Some general functional equations

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Introduction

The purpose of this paper is to find the general solution of the functional equations

$$(1)\qquad \begin{cases} \prod_{i=1}^{n} f_i(x_i) = g\left(\sum_{i=1}^{n} x_i\right) G(x_1-x_n, \dots, x_{n-1}-x_n); \quad x_1, \dots, x_n \in \mathbf{R}, \\ f_i\colon \mathbf{R} \to \mathbf{R} \quad (i=1, \dots, n), \quad g\colon \mathbf{R} \to \mathbf{R}, \quad G\colon \mathbf{R}^{n-1} \to \mathbf{R} \end{cases}$$

and

$$(2)\qquad \begin{cases} \prod_{i=1}^{n} f_i(x_i) = g\left(\sum_{i=1}^{n} x_i\right) G\left(\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right); \quad x_1, \dots, x_n \in \mathbf{R}_+, \\ f_i\colon \mathbf{R}_+ \to \mathbf{R} \quad (i=1, \dots, n), \quad g\colon \mathbf{R}_+ \to \mathbf{R}, \quad G\colon \mathbf{R}_+^{n-1} \to \mathbf{R}, \end{cases}$$

where $\mathbf{R}$ is the set of real numbers and $\mathbf{R}_+ = \{x \mid x>0, x\in\mathbf{R}\}$.

Equations (1) and (2) have applications to the characterization of normal and gamma distributions (see [5]).

1. Preliminary results

We need the following results:

Theorem 1.1. (see [7]). *Suppose that the functions $f, g, h, k\colon \mathbf{R}\to\mathbf{R}$ satisfy the functional equation*

$$(1.1)\qquad f(x)g(y) = h(ax+by)k(cx+dy) \quad (x, y\in\mathbf{R}),$$

where $a, b, c, d \in \mathbf{R}_0 = \mathbf{R}\setminus\{0\}$ are arbitrary rational constants with $\Delta = ad-bc \neq 0$ and there exists a subset $D\subset\mathbf{R}^2$ of positive Lebesgue-measure, such that $f(x)g(y)\neq 0$ for all $(x, y)\in D$, then

$$(1.2)\qquad f(x) = \alpha_1 \exp[A_1(x)+n(x)] \qquad (x\in\mathbf{R}),$$

$$(1.3)\qquad g(x) = \alpha_2 \exp\left[A_2(x) - \frac{bd}{ac}\, n(x)\right] \quad (x\in\mathbf{R}),$$

$$(1.4)\qquad h(x) = \beta_1 f\left(\frac{d}{\Delta}\, x\right) g\left(-\frac{c}{\Delta}\, x\right) \qquad (x\in\mathbf{R}),$$

$$(1.5)\qquad k(x) = \beta_2 f\left(-\frac{b}{\Delta}\, x\right) g\left(\frac{a}{\Delta}\, x\right) \qquad (x\in\mathbf{R}),$$

where

$$(1.6)\qquad \alpha_1\beta_1\alpha_2\beta_2 = 1$$

holds and the functions A_i: $\mathbf{R}\to\mathbf{R}$ $(i=1, 2)$ *and* n: $\mathbf{R}\to\mathbf{R}$ *satisfy the functional equations*

$$A(x+y) = A(x)+A(y) \quad (x, y\in\mathbf{R}) \tag{1.7}$$

and

$$n(x+y)+n(x-y) = 2n(x)+2n(y) \quad (x, y\in\mathbf{R}) \tag{1.8}$$

respectively.

Corollary 1.1 (see [1], [7]). *The general measurable solutions of* (1.1) *are*

$$f(x) = \alpha_1 \exp[a_1 x+b_1 x^2] \qquad (x\in\mathbf{R}), \tag{1.9}$$

$$g(x) = \alpha_2 \exp\left[a_2 x-\frac{bd}{ac} b_1 x^2\right] \qquad (x\in\mathbf{R}), \tag{1.10}$$

$$h(x) = \beta_1\alpha_1\alpha_2 \exp\left[\frac{a_1 d-a_2 c}{\Delta} x+\frac{d}{a} b_1 x^2\right] \quad (x\in\mathbf{R}), \tag{1.11}$$

$$k(x) = \beta_2\alpha_1\alpha_2 \exp\left[\frac{a_2 a-a_1 b}{\Delta} x-\frac{b}{c} b_1 x^2\right] \quad (x\in\mathbf{R}), \tag{1.12}$$

where $a_i, b_1, \alpha_i, \beta_i\in\mathbf{R}$ $(i=1, 2)$ are arbitrary constants satisfying (1.6). (Again the existence of $D\subset\mathbf{R}^2$ with positive measure such that $f(x)g(y)\neq 0$ on D is supposed.)

Theorem 1.2 (see [2] and [6]). *Suppose that the functions* f, g, p, q: $\mathbf{R}_+\to\mathbf{R}$ *satisfy the functional equation*

$$f(x)g(y) = p(x+y)q\left(\frac{x}{y}\right) \quad (x, y\in\mathbf{R}_+) \tag{1.13}$$

and there exist sets $T_1, T_2\subset\mathbf{R}_+$ *of positive Lebesgue-measure such that* $f(x)g(y)\neq 0$ *for all* $x\in T_1$, $y\in T_2$. *Then*

$$f(x) = A \exp[a(x)+m_1(x)] \qquad (x\in\mathbf{R}_+), \tag{1.14}$$

$$g(x) = B \exp[a(x)+m_2(x)] \qquad (x\in\mathbf{R}_+), \tag{1.15}$$

$$p(x) = C \exp[a(x)+m_1(x)+m_2(x)] \qquad (x\in\mathbf{R}_+), \tag{1.16}$$

$$q(x) = D \exp\left[m_1\left(\frac{x}{x+1}\right)-m_2(x+1)\right] \quad (x\in\mathbf{R}_+), \tag{1.17}$$

where

$$AB = CD \tag{1.18}$$

holds and the functions a: $\mathbf{R}\to\mathbf{R}$, m_i: $\mathbf{R}_+\to\mathbf{R}$ $(i=1, 2)$ *satisfy the functional equations* (1.7) *and*

$$m(xy) = m(x)+m(y) \quad (x, y\in\mathbf{R}_+) \tag{1.19}$$

respectively.

We remark that a simple proof of Theorem 1.2 can be based on a theorem of Z. Daróczy—K. Lajkó—L. Székelyhidi (see [3]).

Corollary 1.2 (see [2]). *The general measurable solutions of* (1.13) *are the functions*

$$f(x) = A \exp[ax + b \ln x] \qquad (x \in \mathbf{R}_+), \tag{1.20}$$

$$g(x) = B \exp[ax + c \ln x] \qquad (x \in \mathbf{R}_+), \tag{1.21}$$

$$p(x) = C \exp[ax + (c+b) \ln x] \qquad (x \in \mathbf{R}_+), \tag{1.22}$$

$$q(x) = D \exp\left[b \ln \frac{x}{x+1} - c \ln (x+1)\right] \quad (x \in \mathbf{R}_+), \tag{1.23}$$

where $a, b, c \in \mathbf{R}$ and $A, B, C, D \in \mathbf{R}_0$ are arbitrary constants with (1.18). (Again the existence of $T_1, T_2 \subset \mathbf{R}_+$ with positive measure such that $f(x)g(y) \neq 0$ for all $x \in T_1$, $y \in T_2$ is supposed.)

2. On the functional equation (1)

Lemma 2.1 (see [5]). *Suppose that the functions f_i, g, G satisfy the functional equation* (1) *and there exist sets $A_i \subset \mathbf{R}$ $(i=1, \ldots, n)$ of positive Lebesgue-measure such that $\prod_{i=1}^{n} f_i(x_i) \neq 0$ $(x_i \in A_i;\ i=1, \ldots, n)$. Then $\prod_{i=1}^{n} f_i(x_i) g(u_1) G(u_2, \ldots, u_n) \neq 0$ for all $x_i, u_i \in \mathbf{R}$ $(i=1, \ldots, n)$.*

Using this lemma and the Theorem 1.1, we get the general solution of (1):

Theorem 2.1. *Suppose that the functions f_i, g and G satisfy the conditions of Lemma 2.1. Then we have*

$$f_i(x_i) = \alpha_i \exp[a_i(x_i) + n(x_i)] \qquad (x_i \in \mathbf{R},\ i = 1, \ldots, n), \tag{2.1}$$

$$g(x) = \beta \prod_{i=1}^{n} \alpha_i \exp\left[\frac{1}{n}\left(\sum_{i=1}^{n} a_i(x) + n(x)\right)\right] \quad (x \in \mathbf{R}), \tag{2.2}$$

$$G(x_1, \ldots, x_{n-1}) = \alpha_n \frac{\prod_{i=1}^{n-1} f_i(x_i)}{g\left(\sum_{i=1}^{n-1} x_i\right)} \qquad (x_i \in \mathbf{R},\ i = 1, \ldots, n-1), \tag{2.3}$$

where the functions $a_i \colon \mathbf{R} \to \mathbf{R}$ $(i=1, \ldots, n)$ and $n \colon \mathbf{R} \to \mathbf{R}$ fulfil the functional equations (1.7) and (1.8) respectively, $\alpha_i, \beta \in \mathbf{R}_0$ $(i=1, \ldots, n)$ are arbitrary constants.

Proof. Using Lemma 2.1, it follows that

$$\prod_{i=1}^{n} f_i(x_i)\, g(u_1)\, G(u_2, \ldots, u_n) \neq 0$$

for all $x_i, u_i \in \mathbf{R}$ $(i=1, \ldots, n)$.

Now let $j<n$ be fixed and $x_i=x_n$ if $i\neq j$. From (1) we get

$$f_j(x_j)\prod_{\substack{i=1\\ i\neq j}}^{n} f_i(x_n) = g\big(x_j+(n-1)x_n\big)\,G(0,\,\dots,\,0,\,x_j-x_n,\,0,\,\dots,\,0)$$

for all $x_j, x_n\in\mathbf{R}$. This implies the functional equation

$$f_j(x_j)\bar f_n(x_n) = g\big(x_j+(n-1)x_n\big)\,\bar G(x_j-x_n)\quad (x_j,\,x_n\in\mathbf{R}), \tag{2.4}$$

where

$$\bar f_n(x_n) = \prod_{\substack{i=1\\ i\neq j}}^{n} f_i(x_n)\quad (x_n\in\mathbf{R}), \tag{2.5}$$

$$\bar G(x) = G(0,\,\dots,\,0,\,\underset{j}{x},\,0,\,\dots,\,0)\quad (x\in\mathbf{R}). \tag{2.6}$$

The functional equation (2.4) is a special case of (1.1) with $a=1, b=n-1, c=1, d=-1$. Further $\bar f_n(x_n)\neq 0$ for all $x_n\in\mathbf{R}$. Thus the functions f_j satisfy the conditions of Theorem 1.1. Therefore $f_j, \bar f_n, g$ and $\bar G$ are of the forms

$$f_j(x_j) = \alpha_j \exp\,[a_j(x_j)+n_j(x_j)]\qquad (x_j\in\mathbf{R}), \tag{2.7}$$

$$\bar f_n(x_n) = \alpha_n^j \exp\,[a_n^j(x_n)+(n-1)\,n_j(x_n)]\quad (x_n\in\mathbf{R}), \tag{2.8}$$

$$g(x) = \beta_j^1 f_j\left(\frac{x}{n}\right)\bar f_n\left(\frac{x}{n}\right)\qquad (x\in\mathbf{R}), \tag{2.9}$$

$$\bar G(x) = \beta_j^2 f_j\left(\frac{n-1}{n}\,x\right)\bar f_n\left(-\frac{x}{n}\right)\qquad (x\in\mathbf{R}) \tag{2.10}$$

for any fixed $j<n$, where $\alpha_j, \alpha_n^j, \beta_j^1, \beta_j^2\in\mathbf{R}_0$ are arbitrary constants with $\alpha_j\alpha_n^j\beta_j^1\beta_j^2=1$ and the functions a_j, a_n^j: $\mathbf{R}\to\mathbf{R}$ and n_j: $\mathbf{R}\to\mathbf{R}$ satisfy the functional equations (1.7) and (1.8) respectively.

From (2.5) using (2.7) and (2.8), we get

$$f_n(x_n) = \frac{\alpha_n^j\alpha_j}{\prod\limits_{i=1}^{n-1}\alpha_i}\exp\left[a_j(x_n)+a_n^j(x_n)-\sum_{i=1}^{n-1} a_i(x_n)+nn_j(x_n)-\sum_{i=1}^{n-1} n_i(x_n)\right]$$

for any fixed $j<n$ and for all $x_n\in\mathbf{R}$. Thus $n_j(x)=n(x)$ $(x\in\mathbf{R})$ $(j=1,\,\dots,\,n-1)$. By $a_j(x)+a_n^j(x)=a_k(x)+a_n^k(x)$ $(x\in\mathbf{R})$ and $\alpha_n^j\alpha_j=\alpha_n^k\alpha_k$ the expression $a_j(x)+$ $+a_n^j(x)-\sum\limits_{i=1}^{n-1} a_j(x)$ and $\dfrac{\alpha_n^j\alpha_j}{\sum\limits_{i=1}^{n-1}\alpha_i}$ depend only on n. Denoting these by $a_n(x)$ and α_n respectively, we get that

$$f_n(x_n) = \alpha_n \exp\,[a_n(x_n)+n(x_n)]\quad (x_n\in\mathbf{R}). \tag{2.11}$$

(2.7) and (2.11) gives (2.1).

From (2.9) using (2.5) and (2.1), we get the representation (2.2) for g.

From (1) by $x_n=0$ and using (2.1) and (2.2), we get (2.3).

It is easy to see that functions (2.1)—(2.3) satisfy the functional equation (1). From Theorem 2.1 we can easily obtain

Corollary 2.1. *If the measurable functions f_i, g and G satisfy the conditions of Theorem 2.1, then*

$$f_i(x_i) = \alpha_i \exp[a_i x_i + b x_i^2] \quad (x_i \in \mathbf{R};\ i = 1, \dots, n), \tag{2.12}$$

$$g(x) = \beta \prod_{i=1}^{n} \alpha_i \exp\left[\frac{1}{n}\left(\sum_{i=1}^{n} a_i x + b x^2\right)\right] \quad (x \in \mathbf{R}), \tag{2.13}$$

$$G(x_1, \dots, x_{n-1}) = \alpha_n \frac{\prod_{i=1}^{n-1} f_i(x_i)}{g\left(\sum_{i=1}^{n-1} x_i\right)} \quad (x_i \in \mathbf{R};\ i = 1, \dots, n-1), \tag{2.14}$$

where $a_i, b \in \mathbf{R}$, $\alpha_i, \beta \in \mathbf{R}_0$ $(i=1, \dots, n)$ are arbitrary constants.

Proof. The functions $\bar{f}_n$ and $\bar{G}$ (defined by (2.5) and (2.6) respectively) are measurable. Thus the functions f_j, $\bar{f}_n$, g and $\bar{G}$ satisfy the conditions of Corollary 1.1 if $a=1$, $b=n-1$, $c=1$, $d=-1$, therefore the functions (α_j, n_j in 2.7), (2.8) have the form

$$a_j(x_j) = a_j x_j, \quad n_j(x_j) = b_j x_j^2, \quad a_n^j(x_n) = a_n^j x_n \quad (x_j, x_n \in \mathbf{R})$$

for any fixed $j<n$, where $a_j, b_j, a_n^j \in \mathbf{R}$ are arbitrary constants. Then following the proof of Theorem 2.1, we get that

$$a_i(x_i) = a_i x_i \quad (x_i \in \mathbf{R}) \quad (i = 1, \dots, n); \quad n_j(x) = n(x) = b x^2 \quad (x \in \mathbf{R};\ j = 1, \dots, n-1),$$

where $a_i, b \in \mathbf{R}$ are arbitrary constants and from (2.1)—(2.3) we can deduce the expressions (2.12)—(2.14) for f_i, g, G respectively.

3. On the functional equation (2)

Lemma 3.1. *Suppose that the functions f_i, g and G satisfy the functional equation (2) and there exist sets $T_i \subset \mathbf{R}_+$ $(i=1, \dots, n)$ of positive Lebesgue-measure such that $\prod_{i=1}^{n} f_i(x_i) \neq 0$ for all $x_i \in T_i$ $(i=1, \dots, n)$. Then $\prod_{i=1}^{n} f_i(x_i) g(u_1) G(u_2, \dots, u_n) \neq 0$ for all $x_i, u_i \in \mathbf{R}_+$ $(i=1, \dots, n)$.*

Proof. (2) implies that $g\left(\sum_{i=1}^{n} x_i\right) G\left(\frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n}\right) \neq 0$ for all $x_i \in T_i$ $(i=1, \dots, n)$, that is

$$g(u_1) G(u_2, \dots, u_n) \neq 0 \quad \text{for} \quad u_1 \in \sum_{i=1}^{n} T_i,\ (u_2, \dots, u_n \in \bigcup_{x_n \in T_n} \left[\frac{T_1}{x_n} \times \dots \times \frac{T_{n-1}}{x_n}\right]. \tag{3.1}$$

Substituting

$$u_1 = \sum_{i=1}^{n} x_i, \quad u_{i+1} = \frac{x_i}{x_n} \quad (i = 1, \dots, n-1),$$

we get from (2) the equation

(3.2)

$$\prod_{i=1}^{n-1} f_i\left(\frac{u_1 u_{i+1}}{1+\sum_{j=1}^{n-1} u_{j+1}}\right) f_n\left(\frac{u_1}{1+\sum_{j=1}^{n-1} u_{j+1}}\right) = g(u_1)G(u_2, \dots, u_n) \quad (u_1, \dots, u_n \in \mathbf{R}_+),$$

which together with (3.1) implies that

$$\prod_{i=1}^{n} f_i(x_i) \neq 0 \quad \text{if} \quad \begin{cases} x_i \in A_i = \bigcup_{x_n \in T_n} \left(\dfrac{\left(\sum_{j=1}^{n} T_j\right) T_i}{x_n + \sum_{\substack{j=1\\ j\neq i}}^{n} T_j + T_i}\right) & (i = 1, \dots, n-1) \\ x_n \in A_n = \bigcup_{n_n \in T_n} \left(\dfrac{\sum_{j=1}^{n} T_j}{1+\sum_{j=1}^{n-1} \dfrac{T_j}{x_n}}\right). \end{cases}$$

By a theorem of Steinhaus (see [4]) the sets $A_i\,(i=1, \dots, n)$ contain intervals $I_i^{(0)}=[a_i^{(0)}, b_i^{(0)}]\subset A_i\subset \mathbf{R}_+$ $(i=1, \dots, n)$ such that

$$\prod_{i=1}^{n} f_i(x_i) \neq 0, \quad \text{if} \quad x_i \in I_i^{(0)} \quad (i = 1, \dots, n). \tag{3.3}$$

Repeating this argument (using $I_i^{(0)}$ instead T_i), we have

(3.4)

$$\prod_{i=1}^{n} f_i(x_i) \neq 0, \quad \text{if} \quad \begin{cases} x_i \in B_i = \bigcup_{x_n \in I_n^{(0)}} \left(\sum_{j=1}^{n} I_j^{(0)} \dfrac{I_i^{(0)}}{x_n + \sum_{\substack{j=1\\ j\neq i}}^{n} I_j^{(0)} + I_i^{(0)}}\right) & (i = 1, \dots, n-1) \\ x_n \in B_n = \bigcup_{x_n \in I_n^{(0)}} \left(\dfrac{\sum_{j=1}^{n} I_j^{(0)}}{1+\sum_{j=1}^{n-1} \dfrac{I_j^{(0)}}{x_n}}\right). \end{cases}$$

It is easy to see that the sets B_i in (3.4) are the intervals

$$I_j^{(1)} = \left[\frac{\sum_{j=1}^{n} a_j^{(0)}}{a_i^{(0)} + \sum_{\substack{j=1\\ j\neq i}}^{n} b_j^{(0)}}\, a_i^{(0)},\ \frac{\sum_{j=1}^{n} b_j^{(0)}}{b_i^{(0)} + \sum_{\substack{j=1\\ j\neq i}}^{n} a_j^{(0)}}\, b_i^{(0)}\right] \quad (i = 1, \dots, n).$$

Thus

$$\prod_{i=1}^{n} f_i(x_i) \neq 0, \quad \text{if} \quad x_i \in I_i^{(1)} = [a_i^{(1)}, b_i^{(1)}] \quad (i = 1, \dots, n). \tag{3.5}$$

By induction, we get a sequence $I_i^{(k)}=[a_i^{(k)}, b_i^{(k)}]$ $(i=1, \dots, n;\ k=0, 1, \dots)$ with property

$$(3.6)\qquad I_i^{(k)} = [a_i^{(k)}, b_i^{(k)}] \supseteq \left[\left(\frac{\sum_{j=1}^{n} a_j^{(0)}}{a_i^{(0)} + \sum_{\substack{j=1\\ j\neq i}}^{n} b_j^{(0)}}\right)^{k-1} a_i^{(0)},\ \left(\frac{\sum_{j=1}^{n} b_j^{(0)}}{b_i^{(0)} + \sum_{\substack{j=1\\ j\neq i}}^{n} a_j^{(0)}}\right)^{k-1} b_i^{(0)}\right]$$

$$(i = 1, \dots, n)$$

and

$$\prod_{i=1}^{n} f_i(x_i) \neq 0, \quad \text{if} \quad x_i \in I_i^{(k)} \quad (i = 1, \dots, n;\ k = 0, 1, 2, \dots)$$

From (3.6) one can see that $a_i^{(k)} \to 0$, $b_i^{(k)} \to \infty$ as $k \to \infty$, therefore

$$\prod_{i=1}^{n} f_i(x_i) \neq 0, \quad \text{if} \quad x_i \in \mathbf{R}_+ \quad (i = 1, \dots, n).$$

This and (2) gives that

$$g(u_1)G(u_2, \dots, u_n) \neq 0 \quad \text{if} \quad u_i \in \mathbf{R}_+ \quad (i = 1, \dots, n),$$

which completes the proof of Lemma 3.1.

Now we can easily prove

Theorem 3.1. *Suppose that the functions f_i, g and G satisfy the conditions of Lemma 3.1, then*

$$(3.7)\qquad f_i(x) = A_i \exp[a(x) + m_i(x)] \quad (x \in \mathbf{R}_+;\ i = 1, \dots, n),$$

$$(3.8)\qquad g(x) = B \exp\left[a(x) + \sum_{i=1}^{n} m_i(x)\right] \quad (x \in \mathbf{R})_+,$$

$$(3.9)\qquad G(x_1, \dots, x_{n-1}) = C \exp\left[\sum_{i=1}^{n-1} m_i\left(\frac{x_i}{1 + \sum_{i=1}^{n-1} x_i}\right) - m_n\left(1 + \sum_{i=1}^{n-1} x_i\right)\right]$$

$$(x_i \in \mathbf{R}_+),$$

where the functions a: $\mathbf{R} \to \mathbf{R}$ and m_i: $\mathbf{R}_+ \to \mathbf{R}$ $(i=1, \dots, n)$ fulfil the functional equations (1.7) *and* (1.19) *respectively, A_i, B, $C \in \mathbf{R}_0$ are arbitrary constants with $\prod_{i=1}^{n} A_i = BC$.*

Proof. Using Lemma 3.1, it follows that $\prod_{i=1}^{n} f_i(x_i) g(u_1) G(u_2, \dots, u_n) \neq 0$ for all x_i, $u_i \in \mathbf{R}_+$ $(i=1, \dots, n)$.

Now, let $j<n$ be fixed and $x_i = x_n + x_j$ if $i \neq j, n$. Then we get from (2) that

$$f_n(x_n) f_j(x_j) \prod_{\substack{i=1\\ i\neq j}}^{n-1} f_i(x_n + x_j) = g[(n-1)(x_n + x_j)]\, G\left(1 + \frac{x_j}{x_n}, \dots, \frac{x_j}{x_n}, \dots, 1 + \frac{x_j}{x_n}\right)$$

for all $x_n, x_j \in \mathbf{R}_+$. This implies the functional equation

$$f_n(x_n) f_j(x_j) = p(x_n + x_j) q\left(\frac{x_j}{x_n}\right) \quad (x_n, x_j \in \mathbf{R}_+), \tag{3.10}$$

where

$$p(u) = \frac{g[(n-1)u]}{\prod_{\substack{i=1 \\ i \neq j}}^{n-1} f_i(u)} \quad (u \in \mathbf{R}_+) \tag{3.11}$$

and

$$q(v) = G(1+v, \ldots, \underset{j}{v}, \ldots, 1+v) \quad (v \in \mathbf{R}_+). \tag{3.12}$$

Thus the functions f_n, f_j, p, q satisfy the condition of Theorem 1.2. Therefore f_n and f_j are of the forms

$$f_i(x_i) = A_i \exp[a(x) + m_i(x)] \quad (x \in \mathbf{R}_+;\ i = j, n) \tag{3.13}$$

for any fixed $j < n$, where the functions $a\colon \mathbf{R} \to \mathbf{R}$ and $m_i\colon \mathbf{R}_+ \to \mathbf{R}$ $(i = n, j)$ satisfy the functional equations (1.7) and (1.19) respectively, $A_i \in \mathbf{R}_0$ $(i = n, j)$ are arbitrary constants. Hence (3.7) holds.

From (2) by $x_i = x$ $(i = 1, \ldots, n)$ we get

$$\prod_{i=1}^{n} f_i(x) = g(nx) G(1, \ldots, 1) \quad (x \in \mathbf{R}_+),$$

which together with (3.7) implies (3.8), where $B \in \mathbf{R}_0$ is an arbitrary constant.

Finally from (2) by $x_n = 1$ and using (3.7) and (3.8), we get (3.9) for G, where C is an arbitrary constant.

It is easy to see that the functions (3.7)—(3.9) satisfy the functional equation (2) indeed if $\prod_{i=1}^{n} A_i = BC$.

Now, we can easily obtain

Corollary 3.1. *If the measurable functions f_i, g and G satisfy the conditions of Theorem 3.1, then*

$$f_i(x) = A_i \exp[ax + b_i \ln x] \quad (x \in \mathbf{R}_+,\ i = 1, \ldots, n), \tag{3.14}$$

$$g(x) = B \exp\left[ax + \left(\sum_{i=1}^{n} b_i\right) \ln x\right] \quad (x \in \mathbf{R}_+), \tag{3.15}$$

$$G(x_1, \ldots, x_{n-1}) = C \exp\left[\sum_{i=1}^{n-1} b_i \ln\left(\frac{x_i}{1 + \sum_{i=1}^{n-1} x_i}\right) - b_n \ln\left(1 + \sum_{i=1}^{n-1} x_i\right)\right] \quad (x_i \in \mathbf{R}_+), \tag{3.16}$$

where $a, b_i \in \mathbf{R}$; $A_i, B, C \in \mathbf{R}_0$ $(i = 1, \ldots, n)$ are arbitrary constants with $\prod_{i=1}^{n} A_i = BC$.

The proof of this corollary is similar to that of Corollary 2.1. (Here we need Corollary 1.2 instead of Corollary 1.1.)

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(Received September 15, 1980.)