On the machine interference

By JÁNOS SZTRIK (Debrecen)

1. Introduction

In this paper we deal with a special queueing problem, which is of considerable practical importance. The roots of it date back even to Hincsin, who first raised the question in the thirties. Then it was investigated by several famous mathematicians such as Palm, Naor, Fry, Takács, Kronig, Feller, Benson and Cox. In our case the problem can be formulated as follows.

Let us consider a set of a machines which are serviced by a single repairman. The machines work continuously and independently; however, at any time one of them may break down and need service. Let us suppose that if at time t machine i is in working state, the probability that it will call for service in the time interval $(t, t+\Delta t)$ is $\lambda_i \Delta t + o(\Delta t)$. If a machine breaks down it will be serviced immediately unless the repairman is attending another one, in which case a waiting line is formed. The repairs are carried out in the order of machines' breakdowns. It is supposed that the operative is idle if and only if there is no machine failed. Service times are assumed to be independent and having distribution function $F_i(x)$ for machine i. It should be mentioned that the mathematical model of machine interference often occurs in probabilistic description of multiprogramming computer systems.

Finally, we give some books and papers concerning the problem in question: TAKÁCS [1], COHEN [2], GAVER [3], COX—SMITH [4], PAGE [5], GNEDENKO—KOVALENKO [6].

2. The model

Let the random variable (abbreviated by r.v.) $\nu(t)$ denote the number of machines not working at time t and let $\alpha_1(t), \alpha_2(t), \alpha_{\nu(t)}(t)$ indicate their indices in the order of their breakdowns.

Introduce the stochastic process

$$\underline{y}(t) = \big(\nu(t),\ \alpha_1(t),\ \ldots,\ \alpha_{\nu(t)}(t)\big).$$

The process $(\underline{y}(t), t \geqq 0)$ is not Markovian unless the distribution functions $F_i(x)$ are exponential, $i = 1, 2, \ldots, n$.

Let the r.v. ξ_t denote the attained service time the machine under repair has got till time t.

Putting

$$\underline{x}(t) = (\underline{y}(t),\ \xi_t) = (v(t),\ \alpha_1(t),\ \ldots,\ \alpha_{v(t)}(t);\ \xi_t) \tag{1}$$

the process $(\underline{x}(t), t \geqq 0)$ has already the Markov property.

Let V_k^n denote the set of all variations of order k of the integers $1, 2, \ldots, n$ ordered lexicographically. A possible state of $(\underline{x}(t), t \geqq 0)$ is $(k, i_1, \ldots, i_k; x)$ where $(i_1, \ldots, i_k) \in V_k^n$ and $x \in \mathbf{R}_+$. The process is in this state if k machines are failed, their indices in the order of their breakdowns are $(i_1, \ldots, i_k)$, and the attained service time of machine i_1, which is under repair, is x.

The state when all machines are working will be denoted by $\{0\}$. Thus the state space of $(\underline{x}(t), t \geqq 0)$ is $\left(\bigcup_{k=1}^{n} V_k^n\right) \times \mathbf{R}_+ + \{0\}$.

In order to derive the Kolmogorov equations consider the transitions that can occur in an arbitrary time interval $(t, t+\Delta t)$. The transition probabilities are the following.

(i)
$$P\{x(t+\Delta t) = (i_1, \ldots, i_k;\ x+\Delta t)/x(t) = (i_1, \ldots, i_k;\ x)\} =$$
$$\Big(1 - \sum_{j \neq i_1, \ldots, i_k} \lambda_j \Delta t\Big) \frac{1-F_{i_1}(x+\Delta t)}{1-F_{i_1}(x)} + o(\Delta t),$$

(ii)
$$P\{x(t) = (i_1, \ldots, i_{k+1}, x+\Delta t)/x(t) = (i_1, \ldots, i_k;\ x)\} =$$
$$= \lambda_{i_{k+1}} \Delta t \frac{1-F_{i_1}(x+\Delta t)}{1-F_{i_1}(x)} + o(\Delta t),$$

(iii)
$$P\{x(t+\Delta t) = (i_2, \ldots, i_k;\ o)/x(t) = (i_1, \ldots, i_k;\ x)\} =$$
$$= \frac{F_{i_1}(x+\Delta t) - F_{i_1}(x)}{1-F_{i_1}(x)} + o(\Delta t).$$

Let us introduce some notations.

$$\Lambda_{i_1, \ldots, i_k} = \sum_{j \neq i_1, \ldots, i_k} \lambda_j, \quad \Lambda = \sum_{j=1}^{n} \lambda_j, \quad \beta_i = \int_0^\infty x\, dF_i(x),$$

$$S_{i_1, \ldots, i_k} = \sum_{j=1}^{k} \lambda_{i_j}, \quad \Phi(s; i) = \int_0^\infty e^{-sx}\, dF_i(x),$$

$$\Pi_l^{(i_1, \ldots, i_k)} = \prod_{r=l+2}^{k} \lambda_{i_r} \Big/ \prod_{q=l+2}^{k} S_{i_{l+1}, \ldots, i_q},$$

$$1 \leqq l \leqq k, \quad 1 \leqq k \leqq n-1.$$

For the distribution of (1) consider the functions given below.

$$P_0(t) = P(v(t) = 0),$$

$$P_{i_1, \ldots, i_k}(x, t) = P(v(t) = k, \alpha_1(t) = i_1, \ldots, \alpha_{v(t)}(t) = i_k;\ \xi_t \leqq x),$$

$$(i_1, \ldots, i_k) \in V_k^n, \quad k = 1, 2, \ldots, n.$$

Theorem 1. *If* $\beta_i < \infty$, $i = 1, 2, \ldots, n$ *then the limits*

$$P_0 = \lim_{t\to\infty} P_0(t),$$

$$P_{i_1,\ldots,i_k}(x) = \lim_{t\to\infty} P_{i_1,\ldots,i_k}(x, t),$$

$$(i_1, \ldots, i_k) \in V_k^n, \quad k = 1, 2, \ldots, n, \quad x \in \mathbf{R}_+,$$

exist and satisfy the normalization condition

$$P_0 + \sum_{k=1}^{n} \sum_{V_k^n} \lim_{x\to\infty} P_{i_1,\ldots,i_k}(x) = 1.$$

Proof. Note that $(\underline{x}(t), t \geqq 0)$ is a linear Markov process treated in Gnedenko—Kovalenko [6] in details. Our statement follows from a theorem on page 211 of this monograph.

Our next task is to give a procedure to determine the ergodic probabilities

$$(P_0, P_{i_1,\ldots,i_k}), \quad (i_1, \ldots, i_k) \in V_k^n, \quad k = 1, 2, \ldots, n.$$

To do so, first of all we show that the ergodic functions

$$P_{i_1,\ldots,i_k}(x), \quad (i_1, \ldots, i_k) \in V_k^n, \quad k = 1, 2, \ldots, n,$$

are differentiable at common continuity points of $F_i(x)$. Then we introduce the so-called normed density functions:

$$p^*_{i_1,\ldots,i_k}(x) = \frac{\frac{d}{dx} P_{i_1,\ldots,i_k}(x)}{1 - F_{i_1}(x)}. \tag{2}$$

We derive a system of integro-differential equations for these functions, and by the help of its solution we can give an algorithm for calculating the stationary distribution.

Consider for $(i_1, \ldots, i_k) \in V_k^n$, $k = 1, 2, \ldots, n$, $\tau > 0$, $1 \leqq l < k \leqq n$, the following conditional probability

$$P(v(t+\tau) = k,\ \alpha_j(t+\tau) = i_j,\ 1 \leqq j \leqq k,\ \xi_{t+\tau} = \tau / v(t) = l,\ \alpha_j(t) = i_j,\ 1 \leqq j \leqq l,\ \xi_t = 0).$$

One can easily verify that this quantity is

$$[1 - F_{i_1}(\tau)]\, e^{-\Lambda_{i_1,\ldots,i_k}\tau} \int\limits_{0 < z_{l+1} < \ldots < z_k < \tau} \prod_{j=l+1}^{k} \lambda_j e^{-\lambda_j z_j}\, dz_j.$$

This probability will further be denoted by

$$[1 - F_{i_1}(\tau)]\, V^{i_1,\ldots,i_k}_{i_1,\ldots,i_e}(\tau).$$

In homogeneous case $(\lambda_i \equiv \lambda)$

$$V^{i_1,\ldots,i_k}_{i_1,\ldots,i_r}(\tau) = \frac{1}{(k-r)!} (1 - e^{-\lambda\tau})^{k-r} e^{-(n-k)\tau}.$$

Now we prove the following theorem.

Theorem 2. *The ergodic distribution function* $P_{i_1,\dots,i_k}(x)$ *possesses density function* $p_{i_1,\dots,i_k}(x)$ *for all* $(i_1,\dots,i_k)\in V_k^n$, $k=1,2,\dots,n$ *and almost every* $x\in\mathbf{R}_+$. *In addition, the normed d.f.*

$$p^*_{i_1,\dots,i_k}(x)=\frac{p_{i_1,\dots,i_k}(x)}{1-F_{i_1}(x)}$$

is differentiable at every $x\in\mathbf{R}_+$.

Proof. We first prove the existence of densities $p_{i_1,\dots,i_k}(x)$. Let the s.p. $(\underline{x}(t), t\geqq 0)$ be in state $(i_1,\dots,i_k; x)$ at an arbitrary t. This event occurs if at some epoch $u, t-x<u<t$, the operative completes a repair and immediately starts servising machine i_1. If the indices of machines not working at time u are $(i_1,\dots,i_r)$ then the unexpired service time of i_1 must excess $t-u$ and during the time interval (u,t) machines $(i_{r+1},\dots,i_k)$ should break down in this order.

Consider the sequence of those service completion epochs when the operative has completed a repair and just started to service the machine i_1, while the others with indices $(i_2,\dots,i_r)$ are also waiting for repair. These time instances are regeneration points for the process $(\underline{x}(t), t\geqq 0)$, i.e. they form a renewal process.

If the initial state $(j_1,\dots,j_s; Z)$ differs from $(i_1,\dots,i_r; 0)$ then this renewal process is a so-called delayed one.

Let us denote by $H^{i_1,\dots,i_r}_{j_1,\dots,j_s,z}(t)$ the renewal function of the process considered above.

Denote by

$$(R_0, R_{j_1,\dots,j_s}(z); z\geqq 0, (j_1,\dots,j_s)\in V_s^n, s=1,2,\dots,n)$$

the initial distribution of $(\underline{x}(t), t\geqq 0)$. Keeping in mind the behavior of the recurrent process, by using the theorem of total probability we get

$$P_{i_1,\dots,i_k}(x,t)=\left(\sum_{s=1}^{n}\sum_{V_s^n}\int_0^t dR_{j_1,\dots,j_s}(z)+R_0\right)\cdot$$
$$\cdot\sum_{r=1}^{k}\int_{t-x}^{t}V^{i_1,\dots,i_k}_{i_1,\dots,i_r}(t-u)[1-F_{i_1}(t-u)]\,dH^{i_1,\dots,i_r}_{j_1,\dots,j_s,z}(u).$$

Applying the key renewal theorem of Smith we have

$$P_{i_1,\dots,i_k}(x)=\lim_{t\to\infty}P_{i_1,\dots,i_k}(x,t)=$$
$$=\sum_{r=1}^{k}\frac{1}{m_{i_1,\dots,i_r}}\int_0^x[1-F_{i_1}(u)]V^{i_1,\dots,i_k}_{i_1,\dots,i_r}(u)\,du.$$

where $m_{i_1,\dots,i_r}$ denotes the expected recurrence time into state $(i_1,\dots,i_r; 0)$ which is finite since the process is ergodic. Now it follows that indeed $P_{i_1,\dots,i_k}(x)$ is differentiable at every continuity point of $F_{i_1}(x)$.

This implies that the d.f. $p_{i_1,\dots,i_k}(x)$ is defined almost everywhere and

$$p_{i_1,\dots,i_k}(x)=\sum_{r=1}^{n}\frac{1}{m_{i_1,\dots,i_r}}V^{i_1,\dots,i_k}_{i_1,\dots,i_r}(x)[1-F_{i_1}(x)].$$

Therefore the normed functions are differentiable at every $x\in\mathbf{R}_+$.

Theorem 3. *The stationary density f. of the process* $(\underline{x}(t),\ t \geqq 0)$ *satisfies the following system of differential equations*

$$
\tag{3}
\begin{cases}
\dfrac{dp_{i_1}^*(x)}{dx} + \Lambda_{i_1} p_{i_1}^*(x) = 0, \\[2mm]
\dfrac{dp_{i_1,\dots,i_k}^*(x)}{dx} + \Lambda_{i_1,\dots,i_k} p_{i_1,\dots,i_k}^*(x) = \lambda_{i_k} p_{i_1,\dots,i_{k-1}}^*(x), \\[2mm]
\dfrac{dp_{i_1,\dots,i_n}^*(x)}{dx} = \lambda_{i_n} p_{i_1,\dots,i_{n-1}}^*(x).
\end{cases}
$$

The boundary conditions are

$$
\tag{4}
\begin{cases}
\Lambda P_0 = \sum\limits_{i=1}^{n} \int\limits_0^\infty p_i^*(x)\, dF_i(x) \\[2mm]
p_{i_1}(0) = \lambda_{i_1} P_0 + \sum\limits_{j \neq i_1} \int\limits_0^\infty p_{j,i_1}^*(x)\, dF_j(x), \\[2mm]
p_{i_1,\dots,i_k}(0) = \sum\limits_{j \neq i_1,\dots,i_k} \int\limits_0^\infty p_{j,i_1,\dots,i_k}^*(x)\, dF_j(x), \quad p_{i_1,\dots,i_n}(0) = 0.
\end{cases}
$$

PROOF. The equations are obtained in the manner familiar in the theory of linear Markov processes. For details see SZTRIK [7].

It is quite easy to see that the solution of (3), (4) is

$$
p_{i_1,\dots,i_k}^*(x) = \sum_{l=1}^{k} (-1)^{k-l} c_{i_1,\dots,i_l} e^{-\Lambda i_1,\dots,i_l x} \cdot \Pi_l^{(i_1,\dots,i_k)},
$$

$$
(i_1, \dots, i_k) \in V_k^n, \quad 1 \leqq k \leqq n,
$$

where the constants $c_{i_1,\dots,i_l}$ are to be determined from the boundary condition (4).

In the following we describe an iterative method to calculate these coefficients. Let $\underline{c}_k$ denote the vector

$$
\begin{vmatrix}
c_{1,2,\dots,k} \\
\vdots \\
c_{i_1,\dots,i_k} \\
\vdots \\
c_{n,n-1,\dots,n-k+1}
\end{vmatrix}
$$

of dimension $\binom{n}{k} k!$ The components of $\underline{c}_k$ are listed in the lexicographic order of their indices, $k = 1, 2, \dots, n$. Notice, that the boundary condition $p_{i_1,\dots,i_n}(0) = 0$ is equivalent to equation $\underline{c}_n = A_{n-i}^{(n)} \underline{c}_{n-1} + \dots + A_1^{(n)} \underline{c}_1$ with a suitably chosen $A_k^{(n)}$ matrix of order $n!\, X \binom{n}{k} k!$. The k-th boundary condition, where $2 \leqq k \leqq n-1$

gives the relation

$$\sum_{l=1}^{k}(-1)^{k-l}c_{i_1,\dots,i_l}\Pi_l^{(i_1,\dots,i_k)} =$$
$$\sum_{j\neq i_1,\dots,i_k}\sum_{l=0}^{k}(-1)^{k-l}c_{j,i_1,\dots,i_l}\Pi_{l+1}^{(j,i_1,\dots,i_k)}\int_0^\infty e^{-\Lambda_{j,i_1,\dots,i_l}x}\,dF_j(x).$$

In term of the Laplace—Stieltjes transform this becomes

$$\sum_{l=1}^{k}(-1)^{k-l}c_{i_1,\dots,i_l}\Pi_l^{(i_1,\dots,i_k)} =$$
$$\sum_{j\neq i_1,\dots,i_k}\sum_{l=0}^{k}(-1)^{k-l}c_{j,i_1,\dots,i_l}\cdot\Pi_{l+1}^{(j,i_1,\dots,i_k)}\Phi(\Lambda_{j,i_1,\dots,i_l,j}).$$

More succintly

$$\underline{c}_k = A_{k+1}^{(k)}\underline{c}_{k+1}+\dots+A_1^{(k)}\underline{c}_1.$$

Now we are able to define our algorithm. We have

$$\underline{c}_n = \sum_{j=1}^{n-1}A_j^{(n)}\underline{c}_j,$$

$$(*) \qquad \underline{c}_{n-1} = \sum_{j=1}^{n-2}B_j^{(n-1)}\underline{c}_j,$$

where the matrix $B_j^{(n-1)}$ is defined by

$$B_j^{(n-1)} = (I-A_n^{(n-1)}A_{n-1}^{(n)}-A_{n-1}^{(n-1)})^{-1}\cdot(A_n^{(n-1)}A_j^{(n)}+A_j^{(n-1)}), \quad 1\leqq j\leqq n-2.$$

Similarly

$$(**) \qquad \underline{c}_k = \sum_{j=1}^{k-1}B_j^{(k)}\underline{c}_j,$$

where the matrix $B_j^{(k)}$ is given by

$$B_j^{(k)} = (I-A_{k+1}^{(k)}B_k^{(k+1)}-A_k^{(k)})^{-1}\cdot(A_{k+1}^{(k)}B_j^{(k+1)}+A_j^{(k)}),$$
$$2\leqq k\leqq n-1, \quad 1\leqq j\leqq k-1.$$

For $\underline{c}_1$ we have the equation

$$\underline{c}_1 = A_2^{(1)}\underline{c}_2+A_1^{(1)}\underline{c}_1+P_0\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_n\end{pmatrix}.$$

Using the formula for $\underline{c}_2$ we obtain

$$(I-A_2^{(1)}B_1^{(2)})\underline{c}_1 = P_0\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_n\end{pmatrix}.$$

Hence

$$(***) \qquad \underline{c}_1 = (1-A_2^{(1)}B_1^{(2)})^{-1}\begin{pmatrix}\lambda_1\\ \vdots\\ \lambda_n\end{pmatrix}P_0.$$

Starting with an arbitrary P_0 and using the relations $(*), (**), (***)$ we can determine the vectors $\underline{c}_1, \underline{c}_2, \ldots, \underline{c}_n$ (in this order). Following this procedure we obtain all constants and also the density functions $p^*_{i_1,\ldots,i_k}(x)$.

Let us denote by $P_{i_1,\ldots,i_k}$ the stationary probability that machines with indices $(i_1, \ldots, i_k)$ are not working. Apparently,

$$P_{i_1,\ldots,i_k} = \int_0^\infty p_{i_1,\ldots,i_k}(x)\,dx = \int_0^\infty p^*_{i_1,\ldots,i_k}(x)\big(1-F_{i_1}(x)\big)\,dx.$$

Denote by P_k the steady state probability that k machines are failed. We have

$$P_k = \sum_{V_k^n} P_{i_1,\ldots,i_k}.$$

The value of P_0 can be calculated from the normalization condition

$$P_0 + \sum_{k=1}^n P_k = 1.$$

3. Utility investigations

(i) *Operative utilization.*

It is easy to see that the repairman's activity can be divided into two periods, viz. idle and busy ones. Together they form a cycle. The duration of these cycles are independent and identically distributed random variables.

By the virtue of a renewal consideration it follows that

$$P_0 = \frac{1\Big/\left(\sum_{i=1}^n \lambda_i\right)}{1\Big/\left(\sum_{i=1}^n \lambda_i\right) + M\delta},$$

where $M\delta$ denotes the mean occupation time of the server and $1\Big/\sum_{i=1}^n \lambda_i$ the average idle period length.

If U_0 denotes the utilization of the repairman, which is the long-run fraction of time the server is busy, then

$$U_0 = \frac{M\delta}{1\Big/\left(\sum_{i=1}^n \lambda_i\right) + M\delta}.$$

Consequently

$$M\delta = (1-P_0)/(\Lambda P_0).$$

(ii) *Machine utilization.*

During production the activity of machine i can be divided into three parts: running, waiting and repairing periods. If one consider these periods as a cycle, then in equilibrium for a fixed machine these cycles have identical distribution, but they are not independent.

Let $P^{(i)}$ denote the ergodic probability that machine i is failed and let the average working, waiting, repairing period lengths be $1/\lambda_i$, W_i, β_i, respectively. Consider again the process $(\underline{Y}(t), t \geqq 0)$ with state space $\bigcup_{k=1}^{n} V_k^n + \{0\}$. Let H_i be the event that machine i is not working. Introduce the function

$$Z_{H_i}(t) = \begin{cases} 1 & \text{if } \underline{Y}(t) \in H_i, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4. *For* $P^{(i)}$ *the following relation holds*

$$P^{(i)} = \lim_{T \to \infty} \frac{1}{T} \int_0^T Z_{H_i}(t)\, dt = \frac{W_i + \beta_i}{1/\lambda_i + W_i + \beta_i}.$$

PROOF. The statement is a special case of a theorem concerning mean sojourn time for semi-Markov processes, see TOMKÓ [9] on page 297.

Hence the utilization of machine i is as follows

$$U_i = 1 - P^{(i)} = \frac{1/\lambda_i}{1/\lambda_i + W_i + \beta_i}.$$

Since the probability $P^{(i)}$ can be calculated from the distribution $P_{i_1, \ldots, i_k}$ by

$$P^{(i)} = \sum_{k=1}^{n} \sum_{V_k^n, i \in (i_1, \ldots, i_k)} P_{i_1, \ldots, i_k},$$

the mean waiting time of machine i is $W_i = P^{(i)}/[\lambda_i(1-P^{(i)})] - \beta_i$. Finally, for the total productivity of all machines we get

$$U = \sum_{i=1}^{n} U_i = n - \sum_{i=1}^{n} P^{(i)}.$$

4. Derivation of the Takács-formulae

If the machines are homogeneous, i.e. $\lambda_i \equiv \lambda$ and $F_i(x) = F(x)$, $i = 1, 2, \ldots, n$ we have the famous Takács-model. In this section we are going to show how his results can be obtained from our system.

Let the r.v. $\nu(t)$ denote the number of machines not working at time t and ξ_t the attained service time the machine under repair has got till the time t.

Introduce the stochastic process

$$\underline{x}(t) = (\nu(t), \ \xi_t),$$

which is of Markov-type with state space $\mathscr{X} = \{0\} + \{1, 2, \ldots, n\} \times \mathbf{R}_+$.

Let

$$P_0(t) = P(\nu(t) = 0),$$

$$P_k(x, t) = P(\nu(t) = k, \quad \xi_t \leqq x),$$

and denote the ergodic distribution by

$$P_0 = \lim_{t\to\infty} P_0(t),$$

$$P_k(x) = \lim_{t\to\infty} P_k(x, t).$$

This also possesses stationary density function $p_k(x)$, which are solutions of the following system of differential equations

$$(5) \qquad \begin{cases} \dfrac{dp_1^*(x)}{dx} + (n-1)\lambda p_1^*(x) = 0, \\ \vdots \\ \dfrac{dp_k^*(x)}{dx} + (n-k)\lambda p_k^*(x) = (n-k+1)\lambda p_{k-1}^*(x) \\ \vdots \\ \dfrac{dp_n^*(x)}{dx} = \lambda p_{n-1}^*(x). \end{cases}$$

The boundary conditions are

$$(6) \qquad \begin{cases} n\lambda P_0 = \int\limits_0^\infty p_1^*(x)\, dF(x), \\ p_1(0) = n\lambda P_0 + \int\limits_0^\infty p_2^*(x)\, dF(x), \\ \vdots \\ p_k(0) = \int\limits_0^\infty p_{k+1}^*(x)\, dF(x), \\ \vdots \\ p_n(0) = 0. \end{cases}$$

It is easy to see that the following functions are the solutions of (5) and (6)

$$p_k^*(x) = \sum_{j=1}^{k} (-1)^{k-j} \binom{n-j}{n-k} c_j e^{-(n-j)\lambda x},$$

where the constants c_j are to be determined from (6). This time, however, we can give a closed expression for c_j, namely

$$c_j = \frac{(n-j)\lambda}{1-\Phi_{n-j}} \left[H_{n-j} - \binom{n}{j} \right] P_0, \quad 1 \leqq j \leqq n-1,$$

$$c_n = \frac{1}{\beta} \left[H_0 - \binom{n}{n} \right] P_0,$$

where

$$H_0 = 1 + n\lambda\beta \sum_{j=1}^{n-1} \binom{n-1}{j} \frac{1}{d_j},$$

$$H_r = \frac{n \cdot d_{r-1}}{r} \sum_{l=r-1}^{n-1} \binom{n-1}{l} \frac{1}{d_l},$$

$$d_0 = 1, \quad d_l = \prod_{i=1}^{l} \frac{\Phi_i}{1-\Phi_i}, \quad \Phi_i = \int_0^{\infty} e^{-i\lambda\xi}\, dF(x).$$

Let P_k denote the steady state probability that k machines are failed.

Theorem 5. *For the stationary distribution* $\{P_k\}$, $k=1, 2, \ldots, n$ *we obtain*

$$P_k = \sum_{j=1}^{k} (-1)^{k-j} \binom{n-j}{n-k} \left[B_{n-j}^* - \binom{n}{j} B_n^* \right], \quad 1 \leqq k \leqq n-1,$$

$$P_n = \sum_{j=1}^{n-1} (-1)^{n-j} \left[B_{n-j}^* - \binom{n}{j} B_n^* \right] + \lambda\beta B_1^*,$$

moreover

$$P_0 = H_0^{-1} = \left(1 + n\lambda\beta \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{1}{d_j} \right)^{-1}.$$

In addition

$$P_k = Q_{n-1}^*, \quad k = 1, 2, \ldots, n,$$

where $B_j^* = H_j P_0$ *and* Q_{n-k}^* *is the stationary probability, which has been given by Takács in his model, i.e.*

$$Q_r^* = \sum_{l=1}^{n} (-1)^{l-r} \binom{l}{r} B_l^*.$$

PROOF. It is omitted, further details can be found in SZTRIK [7]. In the following we give some *characteristics* concerning the system. For operative utilization we have

$$U_0 = n\lambda\beta \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{1}{d_j} P_0,$$

while the average busy period is

$$M\delta = \beta \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{1}{d_j}.$$

For utilization of a machine we get

$$U_m = \frac{1}{n\varrho} U_0 = \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{1}{d_j} P_0,$$

where $\varrho = \lambda\beta$ is referred to as traffic intensity. Moreover, the mean waiting time of a failed machine is

$$W = (n-1)\beta - \frac{1}{\lambda} (1 - Q_{n-1}),$$

where

$$Q_{n-1} = \left(\sum_{j=0}^{n-1} \binom{n-1}{j} \frac{1}{d_j} \right)^{-1}.$$

This exactly the famous Hincsin-formula. Finally, for the average virtual waiting time we obtain

$$V = \frac{n\lambda\beta}{n\lambda\beta + Q_{n-1}} \left(\frac{\sigma^2+\beta^2}{2\beta} + (n-1)\beta - \frac{1-Q_{n-1}}{\lambda} \right), \quad (\sigma^2 = D^2(\xi)).$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF DEBRECEN, HUNGARY

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