Preseminormed spaces

By Á. SZÁZ (Debrecen)

Introduction

In this paper, we introduce the notion of preseminormed spaces and initiate a systematic study of them. Preseminormed spaces are, in some sense, equivalent to topological vector spaces, but seem to be more convenient for the purposes of applications and teaching. Seminormed spaces have formerly been used by LARSEN [4] in a quite similar sense.

In § 1, we define a real-valued function p on a vector space X to be a *preseminorm* on X if (a) $\lim_{\lambda \to 0} p(\lambda x)=0$ for all $x \in X$, (b) $p(\lambda x) \leqq p(x)$ for all $|\lambda| \leqq 1$ and $x \in X$, and (c) $p(x+y) \leqq p(x)+p(y)$ for all $x, y \in X$. Moreover, we list the basic properties of preseminorms. Preseminorms are more general and for some purposes more suitable than seminorms, and they have previously been used, under various names, mainly in the proofs of the metrization theorem of topological vector spaces.

In § 2, we define a pair $X(P)=(X, P)$, where P is a nonvoid family of preseminorms on X, to be a *preseminormed space,* and consider X to be equipped with the weakest topology T_P for which all the balls $B_p(x, r)=\{y \in X\colon p(x-y)<r\}$, where $p \in P$, $x \in X$ and $r>0$, are open. Using nets, we prove quite easily that $X(T_P)$ is a topological vector space, and T_P is the weakest translation-invariant topology on X for which each $p \in P$ is continuous. Formerly, it was known that preseminorms can be used to define a vector topology, and each vector topology can be defined by preseminorms [6]. Moreover, the corresponding fact for seminorms has been greatly utilized by several authors. However, our treatment here seems still to be new enough.

In § 3, we define two families of preseminorms to be *equivalent* if they induce the same topology, and investigate the various possible operations on preseminorms from the point of view of this equivalence. The most important fact derived here is that a countable family of preseminorms can always be replaced by a single equivalent one. The results of § 3 are then used in § 4, where we prove the *metrization* and *normality theorems* of preseminormed spaces, which are very similar to those of topological vector spaces, in a surprisingly easy way.

Finally, we remark that in a continuation of this paper the projective and inductive limits of preseminormed spaces by linear relations will be investigated.

§ 1. Preseminorms

Definition 1.1. A real-valued function p on a vector space X over $\mathbf{K}$ $(=\mathbf{R}$ or $\mathbf{C})$ is a *preseminorm* on X if

(a) $\lim_{\lambda\to 0} p(\lambda x)=0$ for all $x\in X$,
(b) $p(\lambda x)\leqq p(x)$ for all $|\lambda|\leqq 1$ and $x\in X$.
(c) $p(x+y)\leqq p(x)+p(y)$ for all $x, y\in X$.

A preseminorm p is a *prenorm* if $p(x)=0$ implies $x=0$.

Remark 1.2. The notion of a preseminorm extends that of a seminorm*) [5, p. 24].

KÖTHE [3, p. 163] and WAELBROECK [6, p. 2] would call such a function an "(F)-seminorm" and a "J-seminorm", respectively.

The condition (a) can be weakened, namely it is enough to suppose only that $\lim_{n\to\infty} p(n^{-1}x)=0$ for all $x\in X$.

Theorem 1.3. *Let p be a preseminorm on X. Then*

(1) $p(0)=0$,

(2) $p(x)\geqq 0$ *for all* $x\in X$,

(3) $p(\lambda x)=p(x)$ *for all* $|\lambda|=1$ *and* $x\in X$,

(4) $p(\lambda x)\leqq p(\mu x)$ *for all* $|\lambda|\leqq|\mu|$ *and* $x\in X$,

(5) $p(nx)\leqq np(x)$ *for all integer* $n\geqq 0$ *and* $x\in X$.

(6) $|p(x)-p(y)|\leqq p(x-y)$ *for all* $x, y\in X$.

PROOF. By (a), $p(0)=\lim_{\lambda\to 0} p(\lambda 0)=0$. If $|\lambda|\leqq|\mu|$ and $\mu\neq 0$, then by (b)

$$p(\lambda x) = p\left(\frac{\lambda}{\mu}(\mu x)\right) \leqq p(\mu x).$$

Properties (2) and (3) follows immediately from (4). Finally, (5) and (6) can be derived from (c) on the usual way.

Corollary 1.4. *Let p be a nonzero preseminorm on a one-dimensional vector space X. Then p is a prenorm on X.*

PROOF. If $x\in X$, then by (2), (4) and (5), we have

$$0 \leqq p(\lambda x) \leqq ([|\lambda|]+1)p(x)$$

for all $\lambda\in\mathbf{K}$, where [ ] denotes the entire part function.

---

*) Meantime, we observed that a little more general functions than preseminorms were formerly used for similar purposes by WILANSKY [Modern Methods in Topological Vektor Spaces, *Mc Graw-Hill, New York, 1978*] under the name "paranorms".

Corollary 1.5. *Let p be a preseminorm on X, $x\in X$ such that $p(x)\neq 0$, and (λ_α) a net in* $\mathbf{K}$. *Then*

(i) $\lim_\alpha p(\lambda_\alpha x)=0$ *and* (ii) $\lim_\alpha |\lambda_\alpha|=0$

are equivalent.

PROOF. By (a), it is clear that (ii) implies (i). Suppose now that (i) holds, and let $\varepsilon>0$. Then, since by Corollary 1.4 $p(\varepsilon x)>0$, there exists α_0 such that $p(\lambda_\alpha x)<p(\varepsilon x)$ for all $\alpha\geqq\alpha_0$. Hence, by (4), it follows that $|\lambda_\alpha|<\varepsilon$ for all $\alpha\geqq\alpha_0$.

Corollary 1.6. *Let p be a preseminorm (prenorm) on X. Then the function d_p defined on $X\times X$ by*

$$d_p(x, y) = p(x-y)$$

is a translation-invariant semimetric (metric) on X.

§ 2. Preseminormed spaces

Definition 2.1. If P is a nonvoid family of preseminorms (seminorms) on X, then the ordered pair

$$X(P) = (X, P)$$

is a *preseminormed (seminormed) space.*

If $X(P)$ is a preseminormed space, then X is considered to be equipped with the weakest topology T_P for which all the balls

$$B_p(x, r) = \{y\in X\colon\ p(x-y) < r\},$$

where $p\in P$, $x\in X$ and $r>0$, are open.

When no confusion seems possible, we shall simply write X instead of $X(P)$ or $X(T_P)$.

Remark 2.2. Seminormed spaces have formerly been used by LARSEN [4] in a quite similar sense.

The next two propositions, whose proofs are left to the reader, can be generalized to "*semimetrized spaces*".

Proposition 2.3. *Let $X(P)$ be a preseminormed space, $x\in X$ and $V\subset X$. Then the following are equivalent:*

(i) *V is a neighborhood of x in $X(T_P)$,*
(ii) *there exists $\{p_k\}_{k=1}^n\subset P$ and $r>0$ such that*

$$\bigcap_{k=1}^{n} B_{p_k}(x, r) \subset V.$$

Remark 2.4. If P is *directed* in the sense that for each $p_1, p_2\in P$ there exists $p\in P$ such that $p_1\leqq p$ and $p_2\leqq p$, then instead of (ii), we may write that there exists $p\in P$ and $r>0$ such that $B\,(x, r)\subset V$.

Proposition 2.5. *Let $X(P)$ be a preseminormed space, (x_α) a net in X, and $x\in X$. Then the following are equivalent:*

(i) $x\in\lim_\alpha x_\alpha$ *in* $X(T_P)$,

(ii) $\lim_\alpha p(x_\alpha-x)=0$ *for all* $p\in P$.

Theorem 2.6. *Let X be a preseminormed space over $\mathbf{K}$. Then*

(i) *the mapping* $(x, y)\mapsto x+y$ *of* $X\times X$ *into* X *is continuous,*

(ii) *the mapping* $(\lambda, x)\mapsto\lambda x$ *of* $\mathbf{K}\times X$ *into* X *is continuous.*

Proof. We shall prove only (ii), the proof of (i) is similar, but simpler. To this end, suppose that $(\lambda, x)\in\mathbf{K}\times X$ and $(\lambda_\alpha, x_\alpha)$ is a net in $\mathbf{K}\times X$ such that

$$(\lambda, x)\in\lim_\alpha(\lambda_\alpha, x_\alpha).$$

Then, by a well-known property of the product topology, we have

$$\lambda\in\lim_\alpha\lambda_\alpha \quad\text{and}\quad x\in\lim_\alpha x_\alpha.$$

Hence, by Proposition 2.5, it follows that

$$\lim_\alpha|\lambda_\alpha-\lambda|=0 \quad\text{and}\quad \lim_\alpha p(x_\alpha-x)=0$$

for all $p\in P$, where P is the family of preseminorms given on X.

Using properties (c), (4) and (5) of preseminorms, we get

$$p(\lambda_\alpha x_\alpha-\lambda x)\leqq p\big((\lambda_\alpha-\lambda)(x_\alpha-x)\big)+p\big((\lambda_\alpha-\lambda)x\big)+p\big(\lambda(x_\alpha-x)\big)\leqq$$
$$\leqq([|\lambda_\alpha-\lambda|]+1)p(x_\alpha-x)+p\big((\lambda_\alpha-\lambda)x\big)+([|\lambda|]+1)p(x_\alpha-x)$$

for all $p\in P$. Hence, it is clear that

$$\lim_\alpha p(\lambda_\alpha x_\alpha-\lambda x)=0$$

for all $p\in P$. Thus, again by Proposition 2.5,

$$\lambda x\in\lim_\alpha\lambda_\alpha x_\alpha,$$

which implies (ii). (This proof is more natural than the one given for seminorms in [5, p. 27].)

Remark 2.7. The above theorem shows that if $X(P)$ is a preseminormed space, then $X(T_P)$ is a topological vector space. On the other hand, it is not very hard to show that each topological vector space can be obtained in this manner. (See for instance, [2, p. 52] or [6, p. 2].) Thus, preseminormed spaces are suitable alternatives for topological vector spaces.

Theorem 2.8. *Let $X(P)$ be a preseminormed space. Then T_P is the weakest translation-invariant topology on X for which each $p\in P$ is continuous.*

PROOF. By (i) in Theorem 2.6, it is clear that T_P is translation-invariant. Moreover, a similar argument as in the proof of Theorem 2.6 shows that each $p \in P$ is continuous for T_P.

Suppose now that T is a translation-invariant topology on X for which each $p \in P$ is continuous. Then

$$B_p(x, r) = x + B_p(0, r) = x + p^{-1}(]-r, r[)$$

belongs to T for all $p \in P$, $x \in X$ and $r > 0$, whence $T_P \subset T$ follows.

Definition 2.9. A family P of preseminorms on X is *separating* if for each $0 \neq x \in X$ there exists $p \in P$ such that $p(x) \neq 0$.

Theorem 2.10. *Let $X(P)$ be a preseminormed space. Then the following are equivalent:*

(i) *T_P is T_0,*
(ii) *P is separating,*
(iii) *T_P is Hausdorff.*

PROOF. An application of Proposition 2.3 and simple calculation with balls.

§ 3. Equivalence of preseminorms

Definition 3.1. If P and Q are nonvoid families of preseminorms on X, then we write

(a) $P \prec Q$ if $T_P \subset T_Q$,
(b) $P \sim Q$ if $T_P = T_Q$.

(The relations $\prec$ and $\sim$ are to be read "is *weaker* than" and "is *equivalent* to", respectively.)

Theorem 3.2. *Let P and Q be nonvoid families of preseminorms on X. Then the following are equivalent:*

(i) *$Q \prec P$,*
(ii) *each $q \in Q$ is continuous for T_P.*

PROOF. This follows at once from Theorem 2.8.

Corollary 3.3. *Let P be a nonvoid family of preseminorms on X, and denote by $\bar{P}$ the family of all preseminorms on X which are continuous for T_P. Then $\bar{P}$ is the largest family of preseminorms on X such that $\bar{P} \sim P$.*

Remark 3.4. A nonvoid family P of preseminorms may be called *total* if $\bar{P} = P$. Note that if P and Q are total families of preseminorms on X, then we have $P \sim Q$ if and only if $P = Q$.

Proposition 3.5. *Let $X(P)$ be a preseminormed space and q be a preseminorm on X. Then the following are equivalent:*

(i) *q is continuous for T_P,*
(ii) *$B_q(0, r)$ is a neighborhood of 0 in $X(T_P)$ for all $r>0$,*
(iii) *if (x_α) is a net in X such that $\lim_\alpha p(x_\alpha)=0$ for all $p\in P$, then $\lim_\alpha q(x_\alpha)=0$.*

PROOF. Simple application of Propositions 2.3 and 2.5.

Theorem 3.6. *Let X be a vector space over $\mathbf{K}$, and denote by P the family of all preseminorms on X. Then the following assertions hold:*

(1) *If $p_1, p_2\in P$ and $p=p_1+p_2$ or $p=\max\{p_1, p_2\}$, then $p\in P$ and $p\sim\{p_1, p_2\}$.*
(2) *If $p\in P$, $c>0$ and $q=cp$ or $q=\min\{p, c\}$, then $q\in P$ and $q\sim p$.*
(3) *If $(p_\alpha)_{\alpha\in\Gamma}$ is a uniformly convergent net in P and $p=\lim_\alpha p_\alpha$, then $p\in P$ and $p\prec\{p_\alpha\}_{\alpha\in\Gamma}$.*
(4) *If $(p_\alpha)_{\alpha\in\Gamma}$ is a family in P such that the series Σp_α converges uniformly, and $p=\sum_{\alpha\in\Gamma} p_\alpha$ then $p\in P$ and $p\sim\{p_\alpha\}_{\alpha\in\Gamma}$.*

PROOF. Routine, but lengthy computation.

Remark 3.7. Note that the second part of (2) is not true for seminorms, and each bounded seminorm is identically zero. These are the main disadvantages of seminorms.

Definition 3.8. A nonvoid family P of preseminorms is *saturated* if $p_1, p_2\in P$ implies that $\max\{p_1, p_2\}\in P$.

Remark 3.9. The importance of this notion lies in the fact that if P is a saturated family of preseminorms on X, then for each $\{p_k\}_{k=1}^n\subset P$ there exists $p\in P$ such that $B_p(x, r)=\bigcap_{k=1}^n B_{p_k}(x, r)$ for all $x\in X$ and $r>0$.

Corollary 3.10. *Let P be a nonvoid family of preseminorms on X, and denote by P^* the family of all* $\max\{p_k\}_{k=1}^n$, *where $\{p_k\}_{k=1}^n\subset P$. Then P^* is the smallest saturated family of preseminorms on X such that $P\subset P^*$, and moreover $P^*\sim P$.*

Remark 3.11. The family P^* will be called the *saturated hull* of P. Note that $P\subset P^*\subset\bar{P}$, and $\bar{P}$ is also saturated.

Corollary 3.12. *Let P be a nonvoid countable family of preseminorms on X. Then there exists a preseminorm p on X such that $p\sim P$.*

PROOF. If $P=\{p_n\}_{n=1}^\infty$, then we may define $p=\sum_{n=1}^\infty q_n$, where $q_n=\min\{p_n, 2^{-n}\}$.

Corollary 3.13. *Let P be a nonvoid finite family of seminorms on X. Then there exists a seminorm p on X such that $p\sim P$.*

§ 4. Metrizability and normality

Theorem 4.1. *Let $X(P)$ be a preseminormed space. Then the following are equivalent:*

(i) *there exists a preseminorm p on X such that $p \sim P$,*
(ii) *there exists a countable base for the neighborhood system of 0 in X.*

PROOF. If (i) holds, then by Proposition 2.3, it is clear that $\left\{B_p\left(0, \frac{1}{n}\right)\right\}_{n=1}^{\infty}$ is a local base at 0 in X.

To prove that (ii) also implies (i), suppose now that $\{V_n\}_{n=1}^{\infty}$ is a local base at 0 in X. Then, by Corollary 3.10 and Remark 2.4, for each n, there exists $p_n \in P^*$ and $r_n > 0$ such that $B_{p_n}(0, r_n) \subset V_n$. Hence, by Proposition 3.5 and Theorem 3.2, it is clear that $\{p_n\}_{n=1}^{\infty} \sim P$. Moreover, by Corollary 3.12, there exists a preseminorm p on X such that $p \sim \{p_k\}_{n=1}^{\infty}$.

Definition 4.2. A subset A of a preseminormed space $X(P)$ is *bounded* if for each $p \in P$ and $r > 0$, there exists a positive integer n such that $A \subset nB_p(0, r)$.

Remark 4.3. If A is a bounded subset of a preseminormed space $X(P)$, then A is also a bounded subset of the semimetrized space $X(D_P)$, where $D_P = \{d_p \colon p \in P\}$. (This means that the d_p-diameter of A is less than $+\infty$ for all $p \in P$.) However, the converse is not, in general, true. Thus, the above concept of boundedness must be handled with care.

Fortunately, if $X(P)$ is a seminormed space, then the two notions of boundedness coincide, since in this case we have $B_p(0, r) = rB_p(0, 1)$ for all $p \in P$ and $r > 0$.

The next proposition, whose proof is again left to the reader, clarifies the above definition further.

Proposition 4.4. *Let X be a preseminormed space over* **K**, *and $A \subset X$. Then the following are equivalent:*

(i) *A is a bounded subset of X,*
(ii) *for every neighborhood V of 0 in X, there exists a positive integer n such that $A \subset nV$,*
(iii) *if (x_α) is a net in A and (λ_α) is a null net in* **K**, *then $(\lambda_\alpha x_\alpha)$ is a null net in X.**)

Remark 4.5. The condition (iii) can be weakened, namely it is enough to suppose only that for any sequence (x_n) in A, $(n^{-1}x_n)$ is a null sequence in X.

Theorem 4.6. *Let $X(P)$ be a seminormed space. Then the following are equivalent:*
(i) *there exists a seminorm p on X such that $p \sim P$,*
(ii) *there exists a bounded neighborhood V of 0 in X.*

PROOF. It is clear that (i) implies (ii). Suppose now that (ii) holds. Then, by Corollary 3.10 and Remark 2.4, there exist $p \in P^*$ and $r > 0$ such that $B_p(0, r) \subset V$.

*) A further equivalent condition is that $\lim_{\lambda \to 0} \sup_{x \in A} p(\lambda x) = 0$ for all $p \in P$ if $A \neq \emptyset$.

Moreover, by Definition 4.2, for any $q \in P$ and $s > 0$, there exists a positive integer n such that $\frac{1}{n} B_p(0, r) \subset B_q(0, s)$, and thus by Proposition 3.5, q is continuous for T_p. Hence, by Theorem 3.2, it is clear that $p \sim P$.

Remark 4.7. If $X(P)$ is a preseminormed space, V is a bounded convex neighborhood of 0 in X, and p is the Minkowski functional of the balanced core of V, then one can show similarly that $p \sim P$.

However, from the point of view of applications, Theorems 4.1 and 4.6 and the above fact are much less important than Corollaries 3.12 and 3.13.

References

[1] J. L. KELLEY, General Topology, Van Nostrand Reinhold, New York, 1955.
[2] J. L. KELLEY—I. NAMIOKA, Linear Topological Spaces, *D. Van Nostrand,* Princeton, 1963.
[3] G. KÖTHE, Topological Vector Spaces I., *Springer, Berlin,* 1969.
[4] R. LARSEN, Functional Analysis, *Marcel Dekker, New York,* 1973.
[5] W. RUDIN, Functional Analysis, *McGraw-Hill, New York,* 1973.
[6] L. WAELBROECK, Topological Vector Spaces and Algebras, *Lecture Notes in Math.* **230**, *Springer Berlin,* 1971.

(Received March 20, 1980.)