A minimax theorem for additive functions

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1. Let $g(n)$ be a nonnegative additive function, assume that tends to zero monotonically on the set of primes. Let

$$\psi(y) = \sum_{p \leqq y} g(p), \tag{1.1}$$

$$\beta(p) = \sup_{\alpha \geqq 1} g(p^\alpha), \tag{1.2}$$

$$\sup_{\alpha \geqq 1} |g(p^\alpha) - g(p)| \leqq t(p). \tag{1.3}$$

Assume that

$$\psi(2y) - \psi(y) = o(1) \quad (y \to \infty) \tag{1.4}$$

$$\psi(y) \to \infty, \tag{1.5}$$

$$\sum_p t(p) < \infty. \tag{1.6}$$

Let $k \geqq 1$,

$$E_k(x) = \max_{n \leqq x} \min_{j=1,\dots,k} g(n+j). \tag{1.7}$$

We should like to determine $E_k(x)$ as $x \to \infty$ for every fixed k. For every integer n let

$$A_k(n) = \prod_{\substack{p^\alpha \| n \\ p \leqq k}} p^\alpha \tag{1.8}$$

$$B_k = \sup_{n \geqq 1} \frac{1}{k} \left\{ \sum_{j=1}^{k} g(A_k(n+j)) \right\}, \quad C_k = \frac{1}{k} \sum_{p > k} (\beta(p) - g(p)). \tag{1.9)–(1.10}$$

We shall prove the following

Theorem 1. *On the assumptions stated above*

$$\lim_{x \to \infty} \left(E_k(x) - \frac{\psi(\log x)}{k} \right) = B_k + C_k - \frac{\psi(k)}{k}. \tag{1.11}$$

Supposing furthermore that $g(p) \leqq g(p^2) \leqq g(p^3) \leqq \dots$, *we have*

$$B_k = \frac{1}{k} \sum_{p \leqq k} \varrho_{p,k}, \tag{1.12}$$

where

$$\varrho_{p,k} = \beta(p) + \sum_{r=1}^{\infty} g(p^r)\left\{\left[\frac{k-1}{p^r}\right] - \left[\frac{k-1}{p^{r+1}}\right]\right\}. \tag{1.13}$$

Remarks. 1. Assuming that $g(n)$ is strongly additive, i.e. $g(p^\alpha)=g(p)$ for $\alpha \geqq 2$, we have

$$\varrho_{p,k} = \begin{cases} \left(1+\left[\dfrac{k}{p}\right]\right) g(p) & \text{if} \quad p \nmid k, \\ \dfrac{k}{p}\, g(p) & \text{if} \quad p|k. \end{cases} \tag{1.14}$$

2. Putting $g(n)=\log \dfrac{\sigma(n)}{n}$, where $\sigma(n)$ is the sum of the divisors of n we have

$$L(k) = \varlimsup_{n\to\infty} (\log\log n)^{1/k} \min\left(\frac{\sigma(n+1)}{n+1}, \dots, \frac{\sigma(n+k)}{n+k}\right) \tag{1.15}$$

where

$$L(k) = \left(\frac{6}{\pi^2} e^\gamma\right)^{1/k} \exp\left(B_k + C_k - \frac{\psi(k)}{k}\right), \tag{1.16}$$

$$\psi(k) = \sum_{p \leqq k} \log\left(1+\frac{1}{p}\right), \quad C_k = \frac{1}{k} \sum_{p>k} \log \frac{1}{1-1/p^2} \tag{1.17}$$

and B_k is computed from (1.12), (1.13) by putting $\beta(p)=\log(1-1/p)^{-1}$, $g(p^r)=$ $=\log(1+p^{-1}+\dots+p^{-r})$.

The proof of (1.14) is an obvious consequence of (1.13). To prove (1.15) we have to observe only that

$$\prod_{p \leqq y} (1+1/p) = (1+o(1)) \frac{6}{\pi^2} e^\gamma \log y.$$

3. Putting $g(n)=-\log \dfrac{\varphi(n)}{n}$, our theorem gives the recent result of M. Hausman [1].

2. Proof of the assertion

Let

$$g_y(n) = \sum_{\substack{p^\alpha \| n \\ p^\alpha \leqq y}} g(p^\alpha); \quad g(n; y) = \sum_{\substack{p^\alpha \| n \\ p^\alpha > y}} g(p^\alpha).$$

First we observe that $g(n; \log n) \to 0$ as $n \to \infty$. Indeed, if $q_1 < q_2 < \dots < q_r$ are the sequence of all prime divisors of n that are greater than $\log n$, then from the monotonity of $g(p)$ we have

$$g(n; \log n) = \sum_{i=1}^{r} \beta(q_i) \leqq \sum_{i=1}^{r} g(p_i) + \sum_{p>y} t(p)$$

where $p_1<p_2<\ldots<p_r$ the smallest r primes greater than $\log n$. Since

$$(\log n)^r \leqq \prod_{i=1}^{r} p_i \leqq \prod_{i=1}^{r} q_i \leqq n,$$

we have $r \leqq (\log n)(\log\log n)^{-1}$, and so $p_r \leqq 4\log n$.

So, by (1.4) we have

$$\sum_{i=1}^{r} g(p_i) \leqq \psi(4\log n)-\psi(\log n) \to 0.$$

Now we give an appropriate upperestimation for $E_k(x)$. It is obvious that

$$E_k(x) \leqq \max_{n\leqq x}\{k^{-1}\{g(n+1)+\ldots+g(n+k)\}\}.$$

From (1.9) we have

$$E_k(x) \leqq B_k+\max_{n\leqq x}\left\{k^{-1}\sum_{j=1}^{k} g(n+j;\,k)\right\}.$$

Since for $p>k$ p divides $n+j$ at most once and $g(n+j, \log x)=\sigma(1)$, we have

$$k^{-1}\sum_{j=1}^{k} g(n+j;\,k) \leqq k^{-1}\sum_{k<p\leqq \log x} \beta(p) = C_k+k^{-1}(\psi(\log x)-\psi(k)),$$

whence in (1.11) the inequality $\leqq$ holds.

Let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ be arbitrary small positive constants. Let n_0 be chosen so that

$$k^{-1}\sum_{j=1}^{k} g(A_k(n_0+j)) \geqq B_k-\varepsilon_1. \tag{2.1}$$

Let $P^{(1)}=\prod_{p\leqq k} p$,

$$P^{(2)} = \text{l.c.m.}\,\{A_k(n_0+j)\ (j=1, \ldots, k)\}.$$

It is obvious that $A_k(n+j)=A_k(b_0+j)$, $j\in[1, k]$ for $n\equiv n_0(\text{mod } P^{(1)}P^{(2)})$. Let y be so large and the exponents β_j for the primes p_j in the interval (k, y) be chosen so that

$$\Sigma g(p_j^{\beta_j}) \geqq kC_k+(\psi(y)-\psi(k))-\varepsilon_2. \tag{2.2}$$

Let $P_3=\prod p_j^{\beta_j}$. Let now

$$P = \prod_{y<p\leqq \frac{1}{2}\log x} p,$$

$Q_1, Q_2, \ldots, Q_k$ be the products of primes from $\left(y, \frac{1}{2}\log x\right)$, so that $P=Q_1\ldots Q_k$ The congruence system

$$n \equiv n_0(\text{mod } P^{(1)}P^{(2)}), \quad n+1 \equiv 0(\text{mod } P^{(3)}),$$

$$n+1 \equiv 0\ (\text{mod } Q_1), \quad n+j \equiv 0\ (\text{mod } Q_j) \quad (j=2, \ldots, k)$$

has a solution in the interval $[1, P^{(1)}P^{(2)}P^{(3)}P]$. From the nonnegativity of g we have

$$\begin{cases} g(n+1) \geqq \gamma_1+\delta+g(Q_1) \\ g(n+j) \geqq \gamma_j+g(Q_j) \quad (j=2, \ldots, k), \end{cases} \tag{2.3}$$

where

$$\gamma_j = g(A_k(n_0+j)), \quad \delta = g(P^{(3)}).$$

We have

$$\psi\left(\frac{1}{2}\log x\right) - \psi(y) = g(P) = g(Q_1) + \ldots + g(Q_k).$$

Let $\Lambda = \psi\left(\frac{1}{2}\log x\right) - \psi(y) + (\gamma_1 + \ldots + \gamma_k + \delta)$. Since $g(p) \to 0$ for $p \to \infty$, we can choose a partition of the primes in $\left(y, \frac{1}{2}\log x\right)$ such that

$$\left|g(Q_1) + (\gamma_1 + \delta) - \frac{\Lambda}{k}\right| \leqq \varepsilon_3, \left|g(Q_j) + \gamma_j - \frac{\Lambda}{k}\right| \leqq \varepsilon_3. \tag{2.4}$$

Taking into account (2.1), (2.2) we get

$$k^{-1}(\gamma_1 + \ldots + \gamma_k + \delta) \geqq B_k + C_k + k^{-1}(\psi(y) - \psi(k)) - k^{-1}(\varepsilon_1 + \varepsilon_2),$$

and from (2.3), (2.4) we deduce

$$\min_{j=1,\ldots,k} g(n+j) \geqq B_k + C_k + k^{-1}\left(\psi\left(\frac{1}{2}\log x\right) - \psi(k)\right) - k^{-1}(\varepsilon_1 + \varepsilon_2) - \varepsilon_3.$$

Since $\psi(\log x) - \psi\left(\frac{1}{2}\log x\right) = \sigma(1)$, we get the desired result.

Reference

[1] M. Hausman, Generalization of a theorem of Landau, *Pacific J. Math.* **84** (1979), 91—95.

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