Relations preserving semi-continuities of relations

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In this note, we give some sufficient conditions under which homeomorphisms or more general realtions of product spaces preserve lower or upper semi-continuity of certain relations.

Definition 1. Let X, Y, Z, W be topological spaces and $\mathcal{S}$ be a family of relations in $X\times Y$. A relation F from $X\times Y$ into $Z\times W$ is said to preserve lower (upper) semi-continuity of relations belonging to $\mathcal{S}$ if $F(S)$ is a lower (upper) semi-continuous relation in $Z\times W$ for each lower (upper) semi-continuous relation $S\in\mathcal{S}$.

Remark 1. Concerning lower and upper semi-continuity of relations, which are usually called multifunctions in this context, the reader is referred to [7]. Note that a relation S in $X\times Y$ have to be called lower (upper) semi-continuous if it is a lower (upper) semi-continuous relation from its domain D_s, which is considered as a subspace of X, into Y in the sense of [7]. (See also [2].)

Theorem 1. *Let X, Y, Z, W be topological spaces, Φ an open (closed) relation from X into Z and Ψ a lower (upper) semi-continuous relation from Y into W. Then the relation F from $X\times Y$ into $Z\times W$ defined by $F(x, y)=\Phi(x)\times\Psi(y)$ preserves lower (upper) semi-continuity of all relations from X into Y.*

PROOF. Let S be a lower (upper) semi-continuous relation from X into Y and $T=F(S)$. Then a straightforward computation shows that $T=\Psi\circ S\circ\Phi^{-1}$. Thus, we have $T^{-1}(V)=\Phi\big(S^{-1}(\Psi^{-1}(V))\big)$ for any $V\subset W$. Hence, it is clear that $T^{-1}(V)$ is open (closed) in Z if V is open (closed) in W, which is a little more than the lower (upper) semi-continuity of T, since the domain of T may be a proper subset of Z.

Corollary 1.1. *Let X, Y, Z, W be topological spaces, φ a continuous function from Z onto X and Ψ a lower (upper) semi-continuous relation from Y into W. Then the relation F from $X\times Y$ into $Z\times W$ defined by $F(x, y)=\varphi^{-1}(x)\times\Psi(y)$ preserves lower (upper) semi-continuity of all relations in $X\times Y$.*

PROOF. Let S be a lower (upper) semi-continuous relation in $X\times Y$ with domain D. Since φ is a continuous function from Z onto X, the restriction $\Phi=\varphi^{-1}|D$ is an open and closed relation from D onto $\varphi^{-1}(D)$. Thus, by Theorem 1, $F(S)$ is a lower (upper) semi-continuous relation in $\varphi^{-1}(D)\times W$, and thus also in $Z\times W$.

Corollary 1.2. *Let X, Y, Z, W be topological spaces, φ a homeomorphism of X onto Z and ψ a homeomorphism of Y onto W. Then the function f defined on $X\times Y$*

by $f(x, y)=(\varphi(x), \psi(y))$ *is a homeomorphism of* $X\times Y$ *onto* $Z\times W$ *which preserves both lower and upper semi-continuity of all relations in* $X\times Y$.

Example 1. Let f be the function defined on $[0, 1]^2$ by $f(x, y)=(y, x)$. Then f is a homeomorphism of $[0, 1]^2$ onto itself which preserve neither lower nor upper semi-continuity of all relations from $[0, 1]$ into itself.

To see this, define the relation S from $[0, 1]$ into itself such that $S(x)=[1/2, 1]$ if $0\leqq x\leqq 1/2$ and $S(x)=[0, 1]$ if $1/2<x\leqq 1$. Then S is lower semi-continuous, but $f(S)=S^{-1}$ is not lower semi-continuous, and S^{-1} is upper semi-continuous, but $f(S^{-1})=S$ is not upper semi-continuous.

Namely, $S^{-1}(A)$ is open and $(S^{-1})^{-1}(A)=S(A)$ is closed in $[0, 1]$ for any $A\subset[0, 1]$, but $S^{-1}(0)=]1/2, 1]$ is not closed and $(S^{-1})^{-1}([0, 1/2[)=S([0, 1/2[)=$ $=[1/2, 1]$ is not open in $[0, 1]$.

Theorem 2. *Let* X, Y, Z, W *be topological spaces and* F *be a lower semi-continuous open relation from* $X\times Y$ *into* $Z\times W$ *such that* $F(\{x\}\times Y)\subset\{z\}\times W$ *whenever* $(z, w)\in F(x, y)$. *Then* F *preserves lower semi-continuity of all relations from* X *into* Y.

Proof. Let S be a lower semi-continuous relation from X into Y and $T=F(S)$. To prove that T is also lower semi-continuous, suppose that $(z_0, w_0)\in T$ and Ω is a neighborhood of w_0 in W. Pick $(x_0, y_0)\in S$ such that $(z_0, w_0)\in F(x_0, y_0)$. Since F is lower semi-continuous, there are neighborhoods U and V of x_0 and y_0 in X and Y, respectively, such that $U\times V\subset F^{-1}(Z\times\Omega)$. Furthermore, since S is lower semi-continuous, there is a neighborhood $U_1\subset U$ of x_0 in X such that $U_1\subset S^{-1}(V)$. Finally, since F is an open relation, there are neighborhoods Σ and Ω_1 of z_0 and w_0 in Z and W, respectively, such that $\Sigma\times\Omega_1\subset F(U_1\times V)$. Thus, it remains only to prove that $\Sigma\subset T^{-1}(\Omega)$, which shows that T is lower semi-continuous at z_0.

If $z\in\Sigma$, then $(z, w_0)\in\Sigma\times\Omega_1\subset F(U_1\times V)$. Thus, there exist $x\in U_1$ and $y\in V$ such that $(z, w_0)\in F(x, y)$. Since $x\in U_1\subset S^{-1}(V)$, there exists $y_1\in V$ such that $x\in S^{-1}(y_1)$, i.e., $(x, y_1)\in S$. Moreover, since $(x, y_1)\in U\times V\subset F^{-1}(Z\times\Omega)$, there exist $z_1\in Z$ and $w_1\in\Omega$ such that $(x, y_1)\in F^{-1}(z_1, w_1)$, i.e., $(z_1, w_1)\in F(x, y_1)$. Hence, since $F(\{x\}\times Y)\subset\{z\}\times W$, it follows that $z_1=z$. Consequently, we have $(z, w_1)=(z_1, w_1)\in F(x, y_1)\subset F(S)=T$, and hence $z\in T^{-1}(w_1)\subset T^{-1}(\Omega)$.

Remark 2. For a relation F from $X\times Y$ into $Z\times W$, the following properties are pairwise equivalent:

(i) $F(\{x\}\times Y)\subset\{z\}\times W$ whenever $(z, w)\in F(x, y)$;

(ii) $\varrho\circ F\circ p^{-1}$, where p and ϱ denote the projections of $X\times Y$ onto X and $Z\times W$ onto Z, respectively, is a function;

(iii) there exist a function φ from X into Z and a relation Ψ_x from Y into W for each $x\in X$ such that $F(x, y)=\varphi(x)\times\Psi_x(y)$ for all $(x, y)\in X\times Y$.

However, we could not use this fact to simplify the above proof.

Corollary 2.1. *Let* X, Y, Z, W *be topological spaces and* f *be a continuous open mapping of* $Z\times W$ *onto* $X\times Y$ *such that* $f^{-1}(\{x\}\times Y)=\{z\}\times W$ *whenever* $f(z, w)=$ $=(x, y)$. *Then the relation* f^{-1} *from* $X\times Y$ *onto* $Z\times W$ *preserves lower semi-continuity of all relations in* $X\times Y$.

Proof. Let S be a lower semi-continuous relation in $X\times Y$ with domain D. Then, by the assumptions on f, it is clear that $f^{-1}(D\times Y)=E\times W$ for some $E\subset Z$ and the restriction $F=f^{-1}|D\times Y$ is a lower semi-continuous open relation from

$D\times Y$ onto $E\times W$ such that $F(\{x\}\times Y)=\{z\}\times W$ whenever $(z, w)\in F(x, y)$. Thus, by Theorem 2, $f^{-1}(S)=F(S)$ is a lower semi-continuous relation in $E\times W$, and thus also in $Z\times W$.

Corollary 2.2. *Let X, Y, Z, W be topological spaces and f be a homeomorphism of $X\times Y$ onto $Z\times W$ such that $f(\{x\}\times Y)=\{z\}\times W$ whenever $f(x, y)=(z, w)$. Then f preserves lower semi-continuity of all relations in $X\times Y$.*

Example 2. Let f be the function defined on $[0, 1]^2$ by $f(x, y)=(x, \psi_x(y))$, where $\psi_x(y)=(x+1/2)y$ if $0\leqq y\leqq 1/2$ and $\psi_x(y)=(3/2-x)y+x-1/2$ if $1/2<y\leqq 1$. Then f is a homeomorphism of $[0, 1]^2$ onto itself such that $f(\{x\}\times[0, 1])=\{x\}\times[0, 1]$ for all $0\leqq x\leqq 1$, but f does not preserve upper semi-continuity of all relations from $[0, 1]$ into itself.

To see this, define the relation S from $[0, 1]$ into itself by $S(x)=[0, 1]\setminus\{1/2\}$. Then S is upper semi-continuous, but $T=f(S)$ is not upper semi-continuous. Namely, we have $T(x)=[0, 1]\setminus\{(1/2)x+1/4\}$ for all $0\leqq x\leqq 1$, whence it is clear that for any $x_0\in[0, 1]$, $T(x_0)$ is a neighborhood of itself in $[0, 1]$, but $T(x)\not\subset T(x_0)$ for each $x\in[0, 1]\setminus\{x_0\}$, and thus T can not be upper semi-continuous at x_0.

Theorem 3. *Let X, Y, Z, W be topological spaces such that Y is regular and W is compact, and let F be a closed relation from $X\times Y$ into $Z\times W$. Then F preserves upper semi-continuity of all closed-valued relations from X into Y.*

Proof. Let S be an upper semi-continuous closed-valued relation from X into Y and $T=F(S)$. Then, by [2, (3)], S is closed in $X\times Y$. Thus, by the assumption, T is closed in $Z\times W$. Hence, by [2, (7)], T is also upper semi-continuous.

Corollary 3.1. *Let X, Y, Z, W be topological spaces such that Y is regular and W is compact, and let f be a continuous function from $Z\times W$ onto $X\times Y$ such that $f^{-1}(\{x\}\times Y)=\{z\}\times W$ whenever $f(z, w)=(x, y)$. Then the relation f^{-1} from $X\times Y$ onto $Z\times W$ preserves upper semicontinuity of all closed-valued relations in $X\times Y$.*

Proof. A similar argument as in the proof of Corollary 2.1 can be applied.

Corollary 3.2. *Let X, Y, Z, W be topological spaces such that Y is regular and W is compact, and let f be a homeomorphism of $X\times Y$ onto $Z\times W$ such that $f(\{x\}\times Y)=\{z\}\times W$ whenever $f(x, y)=(z, w)$. Then f preserves both lower and upper semi-continuity of all closed-valued relations in $X\times Y$.*

Remark 3. Note that, by Example 2, closed-valuedness is an essential requirement in the above assertions.

References

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