On matrix-valued analytic characteristic functions

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1. Introduction

For each point x of the real line R_1 let $f(x)$ be a p by p matrix with entries $f_{jk}(x)$, where $f_{jk}(x)$ is a complex-valued function $(j, k=1, \ldots, p)$. We say f is a matrix-valued function defined on R_1. The normalized trace of f is the scalar function

$$\operatorname{tr} f(x) = \frac{1}{p} \sum_{k=1}^{p} f_{kk}(x).$$

The normalized trace has the following properties (and is determined by them): for any matrix-valued functions $f, g,$ and any complex valued scalars α, β

$$\operatorname{tr}(\alpha f + \beta g) = \alpha \operatorname{tr} f + \beta \operatorname{tr} b.$$

$$\operatorname{tr}(fg) = \operatorname{tr}(gf),$$

$$\operatorname{tr}(U^{-1} f U) = \operatorname{tr} f$$

if $U(x)$ is a unitary matrix-valued function,

$$\operatorname{tr}(ff^*) \geqq 0,$$

f^* is the conjugate of the transpose of f.

$$\operatorname{tr} E = 1,$$

E the unit matrix.

Let the symmetric and positive semidefinite matrix-valued function $F(x)$ with components $a_{jk} F_{jk}(x)$ $(j, k=1, \ldots, p)$ be given, where the p by p matrix A with elements a_{jk} is a stochastic matrix, and $F_{jk}(x)$ is a probability distribution function. We say $F(x)$ to be a symmetric and positive semidefinite matrix-valued distribution function.

If each component function f_{jk} of the matrix-valued function f is integrable, or square integrable with respect to each element of the matrix-valued distribution function F, we shall say that f is integrable (belongs to $L(F)$, or that f is square integrable (belongs to $L_2(F)$) with respect to F.

It follows from definition that a matrix-valued function f is in $L_2(F)$ if and only if the integral

$$\int_{-\infty}^{\infty} \operatorname{tr}(ff^* dF)$$

exists.

The ring of constant p by p matrices with complex elements possesses the natural inner product

$$(a, b) = \operatorname{tr}(ab^*).$$

We can extended this definition to the class of matrix-valued functions in $L_2(F)$ by setting

$$(f, g) = \int_{-\infty}^{\infty} \operatorname{tr}(fg^* dF) = \frac{1}{p} \int_{-\infty}^{\infty} \sum_{j,k,l=1}^{p} f_{jk}\bar{g}_{lk}\, dF_{lj},$$

where f_{jk}, g_{kl}, F_{lj} are components of the matrix-valued functions of f, g and F, respectively. This inner product has the following properties:

$$(f, g) = \overline{(g, f)},$$

$$(f_1 + f_2, g) = (f_1, g) + (f_2, g),$$

$$(\alpha f, g) = \alpha(f, g)$$

with arbitrary complex number α,

$$(f, f) > 0 \quad \text{if} \quad f \neq 0.$$

Let us introduce addition, and multiplication by complex numbers for the set of square integrable matrix-valued functions with respect to a matrix-valued distribution functions. Thus $L_2(F)$ will be a unitary space with norm

$$\|f\| = (f, f)^{1/2} = \Big[\int_{-\infty}^{\infty} \operatorname{tr}(ff^* dF)\Big]^{1/2}.$$

Therefore the Schwartz inequality

$$|(f, g)| \leqq \|f\|\|g\| \tag{1.1}$$

and the triangle inequality

$$\|f + g\| \leqq \|f\| + \|g\|$$

is satisfied in $L_2(F)$. By the help of the Schwartz inequality it is easily to prove that in the ring of scalar matrices inequality

$$\|ab\| \leqq \|a\|\|b\| \tag{1.2}$$

holds, and that in $L_2(F)$

$$\|fg\| \leqq \|f\|\|g\|,$$

provided f and g are commutable.

Let $F(x)$ be a symmetric and prositive semidefinite matrix-valued distribution function. Matrix-valued function

$$\Phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x) \tag{1.3}$$

with elements $a_{hj}\varphi_{hj}(t)$, where

$$\varphi_{hj}(t) = \int_{-\infty}^{\infty} e^{itx}\, dF_{hj}(x) \quad (h, j = 1, \ldots, p) \tag{1.4}$$

is said to be a matrix-valued characteristic function. The matrix-valued characteristic function (1.3) is said to be analytic, if each characteristic function (1.4) is analytic ([3], 130).

If $\Phi(t)$ is a matrix-valued analytic characteristic function, then one is analytic in a horizontal strip, and can be represented in this strip by the Fourier integral (1.3) too. This strip of regularity is either the whole plan, or it has one or two horizontal boundery line ([3], Theorem 7.1.1.). We say $\Phi(t)$ is a matrix-valued entire characteristic function, if one is analytic in the whole plan.

The author has repeatedly dealt with matrix-valued distribution and characteristic functions in his research work. In the present paper the following result of J. Marcinkiewicz ([4]) will be extended for a matrix-valued characteristic function: Let $P_q(t)$ be a polynomial of degree $q>2$, and denote by $f(t)=\exp P_q(t)$. Then $f(t)$ cannot be a characteristic function.

In section 2 we restrict ourselves to extend the following well-known result of the theory of analytic characteristic functions for matrix-valued analytic characteristic functions ([3], 134): Let $f(z)$ be an analytic characteristic function. Then $|f(z)|$ attains its maximum along any horizontal line contained in the interior its strip of regularity in the imaginary axis. This generalized theorem will be used in the proof of the extension of Marcinkiewicz theorem.

In the third section we give the above mentioned generalization of the theorem of Marcinkiewicz. We can find a new characterization of the normal distribution in section 4. The method, which is used in the proof of Marcinkiewicz theorem, is also an extension of the proof of Marcinkiewicz theorem.

2. Preliminary

Let $\Phi(t)$ be a matrix-valued analytic characteristic function with strip of regularity S, and with Fourier integral representation (1.3). Let t be a p by p scalar matrix with complex numbers as its elements. If the eigenvalues of t lie in S, then t can substitute in $\Phi(t)$, and $\Phi(t)$ is equal to the representation (1.3) in this case too.

The following Theorem is a generalization of the well-known one ([2], Theorem 3.2.1.).

Theorem 2.1. *Let $\Phi(t)$ be a matrix-valued analytic characteristic function with strip of regularity S, and with Fourier integral representation*

$$\Phi(z) = \int_{-\infty}^{\infty} e^{izx}\, dF(x), \quad z \in S,$$

where F is a symmetric and positive semidefinite matrix-valued distribution function. Let t be a p by p normal scalar matrix with canonical representation $t=\xi+i\eta$, where ξ and η are Hermitian commutable matrices. Suppose that the eigenvalues of t lie in S. Then

$$|\mathrm{tr}\ \Phi(t)| \leqq (\mathrm{tr}\ A)^{1/2}\|\Phi(i\eta)\|,$$

where $A=F(\infty)$ is a symmetric and positive semidefinite stochastic matrix.

PROOF. Since t is a normal matrix, matrix-valued functions

$$f(x) = \exp\{i\xi x\}, \quad g(x) = \exp\{-\eta x\}$$

are commutable, moreover g is an Hermitian and positive definite, f is a unitary matrix in $L_2(F)$. Thus

$$\Phi(t)= \int_{-\infty}^{\infty} fg\, dF.$$

Taking the identities

$$(f, g) = \mathrm{tr}\ \Phi(t),$$

$$\|f\|^2 = \int_{-\infty}^{\infty} \mathrm{tr}\,(ff^*\,dF) = \mathrm{tr}\ A,$$

$$\|g\|^2 = \int_{-\infty}^{\infty} \mathrm{tr}\,(gg^* dF) = \int_{-\infty}^{\infty} \mathrm{tr} \exp\{-2\eta x\}\, dF(x) = \|\Phi(i\eta)\|^2$$

into consideration, we get the statement of Theorem 2.1. on the basis of the Schwartz inequality (1.1).

Because A is a symmetric and positive semidefinite matrix, inequality

$$\frac{1}{p} \leqq \mathrm{tr}\ A \leqq 1 \tag{2.1}$$

holds.

3. On an extension of the theorem of Marcinkiewicz

Let $a\neq 0$ be a p by p matrix with complex components, and let q be an arbitrary positive integer. Let

$$a = (aa^*)^{1/2} \exp\{iB\} \tag{3.1}$$

be the polar representation of a. The eigenvalues $\lambda_1, \ldots, \lambda_p$ of $(aa^*)^{1/2}$ are non-negative, while at least one of them is positive. B is a Hermitian matrix, thus its eigenvalues $b_1, \dots, b_p$ are real numbers. Let

$$r = \operatorname{tr}(aa^*)^{1/2} = \frac{1}{p} \quad (\lambda_1 + \dots + \lambda_p), \tag{3.2}$$

$$\varrho = \|(aa^*)^{1/2}\| = \left[\frac{1}{p}(\lambda_1^2 + \dots + \lambda_p^2)\right]^{1/2}, \tag{3.3}$$

$$\gamma(s) = \left[\frac{1}{p} \sum_{j=1}^{p} \sin^2 \left\{\frac{1}{q}(2\pi s - b_j)\right\}\right]^{1/2} \tag{3.4}$$

$$(s = 0, 1, \dots, q-1).$$

On the basis of (3.2) and (3.3) we have $r \leqq \varrho$ with equality if and only if $(aa^*)^{1/2}$ is a diagonal matrix with the same diagonal components. Denote by γ the smallest among the numbers (3.4). It obviously that $0 \leqq \gamma \leqq 1$.

Lemma 3.1. *Let* $q \geqq 3$ *be an integer. Then* $0 \leqq \gamma < 1$.

Proof. Namely of $q \geqq 3$, then by fixed j at least one is smaller then 1 among the numbers

$$\sin^2 \left\{\frac{1}{q}(2\pi s - b_j)\right\} \quad (s = 0, 1, \dots, q-1).$$

As an extension of theorem Marcinkiewicz we prove the following.

Theorem 3.1. *Let the polynomial*

$$Q(t) = \sum_{j=1}^{q} a_j t^j$$

of degree $q \geqq 1$ *be given, where the coefficients are* p *by* p *matrices with complex components, in addition let* $a_q = a \neq 0$. *Let*

$$\Phi(t) = \int_{-\infty}^{\infty} e^{itx} \, dF(x)$$

be a matrix-valued characteristic function, where F *is a symmetric and positive semidefinite matrix-valued distribution function. If identity*

$$\Phi(t) = \exp Q(t) \tag{3.5}$$

holds, then we have

$$r \leqq \varrho \gamma^q \tag{3.6}$$

where quantities r, ϱ, γ *are defined by* (3.2), (3.3) *and* (3.4), *respectively.*

Proof. Let the sequence $\{R_n\}_{n=1}^{\infty}$ with positive numbers as its elements be given satisfying condition $R_n \to \infty$, $n \to \infty$. Let

$$t_{ns} = R_n \exp\left\{\frac{i}{q}(2\pi s E - B)\right\} \tag{3.7}$$

$$(s = 0, 1, \dots, q-1)$$

if (3.1) is the polar representation of a. Then for sufficiently large n we have

$$Q(t_{ns}) = (aa^*)^{1/2} R_n^q + O(R_n^{q-1}),$$

where $O(R_n^{q-1})$ is a p by p matrix with components of order $O(R_n^{q-1})$. If we use condition (3.5) we obtain

$$\Phi(t_{ns}) = \exp\{(aa^*)^{1/2} R_n^q + O(R_n^{q-1})\} \tag{3.8}$$

$$(s = 0, 1, \ldots, q-1)$$

for sufficiently large n. If $\lambda_1, \ldots, \lambda_p$ denote the eigenvalues of $(aa^*)^{1/2}$, then

$$p \operatorname{tr} Q(t_{ns}) = (\lambda_1 + \ldots + \lambda_p) R_n^q + O(R_n^{q-1}),$$

i.e. matrix (3.8) has its eigenvalues in the form

$$\exp\{\lambda_j R_n^q + O(R_n^{q-1})\} \quad (j = 1, \ldots, p)$$

for sufficiently large n. In consequence of this

$$\operatorname{tr} \Phi(t_{ns}) = \frac{1}{p} \sum_{j=1}^{p} \exp\{\lambda_j R_n^q + O(R_n^{q-1})\}$$

for sufficiently large n. Thus for sufficiently small $\varepsilon > 0$ we get

$$|\operatorname{tr} \Phi(t_{ns})| \geqq \frac{1}{p} \sum_{j=1}^{p} \exp\{\lambda_j R_n^q - \varepsilon R_n^q\}.$$

Using the inequality of the arithmetic and geometric means we obtain

$$\log |\operatorname{tr} \Phi(t_{ns})| \geqq (r - \varepsilon) R_n^q \tag{3.9}$$

$$(s = 0, 1, \ldots, q-1),$$

for sufficiently large n and sufficiently small ε, where r is defined by (3.2).

We consider next $\Phi(i\eta_{ns})$, where

$$\eta_{ns} = R_n \sin\left\{\frac{1}{q}(2\pi s E - B)\right\}$$

$$(s = 0, 1, \ldots, q-1)$$

on the basis of (3.1), and $\Phi(t)$ is defined by (3.5). For sufficiently large n we have

$$Q(i\eta_{ns}) = (aa^*)^{1/2} \exp\{iB\} i^q \eta_{ns}^q + O(\eta_{ns}^{q-1}).$$

Since identity

$$\|\eta_{ns}\| = R_n \left\|\sin\left\{\frac{1}{q}(2\pi s E - B)\right\}\right\| = R_n \gamma(s)$$

$$s = 0, 1, \ldots, q-1)$$

holds, we obtain for fixed $\varepsilon > 0$ and for sufficiently large n

$$\|Q(i\eta_{ns})\| \leqq (\varrho+\varepsilon) R_n^q \gamma_{(s)}^q \quad (s = 0, 1, \ldots, q-1), \tag{3.10}$$

where ϱ and $\gamma(s)$ are defined by (3.3) and (3.4), respectively.

Using repeatedly the triangle inequality and inequality (1.2) we get

$$\left\|\left[E+\frac{1}{m} Q(i\eta_{ns})\right]^m\right\| \leqq \left[1+\frac{1}{m}\|Q(i\eta_{ns})\|\right]^m \leqq$$

$$\leqq \exp \|Q(i\eta_{ns})\|.$$

Using limit

$$\Phi(i\eta_{ns}) = \lim_{m\to\infty} \left[E+\frac{1}{m} Q(i\eta_{ns})\right]^m,$$

we have

$$\|\Phi(i\eta_{ns})\| \leqq \exp \|Q(i\eta_{ns})\| \leqq \exp \{(\varrho+\varepsilon) R_n^q \gamma_{(s)}^q\} \quad (s = 0, 1, \ldots, q-1) \tag{3.11}$$

on the basis of (3.10) for fixed $\varepsilon > 0$, and for sufficiently large n.

Since $\Phi(t)$, which is defined by (3.5), is a matrix-valued entire characteristic function, and since quantities (3.7) are normal matrices, the conditions of Theorem 2.1. are satisfied. Thus

$$|\mathrm{tr}\, \Phi(t_{ns})| \leqq \|\Phi(i\eta_{ns})\| \quad (s = 0, 1, \ldots, q-1)$$

using inequality (2.1).

Comparing inequalities (3.9) and (3.11) by the help of the last inequality, we get the following result: Let the matrix-valued characteristic function be defined by (3.5). Then inequalities

$$r-\varepsilon \leqq (\varrho+\varepsilon)\gamma_{(s)}^q \quad (s = 0, 1, \ldots, q-1)$$

holds for any $\varepsilon > 0$, i.e. inequality (3.6) is satisfied. Thus the proof of Theorem 3.1. is completed.

The inversion of Theorem 3.1. is the following.

Corollary 3.1. *Let the polynomial $Q(t)$ of degree $q \geqq 1$, and the quantities r, ϱ, γ be defined as in Theorem 3.1. If $r > \varrho\gamma^q$, $\exp Q(t)$ can not be equal to a matrix-valued characteristic function of a symetric and positive semidefinite matrix-valued distribution function.*

Corollary 3.2. *Let the polynomial $Q(t)$ of degree $q \geqq 3$ be defined as in Theorem 3.1. Suppose that $a = \lambda U$, where λ is a positive number and U is a unitary matrix. Then $\exp Q(t)$ can not be a matrix-valued characteristic function of a symmetric and positive semidefinite matrix-valued distribution function.*

PROOF. In this case $r=\varrho=\lambda$. Moreover $0\leqq\gamma<1$ in consequence of Lemma 3.1. Thus $r>\varrho\gamma^q$.

Since in the case of $p=1$ conditions of Corollary 3.2. are satisfied automatically, Theorem 3.1. contains the theorem of Marcinkiewicz too.

4. A new characterization of the normal distributions

By the help of the results of the foregoing section we are proving the following characterization theorem of the normal low.

Theorem 4.1. *Let the elements of the symmetric and positive semidefinite p by p stochastic matrix $A=(a_{jk})$ be positive numbers. Let*

$$Q(t)=a_0+a_1t+\ldots+a_{q-1}t^{q-1}+at^q$$

be a matrix-valued polynomial of degree $q\geqq 1$, where the coefficients are p by p matrices with complex components. Suppose that the coefficients are commutable with one other, and that $a=\lambda U$, where λ is a non-negative number and U is a unitary matrix. Then $\Phi(t)=\exp Q(t)$ is a matrix-valued characteristic function with $\Phi(\infty)=A$ if and only if

$$\Phi(t)=\exp\{i\gamma t-\lambda t^2\}A, \tag{4.1}$$

where γ is a real constant.

PROOF. The sufficiency half of the thecrem is obvious. Now we prove the necessary half of the theorem.

Since $\Phi(t)$ is a matrix-valued characteristic function with $\Phi(\infty)=A$, thus

$$\exp a_0=A,\quad q=2,\quad \gamma(0)=\gamma(1)=1$$

according to Corollary 3.2. In this case the eigenvalues of matrix U can only be the numbers $+1$ and -1, therefore a is a diagonal matrix with elements $+\lambda$ and $-\lambda$. As a and A are commutable and the elements of A are positive numbers, each diagonal elements of a are equal either only to $+\lambda$, or only to $-\lambda$. Since $\Phi(t)$ is a matrix-valued characteristic function, the only solution is $a=-\lambda E, \lambda\geqq 0$.

In the consequence of the commutability relation

$$Aa_1=a_1A \tag{4.2}$$

holds. Taking this into consideration we obtain

$$\Phi(t)=\exp\{-\lambda t^2\}A\exp\{a_1t\} \tag{4.3}$$

where in consequence of (4.2) matrices A and $\exp\{a,t\}$ are commutable.

Since (4.3) holds for all $\lambda\geqq 0$, it is necessary that

$$f(t)=A\exp\{a_1t\}=\left(a_{hj}c_{hj}(t)\right)$$

to be a matrix-valued characteristic function, i.e. that functions $c_{kj}(t)$ to be characteristic functions.

We show that

$$f(t)=A\exp\{i\gamma t\}, \tag{4.4}$$

where γ is a real number. Let

$$f'(0) = a_1 = (a_{hj}\gamma_{hj}), \tag{4.5}$$

where γ_{hj} are real numbers. On the basis of (4.2) we get the relations

$$\sum_{\nu=1}^{p} a_{j\nu} a_{\nu k} \gamma_{\nu k} = \sum_{\nu=0}^{p} a_{j\nu} \gamma_{j\nu} a_{\nu k} \quad (j, k = 1, \ldots, p). \tag{4.6}$$

Let

$$\gamma_j = \sum_{\nu=1}^{p} a_{j\nu} \gamma_{j\nu} (j = 1, \ldots, p).$$

By summation in both side of (4.6) on k, equation system

$$\sum_{\nu=1}^{p} a_{j\nu} \gamma_\nu = \gamma_j \quad (j = 1, \ldots, p) \tag{4.7}$$

holds. Since all elements of A are positive numbers, 1 is an eigenvalue of A with multiplicity one, and the components of the right hand side eigenvector corresponds to the eigenvalue 1, are equal to another ([1], 46, theorem of Perron, and 73). Therefore (4.7) holds if and only if $\gamma_1 = \ldots = \gamma_p = \gamma$, i.e.

$$a_1 \mathscr{E} = i\gamma \mathscr{E} \tag{4.8}$$

on the basis of (4.5), where $\mathscr{E}$ is the column vector of order p with components 1.

From the formula

$$f(t) = A \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} a_1^\nu,$$

and on the basis of equality $A\mathscr{E} = \mathscr{E}$, and of (4.8) we get

$$f(t)\mathscr{E} = \sum_{\nu=0}^{\infty} \frac{t^\nu}{\nu!} (i\gamma)^\nu \mathscr{E} = \exp\{i\gamma t\}\mathscr{E},$$

or writting down in detail

$$\sum_{\nu=1}^{p} a_{j\nu} c_{\nu j}(t) = \exp\{i\gamma t\} \tag{4.9}$$

$$(j = 1, \ldots, p).$$

Let $F_{hj}(x)$ be the distribution function, which corresponds to the characteristic function $c_{hj}(t)$. The equation system (4.9) is equivalent to

$$\sum_{\nu=1}^{p} a_{j\nu} F_{j\nu}(x) = e(x-\gamma) \tag{4.10}$$

$$(j = 1, \ldots, p),$$

where

$$e(x) = \begin{cases} 0 & \\ x & \end{cases} \text{for} \quad x \underset{>}{\leqq} 0.$$

Since A is a stochastic matrix with positive elements, and since functions $F_{hj}(x)$ are distribution functions, (4.10) is satisfied if and only if

$$F_{hj}(x) = e(x-\gamma) \quad (h, j = 1, \ldots, p).$$

i.e. (4.4) and thus (4.1) holds. This completes of the proof of Theorem 4.1.

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