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On right π-duo semigroups and rings

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Abstract. A semigroup or a ring is said to be right π-duo if every right ideal is a radical extension of some ideal. In this paper, we consider the properties of right π-duo semigroups and rings. We give examples of right π-duo semigroups and rings.

1. Introduction

A semigroup is said to be right duo (resp. left duo) if every right ideal (resp. left ideal) is two-sided. A right and left duo semigroup is called a duo semigroup. The class of duo semigroups includes the class of commutative semigroups and many results for commutative semigroups have been extended to those for duo semigroups (see e.g., [1], [2], [5]).

In this paper, as a generalization of right or left duo semigroups, we introduce the concept of right or left π-duo semigroups and we study their properties. We consider primary decompositions of ideals in π-duo semigroups with ascending chain conditions on ideals. We show that a π-duo semigroup is weakly commutative. We also give some examples of right π-duo semigroups. Next we consider the properties of right π-duo rings. We characterize a π-regular right π-duo ring. We show that a right π-duo primitive ring is a division ring. We also show that a right π-duo p.p. ring is reduced and has a right classical quotient ring Q that is strongly regular.

2. Right π-duo semigroups

A semigroup S is right duo if and only if $Sa \subseteq aS$ for all $a \in S$. We generalize this concept. For a subset T of a semigroup S, we set

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$\sqrt{T} = \{a \in S \mid a^n \in T \text{ for some } n \geq 1\}$. We call a semigroup S *right π-duo* if, for any right ideal K of S, there exists an ideal I of S contained in K such that $K \subseteq \sqrt{I}$. Similarly we define a *left π-duo semigroup*. A left and right π-duo ring is called a *π-duo ring.*

Proposition 1. *Let S be a semigroup. Then the following statements are equivalent:*

1) *S is right π-duo.*
2) *For every $a \in S$ there is a natural number $n(a)$ such that $Sa^{n(a)} \subseteq aS$.*

Proof. 1) $\Longrightarrow$ 2). Let $a \in S$. Then there exists an ideal I of S such that $I \subseteq aS$ and $aS \subseteq \sqrt{I}$. Then $a^n \in I$ for some positive integer n. Since I is an ideal of S contained in aS, we have $Sa^n \subseteq aS$.

2) $\Longrightarrow$ 1). Let K be a right ideal of S and let a be an element of K. Then there exists a positive integer n such that $Sa^n \subseteq aS$. Hence Sa^nS is an ideal of S contained in K. Let I be the union of all ideals of S contained in K. Then $Sa^nS \subseteq I$. This implies that $a \in \sqrt{I}$ and hence we obtain $K \subseteq \sqrt{I}$.

A right ideal P in a semigroup S is said to be *right primary* provided $xSy \subseteq P$ and $x \notin P$ implies $y \in \sqrt{P}$. Clearly every prime ideal is right primary. A right ideal K is *completely prime* if, for any $x, y \in S$, $xy \in K$ implies either $x \in K$ or $y \in K$. A right ideal K is *completely semiprime* if, $K = \sqrt{K}$.

Theorem 1. *Let S be a right π-duo semigroup and let K be a right ideal of S. Then:*

(1) $\sqrt{K}$ is an ideal of S and equals the intersection of completely prime ideals containing K.

(2) If K is right primary, then $\sqrt{K}$ is a completely prime ideal of S.

Proof. (1) Let $x \in \sqrt{K}$ and let $s \in S$. Then there exists a positive integer n such that $S(xs)^n \subseteq xsS$. By induction on k we will prove $(xs)^{kn} \in x^kS$. In case $k = 1$ the assertion is clear. So suppose that the assertion is true for k, that is $(xs)^{kn} = x^kt$ for some $t \in S$. Since $S(xs)^n \subseteq xsS$, $t(xs)^n = (xs)u$ for some $u \in S$. Then $(xs)^{(k+1)n} = (xs)^{kn}(xs)^n = x^kt(xs)^n = x^k(xs)u \in x^{k+1}S$. This completes the induction. Hence we obtain $xs \in \sqrt{K}$. Let m be a positive integer such that $Sx^m \subseteq xS$. Then $(sx)^{mn+1} = s(xs)^{mn}x \in Sx^mS \subseteq xS \subseteq K$. Hence $sx \in \sqrt{K}$. These prove that $\sqrt{K}$ is a two-sided ideal of S. Obviously $\sqrt{K}$ is a completely semiprime ideal of S. Hence $\sqrt{K}$ is the intersection of completely prime ideals by [4, Theorem II.3.7].

(2) Assume that K is right primary and let $a,\ b \in S$ with $ab \in \sqrt{K}$. Then $(ab)^k \in K$ for some positive integer k. Since S is right π-duo, $Sa^m \subseteq aS$ and $Sb^n \subseteq bS$ for some positive integers $m,\ n$. Suppose that $a,\ b \notin \sqrt{K}$. Since $(ab)^{k-1}aSb^n \subseteq (ab)^{k-1}abS \subseteq K$, we obtain $(ab)^{k-1}a \in K$. Then $(ab)^{k-1}Sa^m \subseteq (ab)^{k-1}aS \subseteq K$. Hence we obtain $(ab)^{k-1} \in K$. By induction on k, we obtain $ab \in K$. Then $aSb^n \subseteq abS \subseteq K$. Hence we have $a \in K$ or $b^n \in \sqrt{K}$. Therefore $a \in \sqrt{K}$ or $b \in \sqrt{K}$. This is a contradiction.

Corollary 1. *Let S be a right π-duo semigroup. Then every completely semiprime right ideal is an ideal.*

Now we consider primary decompositions of ideals.

Theorem 2. *Let S be a π-duo semigroup with ascending chain condition on ideals. Then every ideal is written as a finite intersection of right primary ideals.*

Proof. We say that an ideal of S is irreducible if it is not the intersection of two strictly larger ideals. Using noetherian induction, we know that every ideal of S is written as a finite intersection of irreducible ideals. Hence it suffices to prove that any irreducible ideal is right primary. So, let I be an irreducible ideal of S. For an element $a \in S$, we set $(I : Sa) = \{s \in S \mid sSa \subseteq I\}$. Suppose that I is not right primary. Then there exist $x \in S \setminus I$ and $y \in S \setminus \sqrt{I}$ such that $xSy \subseteq I$. Since S satisfies the ascending chain condition on ideals, there exists an integer $k > 1$ such that $(I : Sy^i) = (I : Sy^k)$ for all $i \geq k$. Now we claim that $(I : Sy^k) \cap (Sy^{k+1} \cup I) = I$. To prove this, take an arbitrary $z \in (I : Sy^k) \cap (Sy^{k+1} \cup I)$. We may assume that $z \in (I : Sy^k) \cap Sy^{k+1}$. Then we can write $z = sy^{k+1}$ where $s \in S$ and $a \in I$. Then $(sy^{k+1} + a)Sy^k \subseteq I$, and hence $sy^{k+1}Sy^k \subseteq I$. Since S is right π-duo, there exists m such that $Sy^{(k+1)m} \subseteq y^{k+1}S$. Hence $sSy^{(k+1)m}y^k \subseteq I$. This implies that $s \in (I : Sy^{km+k+m}) = (I : Sy^k)$. Thus $z = syy^k \in sSy^k \subseteq I$. This proves our claim. Since S is also left π-duo, there exists n such that $y^{(k+1)n}S \subseteq Sy^{k+1}$, and so we have $(I : Sy^k) \cap (Sy^{(k+1)n}S \cup I) = I$. Since $x \notin (I : Sy^k)$, $(I : Sy^k)$ is strictly larger than I. Since $y \notin \sqrt{I}$, $Sy^{(k+1)n}S \cup I$ is also strictly larger than I. This contradicts the irreducibility of I.

A semigroup S is said to be *right regular* (resp. *left regular*) if $a \in a^2S$ (resp. $a \in Sa^2$) for any element $a \in S$. A semigroup S is said to be *completely regular* if S is right and left regular. For other equivalent conditions for a semigroup S to be completely regular, see e.g. [4, Proposition IV.1.2].

Corollary 2. *Let S be a right π-duo semigroup. Then S is right regular if and only if S is completely regular.*

PROOF. Suppose that S is right regular. Then we can easily see that $K = \sqrt{K}$ for any right ideal of S. Then S is a right duo semigroup by Theorem 1. Then we have $a \in a^2S \subseteq Sa^2$. Hence S is also left regular.

A semigroup S is *weakly commutative* if for any $x,\ y \in S$, $(xy)^n \in ySx$ for some positive integer n.

Proposition 2. *A π-duo semigroup S is weakly commutative.*

PROOF. Since S is left and right π-duo, there are positive integers $m,\ n$ such that $S(yx)^m \subseteq yxS$ and $(yx)^nS \subseteq Syx$. Let $k = mn$. Then $S(yx)^k \subseteq yxS$ and $(yx)^kS \subseteq Syx$. Hence we can write $x(yx)^k = (yx)a$ and $(yx)^ky = b(yx)$ for some $a,\ b \in S$. Then we see that $(xy)^{2k+2} = x(yx)^kyx(yx)^ky = (yx)ayxb(yx) \in ySx$. This implies that S is weakly commutative.

A semigroup S is said to be *archimedean* if for any $a,\ b \in S$, there exists a positive integer n for which $a^n \in SbS$. Combining Proposition 1 with [4, Corollary II.5.6], we obtain the following corollary.

Corollary 3. *Let S be a π-duo semigroup. Then S is a semilattice of archimedean semigroups.*

At this point we give some examples of right π-duo semigroups.

Example 1. Let $A = \{a_i \mid i = 1, 2, 3, \dots\}$ and let F be the free semigroup on A. Let n be a positive integer and let I be the ideal of F generated by $\{x^n \mid x \in F\}$ and let S $= F/I$ (see [3, Section 7 of Chapter I]). Then $S = (F \setminus I) \cup \{0\}$. For any $a \in S$, $a^n = 0$, and hence S is right π-duo. We shall show that $\{0\}$ is a prime ideal of S, but it is not completely prime. Let $x,\ y \in F \setminus I$ and take $a_m \in A$ such that a_m does not appear in the words $x,\ y$. Then $xa_my \neq 0$. Hence $\{0\}$ is a prime ideal of S. However we see that $a_1^{n-1} \notin \{0\}$, but $(a_1^{n-1})^2 \in \{0\}$. Hence $\{0\}$ is not completely prime. We can easily see $\sqrt{\{0\}} = S$.

Example 2. Let $F \subset K$ be two fields and suppose that there exists an automorphism σ of K of order n such that $\sigma(F) \nsubseteq F$. For example, suppose $n \geq 3$ and let $F = \mathbb{Q}(\sqrt[n]{2})$ and let K be the splitting field of $x^n - 2$ over $\mathbb{Q}$. Then $K = \mathbb{Q}(\sqrt[n]{2}, \zeta)$, where ζ denotes a primitive nth root of 1. Then the the automorphism σ of K defined by $\sigma(\sqrt[n]{2}) = \sqrt[n]{2}\zeta$ and $\sigma(\zeta) = \zeta$ is of order n and $\sigma(\mathbb{Q}(\sqrt[n]{2})) = \mathbb{Q}(\sqrt[n]{2}\zeta) \nsubseteq \mathbb{Q}(\sqrt[n]{2})$. Let $R = K[x;\sigma]$ be a skew polynomial ring with $ax = x\sigma(a)$ for $a \in K$. Consider the subsemigroup

$$S = F \cup \bigcup_{i=1}^{\infty} x^i K$$

of the multiplicative semigroup of R. We can easily see that S is right duo. Then, for any $a \in S$, a^n is central. However

$$xS = xF \cup \bigcup_{i=2}^{\infty} x^i K$$

is not a left ideal of S, because $Fx = x\sigma(F) \nsubseteq xF$. Therefore S is π-duo, but is not left duo.

Example 3. Let S_n denote the semigroup constructed in Example 2 using $F = \mathbb{Q}(\sqrt[n]{2})$ and the splitting field K of $x^n - 2$ over $\mathbb{Q}$. Then S_n is a semigroup with zero for each positive integer $n \geq 3$. Let S denote the direct sum of $S_3,\ S_4, \ldots$; then S is right π-duo. But there is no positive integer N such that $Sa^N \subseteq aS$ for all $a \in S$.

3. Applications to ring theory

Throughout this section, R denotes an associative ring with identity. The Jacobson radical of R is denoted by $J(R)$. For a subset F of a ring R, $r_R(F)$ denotes the right annihilator of F in R.

A ring R is said to be *right π-duo* if the multiplicative semigroup of R is right π-duo. Let R be a ring and let K be a right ideal of R. Define the *core* of K (written $\mathrm{Core}(K)$) to be the sum of all those (two-sided) ideals of S contained in K. Thus the core is the unique largest ideal of S contained in K. Then R is right π-duo if and only if for any right ideal K, $K \subseteq \sqrt{\mathrm{Core}(K)}$.

A ring R is said to be reduced if R has no nonzero nilpotent elements.

Theorem 3. *Let R be a right π-duo ring. Then:*

(1) Every idempotent of R is central.

(2) Every completely semiprime right ideal is a two-sided ideal. In particular, every maximal right ideal of the ring R is a two-sided ideal.

(3) Every right primitive factor ring of R is a division ring.

(4) $R/J(R)$ is reduced.

Proof. (1) Let e be an idempotent of R. Then there exists a positive integer n such that $Re^n \subseteq eR$ by Proposition 1. Since e is an idempotent, we have $Re \subseteq eR$. Similarly we obtain $R(1-e) \subseteq (1-e)R$. Clearly these implies that e is central.

(2) Every completely semiprime right ideal of R is an ideal by Corollary 1. Let K be a maximal right ideal of R and let $a \in \sqrt{K}$. Then $a^n \in K$ for some positive integer n. Suppose $a \in K$. Then $aR + K = R$. Hence we can write $1 = ar + b$ for some $r \in R$ and some $b \in K$. Then $(1-b)^m = (ar)^m \in K$ for some m. This deduces $1 \in K$, a contradiction. This proves that K is completely semiprime.

(3) Let P be a (right) primitive ideal of R. Then there is a maximal right ideal K such that $P = \{a \in R \mid Ra \subseteq K\}$. Since K is an ideal by (2), we have $P = K$. Then $R/P = R/K$ is a division ring.

(4) This follows from the fact that $R/J(R)$ is a subdirect sum of primitive rings.

Question 1. Let K be a right ideal of a ring R. Then $\sqrt{K}$ is an ideal of the multiplicative semigroup of R by Theorem 1. Is $\sqrt{K}$ an ideal of the ring R?

Recall that R is said to be *π-regular* if for each element a of R, there exists a positive integer m and an element x of R such that $a^m = a^m x a^m$. A π-regular ring R for which the m in the above can be taken to be 1 for all a is called *regular*.

Proposition 3. *Let R be a π-regular ring. Then the following statements are equivalent:*

1) R is a right π-duo ring.

2) R is a left π-duo ring.

3) Every idempotent of R is central.

Proof. Since the π-regularity of R is left-right symmetric, it suffices to prove the equivalence of 1) and 3).

1) $\Rightarrow$ 3). This follows from Proposition 3 (1).

3) $\Rightarrow$ 1). Let a be an arbitrary element of R. Then there exists a positive integer m and an element $x \in R$ such that $a^m = a^m x a^m$. Then we can easily see that $e = a^m x$ is an idempotent and $a^m R = eR$. By hypothesis, $a^m R = eR$ is a two-sided ideal of R.

A ring is said to be *completely prime* if 0 is a completely prime ideal. We give an example of a π-duo completely prime ring which is not left duo.

Example 4. Let $F \subset K$ be two fields and suppose that there exists an automorphism σ of K of order n such that $\sigma(F) \nsubseteq F$. Let $K[[x;\sigma]]$ be a skew formal power series ring with $\sigma(a)x = xa$ for $a \in K$, and consider the subring $R = \{a_0 + a_1x + a_2x^2 + \cdots \mid a_0 \in F,\ a_i \in K \text{ for all } i > 0\}$ of $K[[x;\sigma]]$. Let $f = a_kx^k + a_{k+1}x^{k+1} + \cdots$ be an element of R with $a_k \neq 0$. Then $g = a_k + a_{k+1}x + \cdots$ has the inverse g^{-1} in $K[[x;\sigma]]$ and $xg^{-1} \in R$. Hence $Rf \ni xg^{-1}f = x^{k+1}$. Hence $Rf = Ra_kx^k + Rx^{k+1} = \{b_0a_kx^k + b_1x^{k+1} + b_2x^{k+2} + \cdots \mid b_0 \in F,\ b_i \in K \text{ for all } i > 0\}$. Then Rf is a two-sided ideal of R if and only if $\sigma^k(F) \subseteq F$. In particular, if $k = 1$, then Rf is not a two-sided ideal of R. Hence R is not left duo. However Rf^n is a two-sided ideal of R for any $f \in R$. Hence R is a left π-duo ring. Also we can easily see that R is right duo. Therefore R is a π-duo completely prime ring, but is not left duo.

A ring R with identity is called a *right* (resp. *left*) *p.p. ring* if every principal right (resp. left) ideal of R is projective. Clearly a completely prime ring is a right and left p.p. ring. We can easily see that R is a right p.p. ring if and only if for any $a \in R$ the right annihilator $r_R(a)$ of a is generated by an idempotent. A reduced regular ring is called a *strongly regular ring*.

Proposition 4. *Let R be a right π-duo, right (or left) p.p. ring. Then R is reduced and has a right classical quotient ring Q that is strongly regular.*

Proof. As saw in the proof of 1) $\Longrightarrow$ 3) of Proposition 5, every idempotent of R is central. Let b be an element of R with $b^2 = 0$. Then $r_R(b) = eR$ for some idempotent e. Since $b \in r_R(b)$, we can write $b = ed$ for some $d \in R$. Then $0 = be = eb = eed = ed = b$. Hence R is reduced. Let $a \in R$. We have $r_R(a) = eR$ for some central idempotent e. Since $aR \cap eR = 0$, we have $aR + eR = (a+e)R$ and $a + e$ is regular (i.e. not a zero divisor). Let c be a regular element of R and let $r \in R$. Then there exists a positive integer n such that $Rc^n \subseteq cR$. Then $rc^n = cs$ for some $s \in R$. Since c^n is regular, this implies that the set of regular elements of R satisfies the right Ore condition. Thus R has a right classical quotient ring Q that is also reduced. Let x be an element of Q. Then $x = ac^{-1}$ for some $a,\ c$ in R with c regular. With e as above we have $xQ = aQ$ and $aQ + eQ = Q$ It follows that Q is von Neumann regular. Since Q is reduced, Q is strongly regular.

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