On a property of finite truncations of the Laurent series of analytic functions

By ALEXANDER ABIAN (Ames, Iowa)

*Abstract.**) Is is showed that there exists a sequence of complex numbers c_n which are zeros of finite truncations of the Laurent series around an isolated essential singularity of an analytic function f such that $\lim_n f(c_n)=0$ provided 0 is not Picard exceptional value of f. It is conjectured that the same conclusion may hold by dropping the above provision.

Let $\sum_{-\infty}^{\infty} a_m(z-a)^m$ be the Laurent series (around a) of an analytic function f and let k and p be nonnegative integers. Then the function whose value at z is given by $\sum_{-k}^{p} a_m(z-a)^m$ is called a *finite truncation* of the corresponding Laurent series of f.

In what follows we prove Theorems 1 and 3 which exhibit some basic properties of the zeros of finite truncations of the Laurent series around an isolated essential singularity of an analytic function f in relation with the zero limiting value of f.

We have presented below both Theorems 1 and 3 regardless of the fact that they have the same hypothesis while the conclusion of Theorem 3 is stronger than that of Theorem 1. The reason for doing this is explained in Remark 3.
Thus, we prove:

Theorem 1. *Let $\sum_{-\infty}^{\infty} a_m z^m$ be the Laurent series of a function f which is analytic in the annulus A given by $0<|z|<r$ and has an essential singularity at 0 and let 0 be not Picard exceptional value of f. Then there exist a sequence of complex numbers c_n and a sequence of finite truncations T_n of the Laurent series of f such that*

(1) $c_n \in A$ *for every* $n\in\omega$ *with* $\lim_n c_n=0$,

(2) $T_n(c_n)=0$ *for every* $n\in\omega$,

(3) $\lim_n f(c_n)=0$.

PROOF. Since 0 is not Picard exceptional value of f and since 0 is an isolated essential singularity of f, there exists, in view of Picard's Great theorem [1, p. 302],

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a sequence of complex numbers h_n such that

(4) $h_n \in A$ *for every* $n \in \omega$ *with* $\lim_n h_n = 0$,

(5) $f(h_n) = 0$ *for every* $n \in \omega$.

Since f is analytic at every h_n and since f is not the zero function, clearly there exists a sequence of closed disks D_n respectively with circumferences C_n and centers h_n such that for every $n \in \omega$

(6) $f(z) \neq 0$ *for every* $z \in C_n$,

(7) $D_n \subseteq A$ *with diameter of* $D_n < 10^{-n}$,

(8) $|f(z)| < 10^{-n}$ *for every* $z \in D_n$.

Since the sequence of (for instance the particular) finite truncations $\sum_{-k}^{k} a_m z^m$ (with $k \in \omega$) of the above-mentioned Laurent series of f converges uniformly on every D_n, from (6) and (5), in view of Hurwitz's theorem [1, p. 148], it follows that there exists a sequence (e.g. the particular one mentioned above) of finite truncations T_n of the Laurent series of f and a sequence of complex numbers c_n such that

(9) $c_n \in D_n$ *for every* $n \in \omega$,

(10) $T_n(c_n) = 0$ *for every* $n \in \omega$.

But then (1) follows from (9), (7) and (4). Similarly, (2) follows from (10) and (3) follows from (8) and (9). Thus, Theorem 1 is proved.

An immediate corollary of Theorem 1 is:

Theorem 2. *Let* $\sum_{-\infty}^{\infty} a_m (z-a)^m$ *be the Laurent series of a function f which is analytic in the annulus* A *given by* $0 < |z-a| < r$ *and has an essential singularity at* a *and let* b *be not Picard exceptional value of f.*

Then there exist a sequence of complex numbers c_n *and a sequence of finite truncations* T_n *of the Laurent series of* f *such that for every* $n \in \omega$ $c_n \in A$ *and* $\lim_n c_n = a$ *and* $T_n(c_n) = b$ *and* $\lim_n f(c_n) = b$.

Proof. It is enough to apply Theorem 1 to the function $f - b$ while changing the origin of the z-plane to a.

Remark 1. As mentioned earlier, from the hypothesis of Theorem 1, we can prove (using basically the proof of Theorem 1) a stronger conclusion than that of Theorem 1.

Thus, we prove the following:

Theorem 3. *Let* $\sum_{-\infty}^{\infty} a_m z^m$ *be the Laurent series of a function f which is analytic in the annulus* A *given by* $0 < |z| < r$ *and has an essential singularity at* 0 *and let* 0 *be not Picard exceptional value of f.*

Let there be preassigned a sequence of finite truncations T_n of the Laurent series of f which converges to f in A. Then there exist a sequence of complex numbers c_n and a nonnegative integer N such that

(11) $c_n \in A$ *for every* $n \in \omega$ *and* $\lim_n c_n = 0$,

(12) $T_{N+n}(c_n) = 0$ *for every* $n \in \omega$,

(13) $\lim_n f(c_n) = 0$.

PROOF. Employing the notations used in the proof of Theorem 1, there exists a nonnegative integer N such that by virtue of Hurwitz's theorem:

$$T_{N+n} \text{ has a zero } c_n \in D_0 \text{ for every } n \in \omega$$

and there exists a nonnegative integer N_1 such that

$$T_{N+N_1+n} \text{ has a zero } c_{N_1+n} \in D_1 \text{ for every } n \in \omega$$

and, in general, there exists a nonnegative integer N_p such that

$$T_{N+N_1+\ldots+N_p+n} \text{ has a zero } c_{N_1+\ldots+N_p+n} \in D_p \text{ for every } n \in \omega.$$

But then, $c_0, c_1, \ldots, c_{N_1}, c_{N_1+1}, \ldots, c_{N_1+\ldots+N_p+n}, \ldots$ is the desired sequence of complex numbers c_n which also satisfies (12).

On the other hand, from Picard's Great theorem it follows that the sequence of complex numbers h_n can be so chosen that $\lim_n h_n = 0$. But then since h_n is the center of D_n, from (7) and (9) it follows that $c_n \in A$ and $\lim_n c_n = 0$ which establishes (11). Clearly, the proof of (3) also establishes (13). Thus, Theorem 3 is proved.

Obviously, Theorem 3 can be generalized the way Theorem 1 is generalized by Theorem 2.

Remark 2. We can further strengthen the conclusion of Theorem 3 via replacing the preassigned sequences of T_n by a preassigned sequence of functions F_n which are analytic in annulus A and which converge uniformly to f on every closed disk contained in A.

Remark 3. As yet it is an open question whether or not the conclusion of Theorem 1 remains valid if in its hypothesis the clause "*let* 0 *be not Picard exceptional value of* f" is dropped. The same is the case with respect to Theorem 3. It is our conjecture that the answer is in the affirmative in connection with Theorem 1. This is the reason why both of the Theorems 1 and 3 are presented in this paper.

Reference

[1] J. B. CONWAY, Functions of one complex variable. *Springer-Verlag, New York*, 1975.

DEPARTMENT OF MATHEMATICS
IOWA STATE UNIVERSITY
AMES, IOWA 50011
U.S.A.

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