Convergence of vector-valued martingales with multidimensional indices

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1. Introduction

There are several theorems concerning almost sure (a.s.) convergence of multi-parameter sequences of scalar random variables. The purpose of this article is to generalize these results to the case of random variables taking values in a Banach space. After some preliminary remarks we give an extension of Gut's theorem concerning reversed martingales (cf. [7]) in section 3. Section 4 deals with Cairoli's theorem on martingales (cf. [2]). Section 5 contains a multidimensional strong law of large numbers for vector-valued random variables (the scalar case is due to Smythe, cf. [12]). Our method is a modification of Chatterji's method (see [4], Prop. 5.2 and Prop. 5.3).

2. Notation and preliminary remarks

Let $(\Omega, \mathcal{A}, P)$ be a probability space, B a real Banach space with norm $|\cdot|$. B^* is the dual of B. $X: \Omega \to B$ will be called a B-valued random variable (r.v.) if X is Bochner measurable. The expectation of X is defined by $EX = \int_\Omega X\,dP$, where the integral is Bochner integral. $L^r = L^r(\mathcal{A}, B)$ (or $L^r(\Omega, \mathcal{A}, P, B)$), $1 \leqq r < \infty$, denotes the Banach space of random variables X, for which $\int_\Omega |X|^r\,dP < \infty$.

Let $(T, \leqq)$ and Z denote a directed set and the set of positive integers respectively.

Definition 2.1. Let $\mathcal{F}_t$ $(t \in T)$ be an increasing sequence of σ-subalgebras of $\mathcal{A}$. $\{X_t, \mathcal{F}_t, t \in T\}$ is called a martingale if $X_t \in L^1(\mathcal{F}_t, B)$ and $E(X_{t_2}|\mathcal{F}_{t_1}) = X_{t_1}$ $(t_1 \leqq t_2)$.
The reversed martingale is defined in similar way.
For suitable references on these subjects see [3], [4], [6], [8] and [11].

Definition 2.2. Let $c(B)$ denote the set of all convergent sequences in B. If $x = (x_1, x_2, \dots) \in c(B)$, let $\|x\|_c = \sup_i (x_i)$. Let $c_0(B)$ denote the set of sequences converging to 0 (the null element of B).

Proposition 2.3. (1) *$c(B)$ is a Banach space with the coordinate-wise addition, scalar multiplication and norm $\|\cdot\|_c$. $c_0(B)$ is a subspace of $c(B)$.*
(2) *If B is separable, then $c(B)$ is also separable.*
(3) *The convergence in $c(B)$ is the coordinate-wise uniform convergence.*

Lemma 2.4. (1) *Let* X_i $(i\in Z)$ *be B-valued random variables. If the sequence* X_i *converges a.s., then*

$$X=(X_1, X_2, \dots)$$

is a $c(B)$*-valued random variable.*

(2) *Let* $X=(X_1, X_2, \dots)$ *be a* $c(B)$*-valued r.v. and let* $\mathcal{F}$ *be a* σ*-subalgebra of* $\mathcal{A}$. *If* $E\|X\|_c<\infty$, *then* $\big(E(X|\mathcal{F})\big)_i=E(X_i|\mathcal{F})$ $(i\in Z)$, *that is the conditional expectation in* $c(B)$ *can be constituted coordinate-wise.*

PROOF. We can suppose that B is separable.

(1) Let $y^0=(y_1^0, y_2^0, \dots)$ be a fixed element in $c(B)$, let $y=(y_1, y_2, \dots)$ denote an arbitrary sequence of elements of B. For $\varepsilon>0$

$$\{y\colon y\in c(B),\ \|y-y^0\|_c\leqq\varepsilon\}=\left(\bigcap_{n=1}^{\infty}\{y\colon |y_n-y_n^0|\leqq\varepsilon\}\right)\cap c(B)=G.$$

$X^{-1}(G)\in\mathcal{A}$, so the inverse image of an arbitrary sphere in $c(B)$ is measurable. The Borel σ-field of $c(B)$ is generated by the spheres of $c(B)$, because $c(B)$ is separable (cf. [10]). Thus X is measurable.

(2) Let $f\in B^*$ and for a fixed $i\in Z$ we define $f_i\in[c(B)]^*$ by the following equation:

$$f_i(a)=f(a_i),\quad a=(a_1, a_2, \dots)\in c(B).$$

$$f\big(\{E(X|\mathcal{F})\}_i\big)=f_i\big(E(X|\mathcal{F})\big)=E\big(f_i(X)|\mathcal{F}\big)=$$
$$=E\big(f(X_i)|\mathcal{F}\big)=f\big(E(X_i|\mathcal{F})\big)\quad\text{a.s.}$$

according to Theorem 2.3 of [11]. That is $f[\{E(X|\mathcal{F})\}_i-E(X_i|\mathcal{F})]=0$ a.s. There is a countable norm-determining set in B^*, because B is separable. Thus $\{E(X|\mathcal{F})\}_i=E(X_i|\mathcal{F})$ a.s.

We need the following results from the theory of scalar martingales.

Lemma 2.5. (1) *If* $\{Z_k, \mathcal{F}_k, k\in Z\}$ *is a scalar positive submartingale, then* $\{Z_k(\log^+ Z_k)^{r-2}, \mathcal{F}_k, k\in Z\}$ *is also a positive submartingale* $(r>2)$.

(2) *Doob's maximal inequalities. If* $\{X_k, \mathcal{F}_k, k=1, 2, \dots, n\}$ *is a real positive submartingale, then*

(a)
$$E\big(\max_{1\leqq k\leqq n} X_k\big)\leqq\frac{e}{e-1}+\frac{e}{e-1}E(X_n\log^+ X_n)$$

and

(b)
$$E\big(\max_{1\leqq k\leqq n} X_k^\alpha\big)\leqq\left(\frac{\alpha}{\alpha-1}\right)^\alpha E(X_n^\alpha),\quad \alpha>1.$$

(3) *Cairoli's inequality:*

$$t(\log^+ t)^{r-2}\log^+[t(\log^+ t)^{r-2}]\leqq(r-1)t(\log^+ t)^{r-1},\quad r\geqq 2,\quad t\geqq 0.$$

PROOF. (1) $t(\log^+ t)^{r-2}$ is a convex increasing function if $r>2$.

(2) See [5], p. 317.

(3) See [2] and [12].

Notations. Let $d\geqq 1$ be an integer and let Z^d denote the positive d-dimensional integer lattice points. The notations $\overline{m}\leqq\overline{n}$, where $\overline{m}=(m_1, m_2, \dots, m_d)$ and

$\bar{n}=(n_1, n_2, \dots, n_d)\in Z^d$, means that $m_i \leqq n_i$, $i=1, 2, \dots, d$. With this partial ordering Z^d is a directed set. $|\bar{n}|$ is used to denote $\prod_{k=1}^{d} n_k$. $\bar{n}\to\infty$ means that $n_i\to\infty$, $i=1, 2, \dots, d$. If $\bar{n}\in Z^d$ and $m\in Z$, then $(n_1, n_2, \dots, n_d, m)\in Z^{d+1}$ is denoted by $(\bar{n}, m)$. $(1, 1, \dots, 1)\in Z^d$ is denoted by $\bar{1}$.

3. Reversed martingales

Theorem 3.1. *Let* $\{X_{\bar{n}}, \mathscr{F}_{\bar{n}}, \bar{n}\in Z^d\}$ *be a B-valued reversed martingale and let*

$$E\big(E(\cdot\,|\mathscr{F}_{\bar{n}})|\mathscr{F}_{\bar{m}}\big)=E(\cdot\,|\mathscr{F}_{\max(\bar{m},\bar{n})}),$$

where $\max(\bar{m}, \bar{n})$ *is the coordinate-wise maximum. Let* $\mathscr{F}=\bigcap_{\bar{m}\in Z^d}\mathscr{F}_{\bar{m}}$.

If $E\big(|X_{\bar{1}}|(\log^+|X_{\bar{1}}|)^{d-1}\big)<\infty$, *then there exists a random variable* X *which is* $\mathscr{F}$*-measurable and* $X_{\bar{n}}\to X$ *a.s. as* $\bar{n}\to\infty$.

Proof. We consider first the case $d=1$ and proceed by induction. For $d=1$ the theorem is a consequence of a theorem of Chatterji (cf. Theorem 4 of [3]). Suppose that the theorem is true for $d-1$. Let $\bar{n}\in Z^{d-1}$ and

$$Y_{\bar{n}}=(X_{(\bar{n},1)},\ X_{(\bar{n},2)}, \dots).$$

We shall prove that $\{Y_{\bar{n}}, \mathscr{F}_{(\bar{n},1)}, \bar{n}\in Z^{d-1}\}$ is a $c(B)$-valued reversed martingale which satisfies the assumptions of the theorem. For every fixed $\bar{n}\in Z^{d-1}$ $\{X_{(\bar{n},j)}, \mathscr{F}_{(\bar{n},j)}, j\geqq 1\}$ is a B-valued reversed martingale and thus from the above mentioned theorem of Chatterji

$$\lim_{j\to\infty} X_{(\bar{n},j)} = E\left\{X_{(\bar{n},1)}\,\middle|\,\bigcap_{j=1}^{\infty}\mathscr{F}_{(\bar{n},j)}\right\} \quad \text{a.s.}$$

According to Lemma 2.4 $Y_{\bar{n}}$ is an $\mathscr{F}_{(\bar{n},1)}$-measurable $c(B)$-valued r.v.

To prove that $Y_{\bar{n}}\in L^1(\mathscr{F}_{(\bar{n},1)}, c(B))$ let $\{V_k, \mathscr{F}_k, k\geqq 1\}$ be the following B-valued reversed martingale:

$$V_1 = X_{\bar{1}} \quad (\bar{1}\in Z^d),$$

$$V_k = X_{(\bar{n},k-1)} \quad \text{for} \quad k=2, 3, \dots \quad (\bar{n}\in Z^{d-1}),$$

and $\mathscr{F}_k$ is the corresponding σ-subalgebra. $\{|V_k|, \mathscr{F}_k, k\geqq 1\}$ is a real submartingale in the reversed ordering. According to Doob's maximal inequality

$$E\{\max_{1\leqq k\leqq m}|V_k|\}\leqq\frac{e}{e-1}+\frac{e}{e-1}E\{|V_1|\log^+|V_1|\}<\infty.$$

From the monotone convergence theorem $E\{\sup_k|V_k|\}<\infty$. Thus $\|Y_{\bar{n}}\|_c=\sup_{1\leqq j<\infty}|X_{(\bar{n},j)}|$ is integrable and this proves that $Y_{\bar{n}}\in L^1(\mathscr{F}_{(\bar{n},1)}, c(B))$. Since $E\{X_{(\bar{n},j)}|\mathscr{F}_{(\bar{k},1)}\}=$ $=X_{(\bar{k},j)}$ for $\bar{k}\geqq\bar{n}$, it follows from Lemma 2.4 that $E\{Y_{\bar{n}}|\mathscr{F}_{(\bar{k},1)}\}=Y_{\bar{k}}$ for $\bar{k}\geqq\bar{n}$, that is $\{Y_{\bar{n}}, \mathscr{F}_{(\bar{n},1)}, \bar{n}\in Z^{d-1}\}$ is a $c(B)$-valued reversed martingale.

Now we show that $Y_{\bar{1}}\in L(\log^+ L)^{d-2}(\bar{1}\in Z^{d-1})$. $\{|X_{(\bar{1},j)}|, \mathscr{F}_{(\bar{1},j)}, j\geqq 1\}$ $(\bar{1}\in Z^{d-1})$ is a positive submartingale. It is a consequence of Lemma 2.5 (1) that

$\{|X_{(\bar{1},j)}|(\log^+|X_{(\bar{1},j)}|)^{d-2},\ \mathscr{F}_{(\bar{1},j)},\ j\geqq 1\}$ is a positive submartingale. By virtue of Lemma 2.5:

$$E\{\|Y_{\bar{1}}\|_c(\log^+\|Y_{\bar{1}}\|_c)^{d-2}\} =$$

$$= E\{\sup_j |X_{(\bar{1},j)}|\,[\log^+(\sup_j |X_{(\bar{1},j)}|)]^{d-2}\} = E\{\sup_j [|X_{(\bar{1},j)}|(\log^+|X_{(\bar{1},j)}|)^{d-2}]\} \leqq$$

$$\leqq \frac{e}{e-1}+\frac{e}{e-1}E\{|X_{(\bar{1},1)}|(\log^+|X_{(\bar{1},1)}|)^{d-2}\log^+[|X_{(\bar{1},1)}|(\log^+|X_{(\bar{1},1)}|)^{d-2}]\} \leqq$$

$$\leqq \frac{e}{e-1}+\frac{e}{e-1}(d-1)E\{|X_{(\bar{1},1)}|(\log^+|X_{(\bar{1},1)}|)^{d-1}\} < \infty.$$

Hence by the induction assumption the $c(B)$-valued reversed martingale $\{Y_{\bar{n}},\ \mathscr{F}_{(\bar{n},1)},\ \bar{n}\in Z^{d-1}\}$ converges to $Y_\infty = E\{Y_{\bar{1}}|\bigcap_{\bar{n}\in Z^{d-1}}\mathscr{F}_{(\bar{n},1)}\}$ a.s. Now using the induction assumption we can show that the components of Y_∞ form the reversed martingale

$$\{E(X_{(\bar{1},n)}|\bigcap_{\bar{m}\in Z^{d-1}}\mathscr{F}_{(\bar{m},1)}),\ \bigcap_{\bar{m}\in Z^{d-1}}\mathscr{F}_{(\bar{m},n)},\quad n=1,2,\ldots\}.$$

This reversed martingale is convergent:

$$\lim_{n\to\infty} E\{X_{(\bar{1},n)}|\bigcap_{\bar{m}\in Z^{d-1}}\mathscr{F}_{(\bar{m},1)}\} =$$

$$E\{E(X_{(\bar{1},1)}|\bigcap_{\bar{m}\in Z^{d-1}}\mathscr{F}_{(\bar{m},1)})|\bigcap_{\substack{\bar{m}\in Z^{d-1}\\ k=1,2,\ldots}}\mathscr{F}_{(\bar{m},k)}\} =$$

$$E\{X_{(\bar{1},1)}|\bigcap_{\substack{\bar{m}\in Z^{d-1}\\ k=1,2,\ldots}}\mathscr{F}_{(\bar{m},k)}\}\quad \text{a.s.}\quad (\bar{1}\in Z^{d-1})$$

and the limit is equal to $E\{X_{\bar{1}}|\bigcap_{\bar{n}\in Z^d}\mathscr{F}_{\bar{n}}\}$, where $\bar{1}\in Z^d$. From here we have that

$$\lim_{\bar{n}\to\infty} X_{\bar{n}} = E(X_{\bar{1}}|\bigcap_{\bar{n}\in Z^d}\mathscr{F}_{\bar{n}})\quad \text{a.s.}$$

Remark. In the preceding theorem the limit can be considered as the last term of the martingale.

Lemma 3.2. *Let $\{X_t,\mathscr{F}_t,t\in T\}$ be a B-valued reversed martingale, where T is a directed set. If $X_{t_0}\in L^r$ $(1\leqq r<\infty)$ for some $t_0\in T$, then X_t is convergent in L^r and in probability.*

Proof. The convergence in L^r is an immediate consequence of Prop. 4.1 of [4] and Lemma V—1—1 of [9]. Accordii g to Chebyshev's inequality convergence in L^r implies convergence in probability.

Theorem 3.3. *Let $\{X_{\bar{n}},\mathscr{F}_{\bar{n}},\ \bar{n}\in Z^d\}$ be a B-valued reversed martingale. If $E(E(\cdot|\mathscr{F}_{\bar{m}})|\mathscr{F}_{\bar{n}})=E(\cdot|\mathscr{F}_{\max(\bar{m},\bar{n})})$ for every $\bar{m},\bar{n}\in Z^d$ and $E|X_{\bar{1}}|^r<\infty$ $(1<r<\infty)$, then $\lim_{\bar{n}\to\infty} X_{\bar{n}}=E(X_{\bar{1}}|\bigcap_{\bar{n}\in Z^d}\mathscr{F}_{\bar{n}})$ a.s. and in L^r.*

PROOF. According to Lemma 3.2 we have to prove only a.s. convergence. For $d=1$ it is a consequence of Prop. 4.1 of [4]. We proceed by induction. Let $\bar{m}$ be an element in Z^{d-1}. Let

$$Y_{\bar{m}} = (X_{(\bar{m},1)}, X_{(\bar{m},2)} \ldots).$$

It follows from Prop. 4.1 of [4] that

$$\lim_{n\to\infty} X_{(\bar{m},n)} = E\left(X_{(\bar{1},1)}\Big|\bigcap_{k=1}^{\infty}\mathscr{F}_{(\bar{m},k)}\right) \quad \text{a.s.} \quad (\bar{1},\ \bar{m}\in Z^{d-1}).$$

Thus $Y_{\bar{m}}$ is a $c(B)$-valued r.v. Using Lemma 2.5 (2b) one can prove that $Y_{\bar{1}}\in L^r(\Omega, \mathscr{A}, P, c(B))$ $(\bar{1}\in Z^{d-1})$.

Hence by the induction assumption the reversed martingale $\{Y_{\bar{m}}, \mathscr{F}_{(\bar{m},1)}, \bar{m}\in Z^{d-1}\}$ converges

$$\lim_{\bar{m}\to\infty} Y_{\bar{m}} = E\big(Y_{\bar{1}}\big|\bigcap_{\bar{m}\in Z^{d-1}}\mathscr{F}_{(\bar{m},1)}\big) \quad \text{a.s.}$$

Finally, the proof can be completed as the proof of Theorem 3.1.

4. Martingales

Lemma 4.1. (*Cf.* [4].) *If* $\{X_n, \mathscr{F}_n, n\geqq 1\}$ *is a B-valued martingale and*

$$\sup_n E\{|X_n|(\log^+|X_n|)^k\} < \infty$$

for a $k\in Z$, *then* $\sup_n |X_n|\in L^1$.

PROOF. Since $\{|X_n|, \mathscr{F}_n, n\geqq 1\}$ is a real positive submartingale, then Lemma 2.4 (2a) is applicable. The monotone convergence theorem implies the required result.

Lemma 4.2. *For every* $X\in L^1(\Omega, \mathscr{A}, P, B)$ *the family of random variables* $E(X|\mathscr{F})$ *obtained when* $\mathscr{F}$ *varies over all the* σ*-subalgebras of* $\mathscr{A}$ *is uniformly integrable.*

PROOF. It is a simple consequence of Lemma IV—2—4 of [9] and the inequality $E(|X|\,|\mathscr{F})\geqq|E(X|\mathscr{F})|$ a.s.

Theorem 4.3. *Let B have RNP (see* [3], [4]*) and let* $\{X_t, \mathscr{F}_t, t\in T\}$ *be a B-valued martingale, where* T *is a directed set. For* X_t *to be of the form* $X_t=E(X|\mathscr{F}_t)$ $(t\in T)$ *for a r.v.* $X\in L^1(\Omega, \mathscr{A}, P, B)$, *it is necessary and sufficient that it be uniformly integrable. In this case* $\lim X_t = E\big(X|\sigma\{\bigcup_{t\in T}\mathscr{F}_t\}\big)$ *in* L^1, *where* $\sigma\{\bigcup_{t\in T}\mathscr{F}_t\}$ *denotes the* σ*-algebra generated by* $\bigcup_{t\in T}\mathscr{F}_t$.

PROOF. This theorem is an analogue of Prop. V—1—2 of [9]. Sufficiency. We prove it under the following weaker condition: "every increasing subsequence $\{X_{t_n}, n\in Z\}$ is uniformly integrable". The proof is the same as that of the above mentioned proposition if we use Chatterji's theorem (Prop. 4.2 (a) of [4]) on B-valued martingales instead of results on real martingales.

The necessity is a simple consequence of Lemma 4.2.

Theorem 4.4. *Let* $\{X_{\bar{m}}, \mathcal{F}_{\bar{m}}, \bar{m}\in Z^d\}$ *be a B-valued martingale. Suppose that* $E\{X_{(\bar{m},\bar{n})}|\mathcal{F}_{(\bar{k},\infty)}\}=X_{(\bar{k},\bar{n})}$ *if* $\bar{k}\leqq\bar{m}$, *where*

$$(\bar{m},\bar{n}) = (m_1, m_2, \ldots, m_j, n_1, n_2, \ldots, n_i)\in Z^d,$$

$$(\bar{k},\bar{n}) = (k_1, k_2, \ldots, k_j, n_1, n_2, \ldots, n_i)\in Z^d,$$

and $\mathcal{F}_{(\bar{k},\infty)}=\sigma\{\bigcup_{\bar{n}\in Z^i}\mathcal{F}_{(\bar{k},\bar{n})}\}$, $(i+j=d;\ i,j\geqq 1)$.

Let B have RNP or let $X_{\bar{m}}$ *be of the form* $X_{\bar{m}}=E\{X|\mathcal{F}_{\bar{m}}\}$ *for a r.v.* $X\in L^1$. *If* $\sup_{\bar{m}\in Z^d} E\{|X_{\bar{m}}|(\log^+|X_{\bar{m}}|)^{d-1}\}<\infty$, *then* $\lim_{\bar{m}\to\infty} X_{\bar{m}}$ *exists a.s. (and in* L^1, *if* $d\geqq 2$).

PROOF. If $d=1$ the theorem is an immediate consequence of propositions 4.1 and 4.2 (a) of [4]. Let $d>1$. Assume the validity of the proposition for $d-1$. Lemma 4.1 and Theorem 4.3 imply that $X_{\bar{m}}=E\{X|\mathcal{F}_{\bar{m}}\}$ $(\bar{m}\in Z^d)$ for a r.v. $X\in L^1$ in both cases and $\lim_{\bar{m}\to\infty} X_{\bar{m}}=X$ in L^1. Let

$$Y_{\bar{n}} = (X_{(\bar{n},1)}, X_{(\bar{n},2)}, \ldots) \quad (\bar{n}\in Z^{d-1}).$$

According to Prop. 4.1 of [4] $\lim_{j\to\infty} X_{(\bar{n},j)}$ exists a.s., thus $Y_{\bar{n}}\in c(B)$. It follows from Lemma 4.1 that $Y_{\bar{n}}\in L^1$. Since the martingale property in $c(B)$ can be proved coordinate-wise $\{Y_{\bar{n}}, \mathcal{F}_{(\bar{n},\infty)}, \bar{n}\in Z^{d-1}\}$ is a $c(B)$-valued martingale.

To show that $Y_{\bar{n}}=E(Y|\mathcal{F}_{(\bar{n},\infty)})(\bar{n}\in Z^{d-1})$ for a suitable $Y\in L^1$ put

$$\begin{aligned} X_{(\infty,n)} &= E\{X|\mathcal{F}_{(\infty,n)}\} \\ &= \lim_{\bar{m}\to\infty} E\{X|\mathcal{F}_{(\bar{m},n)}\} \quad \text{in } L^1 \\ &= \lim_{\bar{m}\to\infty} X_{(\bar{m},n)} \qquad \text{in } L^1, \end{aligned}$$

where

$$\mathcal{F}_{(\infty,n)} = \sigma\{\bigcup_{\bar{m}\in Z^{d-1}}\mathcal{F}_{(\bar{m},n)}\}. \quad \text{Let} \quad Y = (X_{(\infty,1)}, X_{(\infty,2)}, \ldots).$$

Since the components of Y form martingale thus

$$\lim_{n\to\infty} X_{(\infty,n)} = E\left(X|\sigma\left\{\bigcup_{n=1}^{\infty}\mathcal{F}_{(\infty,n)}\right\}\right) \quad \text{a.s.}$$

that is $Y\in c(B)$. Using Lemma 4.1 we can show that $\sup_n |X_{(\infty,n)}|\in L^1$ thus $Y\in L^1$. It follows from our σ-algebra condition that

$$E(Y|\mathcal{F}_{(\bar{m},\infty)}) = Y_{\bar{m}} \quad (\bar{m}\in Z^{d-1}).$$

Furthermore by the Cairoli's inequality

$$\sup_{\bar{m}\in Z^{d-1}} E\{\|Y_{\bar{m}}\|_c(\log^+\|Y_{\bar{m}}\|_c)^{d-2}\} < \infty.$$

Hence the martingale $\{Y_{\bar{m}}, \mathcal{F}_{(\bar{m},\infty)}, \bar{m}\in Z^{d-1}\}$ converges a.s. if the theorem is true for $d-1$. Thus

$$\lim_{\substack{\bar{n}\to\infty\\ \bar{n}\in Z^d}} X_{\bar{n}} = E(X|\sigma\{\bigcup_{\bar{m}\in Z^d}\mathcal{F}_{\bar{m}}\}) \quad \text{a.s.}$$

Lemma 4.5. *Let $\{X_t, \mathscr{F}_t, t\in T\}$ be a B-valued martingale, where T is a directed set. Suppose that $X_t=E(X|\mathscr{F}_t)$, where $X\in L^r\,(1<r<\infty)$ or the following conditions are satisfied: B has RNP and $\sup_{t\in T} E|X_t|^r<\infty$.*

Then $\{X_t, t\in T\}$ is uniformly integrable and $X_t=E(X|\mathscr{F}_t)$, $X\in L^r$. Furthermore X_t converges in L^r.

Proof. In the first case the assertion is trivial. In the second case it follows from Prop. 4.2 (b) of [4], Lemma V—1—1 of [9] and Theorem 4.3.

Theorem 4.6. *Let $\{X_{\bar m}, \mathscr{F}_{\bar m}, \bar m\in Z^d\}$ be a B-valued martingale. Suppose that B has RNP or $X_{\bar m}=E(X|\mathscr{F}_{\bar m})$ $(\bar m\in Z^d)$ for $X\in L^r$ $(1<r<\infty)$. Suppose that*

$$E\{X_{(\bar m,\bar n)}|\mathscr{F}_{(\bar k,\infty)}\}=X_{(\bar k,\bar n)}\quad\text{if}\quad \bar k\leqq\bar m,$$

where $(\bar m,\bar n)$, $(\bar k,\bar n)$ and $\mathscr{F}_{(\bar k,\infty)}$ are defined in Theorem 4.4.

If $\sup_{\bar m\in Z^d} E|X_{\bar m}|^r<\infty$, then $\lim_{\bar m\to\infty} X_{\bar m}$ exists a.s. and in L^r.

The proof is a modification of the proof of Theorem 4.4.

5. A strong law of large numbers

Lemma 5.1. *Let $Y\in L^1(\Omega,\mathscr{A},P,B)$. Let ξ and η be random objects. If (Y,ξ) and η are independent, then*

$$E(Y|\xi,\eta)=E(Y|\xi)\quad a.s.$$

Proof. It follows from the scalar case by the help of linear functionals.

Lemma 5.2. *Let $X_1, X_2, \ldots, X_n$ be independent identically distributed (i.i.d.) B-valued Bochner integrable random variables. If $S_n=\sum_{j=1}^{n} X_j$, then*

$$E(X_k|S_n)=\frac{S_n}{n}\quad a.s.\quad (k=1,2,\ldots,n).$$

If $\mathscr{G}_n=\sigma\{S_n, S_{n+1},\ldots\}$ $(n=1,2,\ldots)$, then $\left\{\frac{S_n}{n}, \mathscr{G}_n,\ n\geqq 1\right\}$ is a B-valued reversed martingale.

Proof. One can use the method of 7.8.1 Lemma and 7.8.3 Theorem of [1] and Lemma 5.1.

Theorem 5.3. *Let $X_{\bar k}$ $(\bar k\in Z^d,\ d\geqq 1)$ be a sequence of i.i.d. B-valued random variables. Then*

$$W_{\bar n}=\frac{S_{\bar n}}{|\bar n|}=\frac{1}{|\bar n|}\sum_{\bar k\leqq\bar n} X_{\bar k}$$

converges a.s. if and only if $E\{|X_{\bar 1}|(\log^+|X_{\bar 1}|)^{d-1}\}<\infty$.

Proof. 1. Let $E\{|X_{\bar 1}|(\log^+|X_{\bar 1}|)^{d-1}\}<\infty$. Suppose that $E(X_{\bar 1})=0$. For $d=1$ our theorem is equivalent to Prop. 3.1 of [4]. We proceed by induction. Let $d>1$

and suppose that the theorem is true for $d-1$. Let $\bar{m}\in Z^{d-1}$ be fixed, let $V_{\bar{m}}^k=\sum_{j=1}^{k} X_{(\bar{m},j)}$, $\mathcal{G}_{\bar{m}}^k=\sigma\{V_{\bar{m}}^k, V_{\bar{m}}^{k+1}, \dots\}$, $(k=1, 2, \dots)$. From the preceding lemma $\left\{\frac{V_{\bar{m}}^k}{k}, \mathcal{G}_{\bar{m}}^k, k\geq 1\right\}$ is a reversed martingale. Since the sequence of the components of the vector

$$Y_{\bar{m}}=\left(\frac{V_{\bar{m}}^1}{1}, \frac{V_{\bar{m}}^2}{2}, \frac{V_{\bar{m}}^3}{3}, \dots\right)$$

converges to 0 a.s. it follows that $Y_{\bar{m}}$ is a $c_0(B)$-valued r.v. The random variables $Y_{\bar{m}}$ $(\bar{m}\in Z^{d-1})$ are i.i.d. and their common expectation is $0\in c_0(B)$. Since the components of $Y_{\bar{1}}$ $(\bar{1}\in Z^{d-1})$ form a reversed martingale thus $Y_{\bar{1}}\in L(\log^+ L)^{d-2}$. By the induction assumption, we then have that

$$\lim_{\substack{\bar{n}\to\infty \\ \bar{n}\in Z^{d-1}}} \frac{1}{|\bar{n}|} \sum_{\substack{\bar{m}\leq\bar{n} \\ \bar{m}\in Z^{d-1}}} Y_{\bar{m}}=0 \quad \text{a.s.} \quad (\text{in } c_0(B)).$$

But $\frac{1}{|(\bar{n}, j)|} S_{(\bar{n},j)}=\frac{1}{|\bar{n}|}\big(\sum_{\bar{m}\leq\bar{n}} Y_{\bar{m}}\big)_j$, where $\bar{n}, \bar{m}\in Z^{d-1}$, $j\in Z$, and $\big(\sum_{\bar{m}\leq\bar{n}} Y_{\bar{m}}\big)_j$ denotes the j-th component of the vector $\sum_{\bar{m}\leq\bar{n}} Y_{\bar{m}}$. From here we have that $\lim_{(\bar{n},j)\to\infty} \frac{1}{|(\bar{n}, j)|} S_{(\bar{n},j)}=0$ a.s.

2. Let $E\{|X_{\bar{1}}|(\log^+|X_{\bar{1}}|)^{d-1}\}=\infty$. From [12] (p. 165) it follows that for $A>0$

$$P\{|W_{\bar{k}}|>A \text{ occurs for arbitrary large indices}\}=1.$$

Thus the sequence $W_{\bar{k}}$ does not converge a.s.

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