cup(E'-g(E-Y))\subset E'-g(E-(X\cup Y))$. In view of these we get $X\cap Y\in \mathfrak{z}_{<'}(A'\cap B')$ and similary $X\cup Y\in \mathfrak{z}_{<'}(A'\cup B')$.

(2.3.2): $g(E)$ is obviously $<'^p$-dense, and $X\in \mathfrak{z}^p_{<'}(A')\Leftrightarrow x'<'E'-g(E-X)$ for any $x'\in A'\Leftrightarrow A'<'^pE'-g(E-X)\Leftrightarrow X\in \mathfrak{z}_{<'^p}(A')$ (see [1], (4.7)).

If $<'$ is perfect, then $<'^p=<'$, therefore $\mathfrak{z}^p_{<'}=\mathfrak{z}_{<'}$. This means that $\mathfrak{z}_{<'}$ is perfect (see (1.5)). ∎

It is easy to show that if $E=E'$ and g is the identity of E, then by the definition (2.1.1) the correspondence described in (0.1) is obtained. After this we shall study the effect of the mapping g having on the properties of the correspondence $<'\to\mathfrak{z}_{<'}$. The connection existing between the behaviour of $<'\to\mathfrak{z}_{<'}$ and g can be formulated in the following general duality theorem:

(2.4) **Theorem.** *If E' consists of at least two elements, then*

(2.4.1) *$<'\to\mathfrak{z}_{<'}$ is surjective iff g is injective;*

(2.4.2) *$<'\to\mathfrak{z}_{<'}$ is injective iff g is surjective;*

(2.4.3) *$<'\to\mathfrak{z}_{<'}$ is one-to-one iff g is of this kind.*

The case of a set E' containing one element x' is trivial, namely the unique semi-topogenous order on E' is $<' = <_{\emptyset, E'}$ and the unique g-mapping is $\mathfrak{z}_{<'}$.

The proof of theorem (2.4) can be read from (2.5), (2.7), (2.12) and (2.13), finally (2.4.1) and (2.4.2) imply (2.4.3).

(2.5) *Examples.* Let E' be a set consisting of at least two elements, and suppose that g *is not an injection.* We shall study two different cases.

(2.5.1) g is not a surjection.

Put $\mathfrak{f}(g(x)) = \{X \subset E : g^{-1}(g(x)) \subset X\}$ for any $x \in E$, and $\mathfrak{f}(x') = \{X \subset E : x_0 \in X\}$ for every $x' \in E' - g(E)$, where x_0 is a fixed element of E such that $g^{-1}(g(x_0))$ contains at least two elements. The filters $\mathfrak{f}(x')$ $(x' \in E')$ satisfy condition (1.6.1), therefore we have a perfect topogenous g-mapping $\mathfrak{z}$ defined by (1.6.1), but this g-mapping *cannot be deduced* from some semi-topogenous order $<'$ on E' such that $g(E)$ is $<'$-dense.

(In fact, assume that $<'$ is such an order. Then $\{x_0\} \in \mathfrak{z}(x')$ for an $x' \in E' - g(E)$. Because of the choice of x_0 this means $x' <' E' - g(E - x_0) = E' - g(E)$, which is impossible, since $g(E)$ is $<'$-dense.

(2.5.2) g is a surjection.

Suppose that x_0 and y_0 are fixed elements of E such that $g^{-1}(g(x_0))$ has at least two members, and $g(x_0) \neq g(y_0)$. Put $\mathfrak{f}(g(x)) = \{X \subset E : g^{-1}(g(x)) \subset X\}$ for $y_0 \neq x \in E$, and $\mathfrak{f}(g(y_0)) = \{X \subset E : V \subset X\}$, where $V = g^{-1}(g(y_0)) \cup \{x_0\}$. Since the filters $\mathfrak{f}(x')$ $(x' \in E')$ have property (1.6.1), we can define a perfect topogenous g-mapping $\mathfrak{z}$ in accordance with (1.6.2). $\mathfrak{z}$ *cannot be deduced* from some semi-topogenous order, that is there does not exist $<'$ on E' such that $\mathfrak{z} = \mathfrak{z}_{<'}$ (in this case $g(E) = E'$ is obviously $<'$-dense). Indeed, $V \in \mathfrak{z}(g(y_0))$, consequently there is a set X' such that $g(y_0) <' X'$ and $g^{-1}(X') \subset V$. At the same time $g^{-1}(X') \in \mathfrak{z}(g(y_0)) = \mathfrak{f}(g(y_0))$, therefore $V \subset g^{-1}(X')$. From these $g^{-1}(X') = V$, but because of the properties of x_0 this is an impossibility. ∎

Further we shall consider a semi-topogenous order on E' for any g-mapping $\mathfrak{z}$, from which $\mathfrak{z}$ can be deduced, provided g is an injection.

(2.6) **Proposition.** *Let $\mathfrak{z}$ be a g-mapping. Then a semi-topogenous order $<_{\mathfrak{t}_\mathfrak{z}}$ can be defined on E' by the following formula:*

(2.6.1) $A' <_{\mathfrak{t}_\mathfrak{z}} B' \Leftrightarrow A' \subset B'$ *and* $g^{-1}(B') \in \mathfrak{z}(A')$.

Proof. $\emptyset <_{\mathfrak{t}_\mathfrak{z}} \emptyset$ and $E' <_{\mathfrak{t}_\mathfrak{z}} E'$ obviously hold. By the definition $A' <_{\mathfrak{t}_\mathfrak{z}} B'$ implies $A' \subset B'$. Finally let us suppose $A' \subset A_1' <_{\mathfrak{t}_\mathfrak{z}} B_1' \subset B'$. Then $A' \subset A_1' \subset B_1' \subset B'$ and $g^{-1}(B') \supset g^{-1}(B_1') \in \mathfrak{z}(A_1') \subset \mathfrak{z}(A')$ (see (M3)), hence $A' \subset B'$ and $g^{-1}(B') \in \mathfrak{z}(A')$ (cf. (M0)), that is $A' <_{\mathfrak{t}_\mathfrak{z}} B'$. ∎

(2.7) **Theorem.** *If $\mathfrak{z}$ is an arbitrary g-mapping, then $<_{\mathfrak{t}_\mathfrak{z}}$ is the finest of all semi-topogenous orders $<'$ on E' such that $g(E)$ is $<'$-dense and $\mathfrak{z}_{<'} \sqsubset \mathfrak{z}$. In particular, if g is an injection, then $\mathfrak{z}_{<_{\mathfrak{t}_\mathfrak{z}}} = \mathfrak{z}$, i.e. $\mathfrak{z}$ can be deduced from $<_{\mathfrak{t}_\mathfrak{z}}$.*

Proof. If $x' \in E'$ and $x' <_{\mathfrak{t}_\mathfrak{z}} B'$, then $g^{-1}(B') \in \mathfrak{z}(x')$, hence by (M1) $\emptyset \neq g(g^{-1}(B')) = g(E) \cap B'$. This shows that $g(E)$ is $<_{\mathfrak{t}_\mathfrak{z}}$-dense. If $A' <_{\mathfrak{t}_\mathfrak{z}} X'$ and

$g^{-1}(X')\subset X$, then we have $g^{-1}(X')\in\mathfrak{z}(A')$, and from (M0) $X\in\mathfrak{z}(A')$ follows that is $\mathfrak{z}_{<_{\uparrow\mathfrak{z}}}\sqsubset\mathfrak{z}$. Let $<'$ be a semi-topogenous order on E' such that $g(E)$ is $<'$-dense, and $\mathfrak{z}_{<'}\sqsubset\mathfrak{z}$. If $A'<'B'$, then $A'\subset B'$ and $g^{-1}(B')\in\mathfrak{z}_{<'}(A')\subset\mathfrak{z}(A')$, therefore $A'<_{\uparrow\mathfrak{z}}B'$, so that $<'\sqsubset<_{\uparrow\mathfrak{z}}$. Finally assume that g is an injection, and $X\in\mathfrak{z}(A')$. Then for $X'=A'\cup g(X)$ we have $A'\subset X'$ and $g^{-1}(X')=X$, since $g^{-1}(A')\subset X$. This gives $A'<_{\uparrow\mathfrak{z}}X'$, thus $X\in\mathfrak{z}_{<_{\uparrow\mathfrak{z}}}(A')$. ∎

(2.8) **Proposition.** *Let* $\mathfrak{z}$ *be a g-mapping.*

(2.8.1) *If* $\mathfrak{z}$ *is topogenous, then* $<_{\uparrow\mathfrak{z}}$ *is also topogenous.*

(2.8.2) $<^{p}_{\uparrow\mathfrak{z}}=<_{\uparrow\mathfrak{z}^{p}}$, *in particular* $<_{\uparrow\mathfrak{z}}$ *is perfect for every perfect* $\mathfrak{z}$.

Proof. (2.8.1): Suppose $A'<_{\uparrow\mathfrak{z}}B'$ and $C'<_{\uparrow\mathfrak{z}}D'$. Then $A'\subset B'$, $C'\subset D'$, $g^{-1}(B')\in\mathfrak{z}(A')$ and $g^{-1}(D')\in\mathfrak{z}(C')$. Obviously $A'\cap C'\subset B'\cap D'$, $A'\cup C'\subset B'\cup D'$, and from the topogenity of $\mathfrak{z}$ the relations $g^{-1}(B'\cap D')=g^{-1}(B')\cap g^{-1}(D')\in\mathfrak{z}(A'\cap C')$, $g^{-1}(B'\cup D')=g^{-1}(B')\cup g^{-1}(D')\in\mathfrak{z}(A'\cup C')$ follow. Thus we have $A'\cap C'<_{\uparrow\mathfrak{z}}B'\cap D'$, $A'\cup C'<_{\uparrow\mathfrak{z}}B'\cup D'$, consequently $<_{\uparrow\mathfrak{z}}$ is topogenous.
(2.8.2): $A'<^{p}_{\uparrow\mathfrak{z}}B'\Leftrightarrow x'<_{\uparrow\mathfrak{z}}B'$ for any $x'\in A'$ (see [1], (4.7)) $\Leftrightarrow x'\in B'$ and $g^{-1}(B')\in\mathfrak{z}(x')$ for every $x'\in A'\Leftrightarrow A'\subset B'$ and $g^{-1}(B')\in\mathfrak{z}^{p}(A')\Leftrightarrow A'<_{\uparrow\mathfrak{z}^{p}}B'$. Therefore if $\mathfrak{z}$ is perfect, then by (1.5) $\mathfrak{z}=\mathfrak{z}^{p}$, and thus $<^{p}_{\uparrow\mathfrak{z}}=<_{\uparrow\mathfrak{z}}$. In view of [1], (4.9) this means that $<_{\uparrow\mathfrak{z}}$ is perfect. ∎

The semi-topogenous order $<_{\uparrow\mathfrak{z}}$ was defined on E'. In another way one can determine a semi-topogenous order $<_{\downarrow\mathfrak{z}}$ on E for an arbitrary g-mapping $\mathfrak{z}$. This order will be used for the examination of the case of a surjective g.

(2.9) **Proposition.** *If* $\mathfrak{z}$ *is a g-mapping, we have a semi-topogenous order* $<_{\downarrow\mathfrak{z}}$ *on* E *given by the following definition:*

(2.9.1) $\quad A<_{\downarrow\mathfrak{z}}B\Leftrightarrow B\in\mathfrak{z}(\mathrm{g}(A))$.

Proof. $\emptyset<_{\downarrow\mathfrak{z}}\emptyset$ and $E<_{\downarrow\mathfrak{z}}E$ are obvious. If $A<_{\downarrow\mathfrak{z}}B$, then $B\in\mathfrak{z}(\mathrm{g}(A))$, therefore by (M2) $A\subset g^{-1}(g(A))\subset B$. Finally suppose $A\subset A_1<_{\downarrow\mathfrak{z}}B_1\subset B$. Then $B\supset B_1\in\mathfrak{z}(g(A_1))$, $\mathrm{g}(A)\subset\mathrm{g}(A_1)$, consequently because of (M0) and (M3) $B\in\mathfrak{z}(g(A))$, that is $A<_{\downarrow\mathfrak{z}}B$ holds. ∎

Let us observe that if g is the identity of $E=E'$, then $<_{\downarrow\mathfrak{z}}=<_{\uparrow\mathfrak{z}}$ for any g-mapping $\mathfrak{z}$, and the correspondence $\mathfrak{z}\to<_{\downarrow\mathfrak{z}}$ is identical with (0.2).

(2.10) **Theorem.** *Let* $<'$ *be a semi-topogenous order on* E', *and* $g(E)$ *be* $<'$*-dense. Then we have* $g^{-1}(<')=<_{\downarrow\mathfrak{z}_{<'}}$.

Proof. In fact, $A\,g^{-1}(<')B\Leftrightarrow\mathrm{g}(A)<'E'-\mathrm{g}(E-B)$ (see [1], (5.1)) $\Leftrightarrow B\in\mathfrak{z}_{<'}(g(A))\Leftrightarrow A<_{\downarrow\mathfrak{z}_{<'}}B$. ∎

(2.11) **Proposition.** *If* $\mathfrak{z}$ *is a g-mapping, then*

(2.11.1) $<_{\downarrow\mathfrak{z}}$ *is topogenous, if* $\mathfrak{z}$ *is one.*

(2.11.2) $<^{p}_{\downarrow\mathfrak{z}}=<_{\downarrow\mathfrak{z}^{p}}$, *therefore if* $\mathfrak{z}$ *is perfect, then* $<_{\downarrow\mathfrak{z}}$ *is perfect, too.*

Proof. (2.11.1): Supposing $A<_{\downarrow\mathfrak{z}}B$ and $C<_{\downarrow\mathfrak{z}}D$, we get $B\in\mathfrak{z}(g(A))$ and $D\in\mathfrak{z}(\mathrm{g}(C))$. $\mathrm{g}(A\cap C)\subset\mathrm{g}(A)\cap g(C)$ and $g(A\cup C)=g(A)\cup g(C)$, thus $B\cap D\in$

$\in \mathfrak{z}(g(A\cap C))$ (cf. (M3)), and $B\cup D \in \mathfrak{z}(g(A\cup C))$. This means that $A\cap C <_{\mathfrak{i}\mathfrak{z}} B\cap D$ and $A\cup C <_{\mathfrak{i}\mathfrak{z}} B\cup D$.

(2.11.2): $A <^p_{\mathfrak{i}\mathfrak{z}} B \Leftrightarrow x <_{\mathfrak{i}\mathfrak{z}} B$ for every $x\in A \Leftrightarrow B\in \mathfrak{z}(g(x))$ for any $x\in A \Leftrightarrow B\in \mathfrak{z}^p(g(A)) \Leftrightarrow$ $\Leftrightarrow A <_{\mathfrak{i}\mathfrak{z}^p} B$. If $\mathfrak{z}$ is perfect, then $\mathfrak{z}=\mathfrak{z}^p$, consequently $<^p_{\mathfrak{i}\mathfrak{z}} = <_{\mathfrak{i}\mathfrak{z}}$ implies that $<_{\mathfrak{i}\mathfrak{z}}$ is perfect. ∎

(2.12) **Theorem.** *Let g be a surjection and $\mathfrak{z}$ be a g-mapping. If there exists a semi-topogenous order $<'$ on E' such that $\mathfrak{z}=\mathfrak{z}_{<'}$, then $<'=g(<_{\mathfrak{i}\mathfrak{z}})$.*

Proof. If $\mathfrak{z}=\mathfrak{z}_{<'}$, then $<_{\mathfrak{i}\mathfrak{z}} = g^{-1}(<')$, in accordance with (2.10). By [1], (6.36) this implies

$$<' = g(g^{-1}(<')) = g(<_{\mathfrak{i}\mathfrak{z}}). \quad ∎$$

(2.13) *Example.* Suppose that E' is a set having at least two elements, and g is not a surjection. Put $<'_1 = <_{\emptyset, E'}$, and let $<'_2$ be the biperfect topogenous order generated by the system $\mathfrak{S}=\{\emptyset, g(E), E'\}$ (see [1], (2.1)). Then $<'_1 \neq <'_2$, $g(E)$ is $<'_1$- and $<'_2$-dense, and obviously $\mathfrak{z}_{<'_1}=\mathfrak{z}_{<'_2}$. This gives that in this situation the correspondence $<' \to \mathfrak{z}_{<'}$ cannot be injective. ∎

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(Received August 7, 1980)