On the measurable homomorphisms

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In [3] R. SIKORSKI examined the necessaey and sufficient conditions that every σ-homomorphism of a σ-field $\mathscr{B}$ (of subsets of Y) into a σ-quotient algebra $\mathscr{A}|\mathscr{I}$ be induced by a function of X into Y, where $(X, \mathscr{A})$ is a measurable space. According to the definition of Sikorski we say that a σ-homomorphism x of $\mathscr{B}$ into a σ-quotient algebra $\mathscr{A}|\mathscr{I}$ is induced by a function f of X into Y if $f^{-1}(E)\in\mathscr{A}$ and $[f^{-1}(E)]=x(E)$ for all $E\in\mathscr{B}$. $[f^{-1}(E)]$ denotes here the element of $\mathscr{A}|\mathscr{I}$ which contains the set $f^{-1}(E)$. This examination was done by Sikorski under some more special properties of $\mathscr{B}$, too. We prove the analogous theorems exchanged $\mathscr{A}|\mathscr{I}$ by an abstract σ-orthomodular poset.

A partially ordered set (poset) $(\mathscr{L}, \leqq)$ is said to be an σ-orthomodular poset if (i) there are two elements 0 and 1 in $\mathscr{L}$ such that $0\leqq a\leqq 1$ for all $a\in\mathscr{L}$, (ii) there exists a mapping $'$: $a\to a'$ of $\mathscr{L}$ into itself such that $(a^\perp)^\perp=a$, $a\vee a^\perp=1$ for all $a\in\mathscr{L}$, and if $a\leqq b$, then $b^\perp\leqq a^\perp$, (iii) if $a_i\leqq a_j^\perp$, then $\bigvee_{i=1}^{\infty} a_i$ exists in $\mathscr{L}$, (iv) if $a\leqq b$, then there is $d\in\mathscr{L}$ such that $a\leqq d^\perp$ and $a\vee d=b$.

Let $(Y, \mathscr{B})$ be an arbitrary measurable space and suppose $\mathscr{L}=\mathscr{L}(\vee, \wedge, ', 0, 1)$ is a σ-orthomodular poset with first and last elements 0, 1 respectively. Then for all σ-homomorphism $x\colon \mathscr{B}\to\mathscr{L}$ $x(\mathscr{B})=\{x(b), b\in\mathscr{B}\}$ is a sub σ-Boolean algebra of $\mathscr{L}$.

Definition 1. The σ-homomorphism x is called *measurable* if for every such measurable space $(\Omega, \mathscr{F})$ that there exists a σ-homomorphism h of $\mathscr{F}$ onto $x(\mathscr{B})$ one can find also an $\mathscr{F}$-measurable function $f\colon \Omega\to Y$ so that

$$x(E) = h\big(f^{-1}(E)\big) \quad \text{for all} \quad E\in\mathscr{B}.$$

For brevity let $\operatorname{Hom}(\mathscr{B}, \mathscr{L})$ denote the collection of all σ-homomorphism of $\mathscr{B}$ into $\mathscr{L}$.

Theorem 1. *Let $(Y, \mathscr{B})$ be a measurable space. Then for every σ-orthomodular poset $\mathscr{L}$ every $x\in\operatorname{Hom}(\mathscr{B}, \mathscr{L})$ is measurable if and only if for every σ-quotient algebra $\mathscr{F}|\mathscr{I}$ (of a set Ω) every element of $\operatorname{Hom}(\mathscr{B}, \mathscr{F}|\mathscr{I})$ is induced by a function of Ω into Y.*

PROOF. Let $(\Omega, \mathscr{F})$, h be the same as in Definition 1. Suppose that every σ-homomorphism of $\mathscr{B}$ into every σ-quotient algebra $\mathscr{F}|\mathscr{I}$ (of the set Ω) is induced by a function from Ω into Y. Let $x(\mathscr{B})$ be isomorphic to a σ-quotient algebra $\mathscr{F}|\mathscr{I}$, where $\mathscr{I}$ is a σ-ideal of $\mathscr{F}$. (The isomorphism: $l\colon \mathscr{F}|\mathscr{I}\to x(\mathscr{B})$). Let us denote the natural σ-homomorphism of $\mathscr{F}$ onto $\mathscr{F}|\mathscr{I}$ by h' (See Fig. 1.) Trivially $h=l\circ h'$.

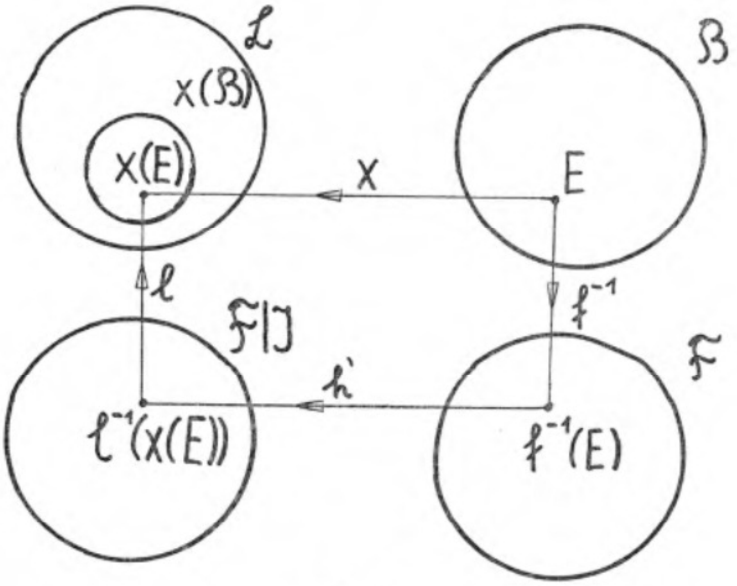

Fig. 1

Now $l^{-1}\circ x$ is a σ-homomorphism of $\mathscr{B}$ onto $\mathscr{F}|\mathscr{I}$, so there exists an $\mathscr{F}$-measurable mapping $f\colon \Omega\to Y$ such that

$$l^{-1}(x(E)) = h'(f^{-1}(E)) \quad \text{for all } E\in\mathscr{B}. \quad \text{Hence}$$
$$x(E) = l\circ h'(f^{-1}(E)) = h(f^{-1}(E)).$$

Necessity. Let $x\in\mathrm{Hom}(\mathscr{B},\mathscr{L})$ be measurable for every σ-orthomodular poset $\mathscr{L}$ and for all $x\in\mathrm{Hom}(\mathscr{B},\mathscr{L})$. Suppose $(X,\mathscr{G})$ is a measurable space, $\mathscr{G}|\mathscr{J}$ is a σ-quotient algebra and $y\in\mathrm{Hom}(\mathscr{B},\mathscr{G}|\mathscr{J})$. Then there exists a $\mathscr{G}$-measurable function g of X into Y such that

$$n(g^{-1}(E)) = y(E) \quad \text{for all} \quad E\in\mathscr{B},$$

where n is the natural homomorphism of $\mathscr{G}$ onto $\mathscr{G}|\mathscr{J}$. So y is measurable. Q.e.d.

Let us consider Theorems 3.2, 4.3 and 4.4 of R. Sikorski [3]. With the help of our Theorem 1 we have the following three theorems.

Theorem 2. Let $\emptyset\neq E\subseteq R$. In order that every σ-homomorphism x of the Borel σ-field $\mathscr{B}(E)$ into an arbitrary σ-orthomodular poset $\mathscr{L}$ be measurable, it is necessary and sufficient that $E\in\mathscr{B}(R)$, where $\mathscr{B}(R)$ is the Borel σ-field ot the real line R.

Theorem 3. Suppose that there exists an enumerable sequence $\{E_i\}$ in $\mathscr{B}$ such that $\mathscr{B}$ is the least σ-field containing all the sets $E_i\ (i=1, 2, \dots)$. Then in order that every $x\in\mathrm{Hom}(\mathscr{B},\mathscr{L})$ be measurable for every σ-orthomodular poset $\mathscr{L}$, it is sufficient and necessary that $\mathscr{B}$ be isomorphic to the Borel σ-field $\mathscr{B}(\tau)$ of a Borel space τ.

Definition 2. The topological space τ is a Borel space iff $\mathscr{B}$ is homeomorphic to a Borel set of the Hilbert cube.

Theorem 4. Let Y be a separable metric space and $\mathscr{B}(Y)$ its Borel σ-field. Then in order that every $x \in \mathrm{Hom}(\mathscr{B}, \mathscr{L})$ be measurable for arbitrary σ-orthomodular poset $\mathscr{L}$, it is necessary and sufficient that $\mathscr{B}$ be a Borel space.

Finally we shall prove two corollaries of our theorems*.

Corollary 1. *Let Y be a separable complete metric space and denote $\mathscr{B}(Y)$ its Borel sets. Let $\mathscr{L}$ be a σ-orthomodular poset. Then every σ-homomorphism x from $\mathscr{B}(Y)$ into $\mathscr{L}$ is measurable.*

PROOF. Since every separable complete metric space is a Borel space (see [1], IX. 6.1. Corollary 1. of Theorem 1., for example), our statement folows from Theorem 4.

In theoretical physics it is almost the case that a measurable quantity takes its values in a locally compact second countable Hausdorff space. This fact makes the following theorem interesting.

Corollary 2. *If Y is locally compact second countable Hausdorff space, then every $x \in \mathrm{Hom}(\mathscr{B}(Y), \mathscr{L})$ is measurable for every σ-orthomodular poset $\mathscr{L}$.*

PROOF. It follows from [1], IX. 2.9. Corollary of Proposition 16., and IX. 6.1. Corollary of Proposition 2. that every locally compact second countable Hausdorff space is metrisable and what is more, it is a separable complete metric space. So our Corollary 1. implies Corollary 2.

References

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