Associative functions and Abel—Schröder systems

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1. Introduction

The principal aim of this paper is to investigate questions of the existence and uniqueness of solutions to the associativity equation,

$$T(T(x, y), z) = T(x, T(y, z)),$$

given the restriction of T to simple one-dimensional subsets of its domain. More specifically, for functions T defined on certain pairs of line segments in $(0, 1)^2$, we shall establish uniqueness and obtain conditions for existence of an extension of T to an associative function on $[0, 1]^2$.

Our interest in these questions stems in part from their reformulation, via the celebrated representation theorem, as questions about common solutions of two classical functional equations in a single variable, the Abel and Schröder equations. While there is an extensive literature devoted to these equations individually, Abel—Schröder systems seem to have been neglected, and merit study apart from any connection with associativity.

The associative functions to be considered are the so-called *Archimedean* T on $[0, 1]^2$: continuous, increasing in each place, satisfying the boundary conditions

$$T(0, x) = T(x, 0) = 0, \quad T(1, x) = T(x, 1) = x,$$

and for which

$$T(x, x) < x \quad \text{when} \quad 0 < x < 1.$$

To motivate both the original problems and their reformulation, we begin with the well-known version of the representation theorem for those Archimedean T that are strictly increasing in each place over $(0, 1]^2$ [1, 6]. Each T in this class admits the representation

$$\text{(R)} \qquad T(x, y) = g^{-1}\big(g(x) + g(y)\big),$$

where g is a strictly decreasing function from $[0, 1]$ onto $[0, \infty]$; and conversely, any such g generates a member of this class via (R).

In two subsequent improvements of this theorem, a prominent role is played by the *diagonal* δ of T, i.e., the function

$$\delta(x) = T(x, x).$$

LING [5] extended the representation (R) to include all Archimedean T, provided that g^{-1} is appropriately modified in case the range of g is bounded. And recently KRAUSE [3] strengthened Ling's result by showing that the theorem remains valid when some of the assumptions (but not those concerning δ) are weakened.

It is thus reasonable to ask how much information about T is contained in its diagonal. As was pointed out in [2], one can easily use (R) to generate many distinct T with a prescribed diagonal δ by constructing solutions of the Schröder equation

$$g(\delta(x)) = 2g(x). \tag{S}$$

In a similar fashion, for any fixed s in (0, 1), solutions of the Abel equation

$$g(\sigma(x)) = g(x) + g(s) \tag{A}$$

generate associative T with a given *s-section*

$$\sigma(x) = T(x, s).$$

And with minor technical adjustments, these statements hold for all Archimedean T.

Hence a diagonal δ and any section σ, each a function defined on a line segment in $[0, 1]^2$, can be extended individually to many Archimedean T. But what can be said if *both* δ and σ are specified? Does there exist a common solution of (S) and (A) and thus an extension to T on $[0, 1]^2$? If so, is T unique?

In the sequel we develop necessary and sufficient conditions to settle the first question and show that the answer to the second one is affirmative in case one of δ and σ and certain restrictions of the other are given.

An important first step is recognizing that, for our purposes, it is possible to reduce the system (A), (S) to certain simpler Abel—Schröder systems that are more amenable to solution.

Some definitions, elementary results, and background needed throughout the paper are gathered in Section 2.

The subject of Section 3 is Abel—Schröder systems. There we derive necessary and sufficient conditions for the existence and uniqueness of solutions for the special systems mentioned above.

The next two sections are devoted to using these results to solve the problems we have posed concerning Archimedean T—Section 4 to uniqueness and Section 5 to existence.

The paper concludes with some observations about our results and suggestions for further study (Section 6).

We end this introduction with a few remarks about notation. Parentheses will regularly be omitted in expressions involving composites of several functions; thus $f(g(h(x)))$ may appear as $fgh(x)$. For $n=0, 1, 2, \ldots, f^n$ will always denote the n^{th} iterate of f, defined recursively by

$$f^0(x) = x, \quad f^n = ff^{n-1};$$

and f^{-n} means $(f^{-1})^n$ when f is invertibile.

2. Preliminaries

In this section we present some notation and terminology as well as develop some basic facts about generators, diagonals, and sections of the Archimedean T defined in the introduction.

According to [5], a function $T: [0, 1]^2 \to [0, 1]$ is Archimedean if and only if there exists a *generator* g, a strictly decreasing function from $[0, 1]$ onto $[0, a]$, such that

$$\text{(R)} \qquad T(x, y) = \begin{cases} g^{-1}(g(x)+g(y)), & \text{when} \quad 0 \leqq g(x)+g(y) < a, \\ 0, & \text{when} \quad a \leqq g(x)+g(y), \end{cases}$$

for all x, y in $[0, 1]$. Here $0 < a \leqq +\infty$. T is said to be *strict* when $a = +\infty$ because it is then strictly increasing in each place over $(0, 1]^2$.

In problems of extension and uniqueness, a second Archimedean T_0 with generator g_0: $[0, 1] \to [0, a_0]$ will arise. Let

$$\varphi = g_0 g^{-1}.$$

Clearly φ is strictly increasing from $[0, a]$ onto $[0, a_0]$, and $g_0 = \varphi g$. Depending on how we shall relate T_0 to T, φ will figure in one or more functional equations.

For example, suppose $T_0 = T$. Then, as is easy to see, φ must satisfy the Cauchy equation

$$\varphi(u+v) = \varphi(u) + \varphi(v)$$

when $0 \leqq u+v \leqq a$. Since φ is continuous, there exists a k such that $\varphi(u) = ku$ when $0 \leqq u \leqq a$, and so we immediately get the following well-known fact.

Theorem 2.1. *$T_0 = T$ if and only if $g_0 = kg$ for some $k > 0$.*

The *diagonal* of T is the function δ: $[0, 1] \to [0, 1]$ defined by

$$\delta(x) = T(x, x)$$

for all x in $[0, 1]$. Note that $\delta(x) = 0$ if and only if $2g(x) \geqq a$. Let $d = g^{-1}(a/2)$. Then by virtue of (R),

$$\delta(x) = \begin{cases} 0, & \text{when} \quad 0 \leqq x \leqq d, \\ g^{-1}(2g(x)), & \text{when} \quad d \leqq x \leqq 1. \end{cases}$$

Theorem 2.2. *A function δ: $[0, 1] \to [0, 1]$ is the diagonal of some Archimedean T if and only if there is a d in $[0, 1)$ such that*

(a) $\delta(x) = 0$ *when* $0 \leqq x \leqq d$,
(b) *the restriction of δ to $[d, 1]$ is strictly increasing onto $[0, 1]$,*
(c) $\delta(x) < x$ *when* $0 < x < 1$.

Moreover, when δ is the diagonal of T, $d = 0$ if and only if $a = +\infty$.

Necessity and the last sentence can be verified in a straightforward manner. To find an Archimedean T whose diagonal is a function δ satisfying (a), (b), (c), it suffices to find a strictly decreasing g on $[0, 1]$ with $g(1) = 0$ that satisfies the Schröder equation

$$g\delta(x) = 2g(x)$$

for $d \leqq x \leqq 1$. We adapt the standard construction given in [4] to fit the current conditions. If $d>0$, there exists by (b) a unique x_0 in $(d, 1)$ for which $\delta(x_0)=d$; if $d=0$, choose x_0 arbitrarily in $(0, 1)$. Let $x_n=\delta^n(x_0)$ for all integers $n>0$. Notice that when $d>0$, $x_n=0$ for all $n>1$; otherwise from (c) it follows that the x_n form a strictly decreasing sequence convergent to zero. For each integer $n<0$ there is a unique x_n in $(x_0, 1)$ for which $\delta^{-n}(x_n)=x_0$. These x_n form a strictly increasing sequence that converges to 1 as n goes to $-\infty$. Now choose any strictly decreasing $\bar{g}$ from $[x, x_0]$ onto $[1, 2]$. Extend $\bar{g}$ over $[0, 1]$ by defining $g(x)=2^n\bar{g}(\delta^{-n}(x))$ when $x_{n+1} \leqq x < x_n$. Let $g(1)=0$, and let $g(0)=+\infty$ in case $d=0$. This yields the required g.

Theorem 2.2 and the following result allow us to determine, among other things, all T having a prescribed diagonal.

Theorem 2.3. *Consider Archimedean T_0 and T for which $d_0=d$, and let D be any subset of $[d, 1]$. Then $\delta_0(x)=\delta(x)$ for all x in D if and only if φ satisfies the Schröder equation*

$$\varphi(2u) = 2\varphi(u)$$

for all u in $g(D)$.

The proof of this is straightforward upon observing that $\delta_0(x)=g_0^{-1}(2g_0(x))$ and $\delta(x)=g^{-1}(2g(x))$ when $d \leqq x \leqq 1$.

For $0<s<1$, the *s-section* of T is the function $\sigma\colon [0, 1] \to [0, 1]$ defined by

$$\sigma(x) = T(x, s)$$

for all x in $[0, 1]$. Since the generator g of T is, according to Theorem 1.1, determined up to a positive multiple, g can be chosen so that

$$g(s) = 1.$$

Unless otherwise specified, we shall, in the presence of an s-section, tacitly assume that this has been done. Note that $\sigma(x)=0$ if and only if $g(x)+1 \geqq a$. Let $c=g^{-1}(a-1)$. Then by virtue of (R),

$$\sigma(x) = \begin{cases} 0, & \text{when } 0 \leqq x \leqq c, \\ g^{-1}(g(x)+1), & \text{when } c \leqq x \leqq 1. \end{cases}$$

Observe that, since T is commutative, the "horizontal" and "vertical" s-sections are identical.

The proofs of the next two theorems are analogues of the proofs of the preceding two.

Theorem 2.4. *A function $\sigma\colon [0, 1] \to [0, 1]$ is the s-section of some Archimedean T if and only if there is a c in $[0, 1)$ such that*

(a) $\sigma(x)=0$ *when* $0 \leqq x \leqq c$,
(b) *the restriction of σ to $[c, 1]$ is strictly increasing onto $[0, s]$,*
(c) $\sigma(x)<x$ *when* $0<x \leqq 1$.

Moreover, when σ is the s-section of T, $c=0$ if and only if $a=+\infty$.

To obtain a T with prescribed s-section σ, it suffices to construct a generator g satisfying the Abel equation

$$g\sigma(x) = g(x)+1$$

for $c \leqq x \leqq 1$. Let $y_n = \sigma^n(s)$ for integers $n \geqq -1$. Notice that the y_n converge monotonically to zero when $c=0$; and they eventually reach 0 when $c>0$. Choose any strictly decreasing $\bar{g}$ from $[s, 1]$ onto $[0, 1]$. Extend $\bar{g}$ over $[0, 1]$ by defining $g(x) = n + \bar{g}(\sigma^{-n}(x))$ when $y_n < x \leqq y_{n-1}$. Let $g(0) = +\infty$ in case $c=0$, and $g(0)=2$ in case $c>0$.

Theorem 2.5. *For fixed s in $(0, 1)$, consider Archimedean T_0 and T for which $c_0 = c$, and let S be any subset of $[c, 1]$. Then $\sigma_0(x) = \sigma(x)$ for all x in S if and only if φ satisfies the Abel equation*

$$\varphi(u+1) = \varphi(u)+1$$

for all u in $g(S)$.

Some elementary relations between diagonal and sections are collected below.

Theorem 2.6. *For Archimedean T with diagonal δ and s-section σ*

(a) $\sigma(s) = \delta(s)$
(b) $\delta(x) \leqq \sigma(x)$ *when* $0 \leqq x < s$
(c) $\sigma(x) \leqq \delta(x)$ *when* $s < x \leqq 1$
(d) *either* $s < d < c$, $s = d = c$, *or* $s > d > c$.

The inequalities in (b) *and* (c) *are strict when the right sides are positive.*

3. Abel and Schröder Jointly

In this section we present some results on the existence of a common solution φ to an Abel equation and a Schröder equation

$$\text{(A)} \qquad \varphi\xi(x) = 1 + \varphi(x)$$

$$\text{(S)} \qquad \varphi\eta(x) = 2\varphi(x).$$

Two special cases needed in the following sections will be considered.

Case 1. Let ξ be strictly increasing from $[0, 1]$ onto $[1, 2]$, and let $\eta(x) = 2x$ for all x. When does there exist a strictly increasing φ from $[0, 2]$ onto $[0, 2]$ satisfying both (A) and (S) for all x in $[0, 1]$?

Case 2. Let η be strictly increasing from $[1, +\infty]$ onto $[2, +\infty]$, and let $\xi(x) = 1 + x$ for all x. When does there exist a strictly increasing φ from ($[1, +\infty]$ onto $[1, +\infty]$ satisfying both (A) and (S) for all x in $[1, +\infty]$?

We begin with case 1. It can quickly be seen that any solution φ: $[0, 2] \to \mathbf{R}$ whatever to

$$\text{(A}_1\text{)} \qquad \varphi\xi(x) = 1 + \varphi(x)$$

$$\text{(S}_1\text{)} \qquad \varphi(2x) = 2\varphi(x)$$

for all x in $[0, 1]$ must assume the values $\varphi(0)=0$, $\varphi(1)=1$, $\varphi(2)=2$ and must satisfy the equation $\varphi\left(\frac{y}{2^n}\right)=\frac{1}{2^n}\varphi(y)$ for all $n>0$ when $1\leqq y\leqq 2$. Consequently, by way of (A_1),

$$\varphi\xi\left(\frac{y}{2^n}\right) = 1+\frac{1}{2^n}\varphi(y)$$

for all y in $[1, 2]$ and all $n>0$. Let f be the restriction of φ to $[1, 2]$. Then $f(1)=1$, $f(2)=2$, and for all $n>0$

$$f\xi\left(\frac{y}{2^n}\right) = 1+\frac{1}{2^n}f(y) \tag{AS$_n$}$$

when $1\leqq y\leqq 2$. Conversely, suppose that f is a function on $[1, 2]$ with $f(1)=1$ and $f(2)=2$ which satisfies $(AS)_n$ for all $n>0$ when $1\leqq y\leqq 2$. Let $\varphi(0)=0$, and

$$\varphi(z) = \frac{1}{2^n}f(2^n z) \quad \text{for} \quad \frac{1}{2^n} < z \leqq \frac{1}{2^{n-1}} \tag{1}$$

when $n>0$. Then it is easily checked that φ satisfies (A_1) and (S_1) for all x in $[0, 1]$. This yields

Theorem 3.1. *By way of* (1) *there is a one-to-one correspondence between the simultaneous solutions* φ *of* (A_1) *and* (S_1) *on* $[0, 1]$ *and the simultaneous solutions* f *on* $[1, 2]$ *of the* $(AS)_n$, *for* $n>0$, *with* $f(1)=1$ *and* $f(2)=2$. *Moreover,* φ *is continuous, increasing, or strictly increasing on* $[0, 2]$ *precisely when* f *is (respectively) continuous, increasing, or strictly increasing on* $[1, 2]$.

For $n\geqq 0$ let

$$t_n = \xi\left(\frac{1}{2^n}\right).$$

Then $t_0=2$ and t_n decreases strictly to 1 as n goes to $+\infty$. For any f on $[1, 2]$ with $f(1)=1$ and $f(2)=2$, let $\hat{f}$ be the function on $[1, 2]$ defined by

$$\hat{f}(y) = 1+\frac{1}{2^n}f\big(2^n\xi^{-1}(y)\big) \quad \text{for} \quad t_n < y \leqq t_{n-1} \quad \text{and} \quad \hat{f}(1) = 1. \tag{2}$$

Theorem 3.2. *Let* f_0 *be a bounded function on* $[1, 2]$ *with* $f_0(1)=1$ *and* $f_0(2)=2$. *Using* (2), *define* f_m *on* $[1, 2]$ *recursively by* $f_m=\hat{f}_{m-1}$ *for* $m>0$. *Then* f_m *converges uniformly on* $[1, 2]$ *to a bounded function* f *on* $[1, 2]$, *with* $f(1)=1$ *and* $f(2)=2$, *that satisfies* $(AS)_n$ *on* $[1, 2]$ *for all* $n>0$. *Moreover,* f *is continuous or increasing when* f_0 *is continuous or increasing, respectively.*

To show the uniform convergence of the f_m, consider any y in $(1, 2]$ and the corresponding $n>0$ for which $t_n<y\leqq t_{n-1}$. Given $m>0$,

$$|f_{m+1}(y)-f_m(y)| = |\hat{f}_m(y)-\hat{f}_{m-1}(y)| =$$

$$= \frac{1}{2^n}\left|f_m(2^n\xi^{-1}(y))-f_{m-1}(2^n\xi^{-1}(y))\right| \leqq \frac{1}{2}|f_m(y_1)-f_{m-1}(y_1)|$$

for some y_1 in (1, 2]. Repetition of this argument for y_1, and so on, yields

$$|f_{m+1}(y)-f_m(y)| \leqq \frac{1}{2^m}|f_1(y_m)-f_0(y_m)|$$

for some y_m in (1, 2]. Since f_0 is bounded, there exists an M such that

$$|f_{m+1}(y)-f_m(y)| \leqq \frac{M}{2^m} \quad \text{when} \quad 1 \leqq y \leqq 2 \quad \text{and} \quad m > 0.$$

Consequently, for all $m, k > 0$ and all y in [1, 2],

$$|f_{m+k}(y)-f_m(y)| \leqq \sum_{j=m}^{m+k-1} |f_{j+1}(y)-f_j(y)| \leqq \sum_{j=m}^{m+k-1} \frac{M}{2^j} \leqq \frac{M}{2^{m-1}},$$

whence the f_m converge uniformly on [1, 2] to some f on [1, 2]. Now fix $n>0$. By (2),

$$f_{m+1}\xi\left(\frac{y}{2^n}\right) = \hat{f}_m\xi\left(\frac{y}{2^n}\right) = 1+\frac{1}{2^n}f_m(y)$$

when $1 \leqq y \leqq 2$, and so in the limit, as $m \to +\infty$, f satisfies $(AS)_n$ on [1, 2]. What is left to prove follows at once by virtue of inheritance through uniform convergence and the hat operation.

For questions of uniqueness it is useful to introduce, for each $n>0$, the strictly increasing function ξ_n from [1, 2] onto $[t_n, t_{n-1}]$ defined by

$$\xi_n(y) = \xi\left(\frac{y}{2^n}\right)$$

for $1 \leqq y \leqq 2$. The composites $\xi_{n_1}\xi_{n_2}\dots\xi_{n_k}$ are critical because

$$f(\xi_{n_1}\dots\xi_{n_k}(y)) = 1+\frac{1}{2^{n_1}}+\frac{1}{2^{n_1+n_2}}+\dots+\frac{f(y)}{2^{n_1+\dots+n_k}} \tag{3}$$

for $1 \leqq y \leqq 2$ whenever f satisfies $(AS)_n$ for all n. The verification of (3) requires an easy induction.

Let R be the set of all dyadic rationals in (1, 2), that is, all

$$[n_1, n_2, \dots, n_k] = 1+\frac{1}{2^{n_1}}+\frac{1}{2^{n_1+n_2}}+\dots+\frac{1}{2^{n_1+n_2+\dots+n_k}}$$

for positive integers $n_1, n_2, \dots, n_k$. And let

$$\theta([n_1, \dots, n_k]) = \xi_{n_1}\dots\xi_{n_k}(1)$$

for each $[n_1, \dots, n_k]$. Let K be the range of θ, a subset of (1, 2). It is not difficult to show that θ is a one-to-one, order preserving mapping of R onto K. Therefore, if f is a solution to the $(AS)_n$ on [1, 2] with $f(1)=1$ and $f(2)=2$, then (3) immediately implies that $f(\theta(r))=r$ for all r in R. The restriction of f to K is thus also a one-to-one, order preserving mapping of K onto R. Upon combining these facts with Theorems 3.1 and 3.2, we obtain the proof of

Theorem 3.3. *When K is dense in* $[1, 2]$ *there is a unique continuous solution* φ *to* (A_1) *and* (S_1) *simultaneously on* $[0, 1]$, *which of necessity is strictly increasing.*

Assume, now, that f is continuous and increasing from $[1, 2]$ onto $[1, 2]$ and that f satisfies $(AS)_n$ on $[1, 2]$ for all $n>0$. Let (a, b) be an open interval in $[1, 2]$ contiguous to K — that is, a component of the complement in $[1, 2]$ of $\bar{K}$, the closure of K. Note that 1 and 2 are in $\bar{K}$. Since the restriction of f to K is one-to-one and order preserving onto R, it follows that $f(a)=f(b)$, whence f is constant on $[a, b]$. But this means that f is uniquely determined, and we have proved

Theorem 3.4. *There is a unique continuous and increasing solution* φ *to* (A_1) *and* (S_1) *simultaneously on* $[0, 1]$. *Moreover,* φ *is strictly increasing if and only if* K *is dense in* $[1, 2]$.

The following result provides a useful test for the density of K. To simplify its statement and proof, some *ad hoc* terminology will be adopted: we say that an infinite sequence $n_1, n_2, \ldots, n_k, \ldots$ of positive integers is *admissible* if it is not eventually constantly 1, in which case we let $n_{k_1}, n_{k_2}, \ldots, n_{k_j}, \ldots$ be the subsequence obtained by deleting all occurrences of 1.

Theorem 3.5. K *is dense in* $[1, 2]$ *if and only if, for each admissible sequence* $n_1, n_2, \ldots, n_k, \ldots$ *of positive integers,*

$$\xi_{n_1}\xi_{n_2}\cdots\xi_{n_{k_j}}(2)-\xi_{n_1}\xi_{n_2}\cdots\xi_{n_{k_j}}(1)\to 0$$

as $j\to\infty$.

We begin the proof by observing that the equation

$$v = 1+\frac{1}{2^{n_1}}+\frac{1}{2^{n_1+n_2}}+\frac{1}{2^{n_1+\ldots+n_k}}+\ldots$$

establishes a one-to-one correspondence between the real numbers v in $(1, 2)$ that are not in R and the admissible sequences. Clearly $[n_1, \ldots, n_k]$ increases to v as $k\to+\infty$, and $[n_1, \ldots, n_{(k_j-1)}]$ decreases to v as $j\to+\infty$. Since $\xi_{n_{(k_j-1)}}(1)=\xi_{n_{k_j}}(2)$ and since θ is one-to-one and order preserving from R onto K it follows that

$$\xi_{n_1}\cdots\xi_{n_{k_j}}(1)<\xi_{n_1}\cdots\xi_{n_{k_j}}(2), \tag{4}$$

with the left side increasing and the right side decreasing as $j\to+\infty$.

Suppose K is not dense in $[1, 2]$. Then there is a subinterval (u, w) of $[1, 2]$ disjoint from K. Let $R^-=\{r \text{ in } R\colon \theta(r)<u\}$ and $R^+=\{r \text{ in } R\colon w<\theta(r)\}$. Since $R^-\cup R^+=R$ and since $x<y$ when x is in R^- and y is in R^+, there exists a v in $(1, 2)$ for which $\sup R^-=v=\inf R^+$. Moreover v is not in R so that

$$v = 1+\frac{1}{2^{n_1}}+\ldots+\frac{1}{2^{n_1+\ldots+n_k}}+\ldots$$

for some admissible sequence $n_1, \ldots, n_k, \ldots$. As a consequence $\theta([n_1, \ldots, n_{k_j}])<u$

and $\theta([n_1, ..., n_{(k_j-1)}])>w$, whereupon

$$\xi_{n_1} \dots \xi_{n_{k_j}}(2) - \xi_{n_1} \dots \xi_{n_{k_j}}(1) \geqq w-u > 0$$

for all j and cannot converge to zero.

Conversely, suppose K is dense in [1, 2]. Let $n_1, ..., n_k, ...$ be any admissible sequence, and let v be as above. Let u and w be the limits of the sequences on the left and right sides of (4), respectively. Suppose $u<w$. The denseness of K guarantees u', w' in K such that $u<u'<w'<w$. But then $[n_1, ..., n_{k_j}]<\theta^{-1}(u')<\theta^{-1}(w')<$ $<[n_1, ..., n_{k_j-1}]$ for all j so that $v\leqq\theta^{-1}(u')<\theta^{-1}(w')\leqq v$, which is impossible and completes the proof.

We now consider case 2. It is readily checked that any solution φ: $[1, +\infty]\to\mathbf{R}$ whatever to

$$(A_2) \qquad \varphi(x+1) = 1+\varphi(x)$$

$$(S_2) \qquad \varphi\eta(x) = 2\varphi(x)$$

for all x in $[1, +\infty]$ satisfies $\varphi(n)=n$ for all integers $n\geqq 1$ and $\varphi(x+n)=\varphi(x)+n$ for all $x\geqq 1$ and $n\geqq 0$. Thus, φ is determined by its resctriction f to [1, 2].

Our chief concern is with the existence of a strictly increasing solution φ from $[1, +\infty]$ onto $[1, +\infty]$. If such a φ exists, then $\eta(x)=\varphi^{-1}(2\varphi(x))$ when $x\geqq 1$ by (S_2), from which we can easily obtain the further relation

$$(*) \qquad \eta(x+n) = \eta(x)+2n$$

for all $x\geqq 1$ and integers $n\geqq 0$. In particular, note that $\eta(n)=2n$ for all integers $n\geqq 1$. *For the rest of this section, assume that (*) holds.*

In view of the above observations, the problem of finding solutions φ to (A_2) and (S_2) is equivalent to determining those f on on [1, 2] with $f(1)=1$ and $f(2)=2$ for which

$$(AS)_* \qquad 2f(x) = \begin{cases} f(\eta(x)-1)+1, & \text{when} \quad 2 \leqq \eta(x) \leqq 3, \\ f(\eta(x)-2)+2, & \text{when} \quad 3 \leqq \eta(x) \leqq 4. \end{cases}$$

Clearly φ will be strictly increasing onto $[1, +\infty]$ just when f is strictly increasing onto [1, 2].

For any bounded f on [1, 2] with $f(1)=1$ and $f(2)=2$, let

$$(5) \qquad \hat{f}(x) = \begin{cases} \dfrac{f(\eta(x)-1)+1}{2}, & \text{when} \quad 1 \leqq x \leqq \eta^{-1}(3), \\ \dfrac{f(\eta(x)-2)+2}{2}, & \text{when} \quad \eta^{-1}(3) \leqq x \leqq 2. \end{cases}$$

Then $\hat{f}$ is well-defined on [1, 2] with $\hat{f}(1)=1$, $\hat{f}(2)=2$, and $\hat{f}\eta^{-1}(3)=3/2$.

Theorem 3.6. *Let f_0 be a bounded function on* [1, 2] *with $f_0(1)=1$ and $f_0(2)=2$. Using* (5), *define f_m on* [1, 2] *recursively by $f_m=\hat{f}_{m-1}$ for $m>0$. Then f_m converges uniformly on* [1, 2] *to a bounded function f on* [1, 2], *with $f(1)=1$ and $f(2)=2$, that satisfies* $(AS)_*$. *Moreover, f is continuous or increasing when f_0 is continuous or increasing, respectively.*

The proof is not essentially different from that of Theorem 3.2, its counterpart for case 1. Observe that instead of the intervals $(t_n, t_{n+1}]$ in the proof of the former, there are here just the two intervals $[1, \eta^{-1}(3)]$ and $[\eta^{-1}(3), 2]$. We omit the details.

To establish uniqueness in case 2, we need a representation for the dyadic rationals R in [1, 2] different from the one used in case 1.

Let

$$\langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k \rangle = \frac{3}{2} + \frac{(-1)^{\varepsilon_1}}{2^2} + \ldots + \frac{(-1)^{\varepsilon_k}}{2^{k+1}},$$

where the ε_i are the integers 1 and 2. Each member of R except $\frac{3}{2}$ has exactly one such form. For $\varepsilon=1$ and 2 let

$$\eta_\varepsilon(x) = \eta^{-1}(x+\varepsilon) \quad \text{for} \quad 1 \leqq x \leqq 2.$$

From the strictly increasing character of η, together with the fact that $\eta(2)=4$, we get the useful inequalities

$$1 < \eta_1(x) < \eta^{-1}(3) < \eta_2(x) < 2 \quad \text{when} \quad 1 < x < 2. \tag{6}$$

Let θ_*: $R \to [1, 2]$ be defined as follows:

$$\theta_*\left(\frac{3}{2}\right) = \eta^{-1}(3),$$

$$\theta_*(\langle \varepsilon_1, \ldots, \varepsilon_k \rangle) = \eta_{\varepsilon_1}\eta_{\varepsilon_2}\ldots\eta_{\varepsilon_k}(\eta^{-1}(3)).$$

And let K_* be the range of θ_*.

We shall show that θ_* is one-to-one and order preserving. Suppose r, r' in R are such that $r<r'$. When $r=\frac{3}{2}$ and $r'=\langle \varepsilon_1, \ldots, \varepsilon_k \rangle$, then $\varepsilon_1=2$ and at once from (6) we have $\theta_*(r)=\eta^{-1}(3)<\theta_*(r')$. A similar argument holds when $r'=\frac{3}{2}$. Suppose, now, that $r=\langle \varepsilon_1, \ldots, \varepsilon_k \rangle$ and $r'=\langle \varepsilon_1', \ldots, \varepsilon_l' \rangle$. Since $r<r'$, $\varepsilon_1 \leqq \varepsilon_1'$. If $\varepsilon_1<\varepsilon_1'$ (i.e., $\varepsilon_1=1$ and $\varepsilon_1'=2$), it is clear from (6) that $\theta_*(r)<\eta^{-1}(3)<\theta_*(r')$. If $\varepsilon_1=\varepsilon_1'$, there is a largest integer $j \geqq 1$ for which $\varepsilon_i=\varepsilon_i'$ when $i \leqq j$. Three cases must be considered:

$$j = l \quad \text{with} \quad \varepsilon_{j+1} = 1,$$

$$j = k \quad \text{with} \quad \varepsilon_{j+1}' = 2,$$

$$j < \min\{k, l\} \quad \text{with} \quad \varepsilon_{j+1} = 1 \quad \text{and} \quad \varepsilon_{j+1}' = 2.$$

From (6) and the definition of θ_* we get three corresponding inequalities:

$$\theta_*(\langle \varepsilon_{j+1}, \ldots, \varepsilon_k \rangle) < \theta_*\left(\frac{3}{2}\right),$$

$$\theta_*\left(\frac{3}{2}\right) < \theta_*(\langle \varepsilon_{j+1}', \ldots, \varepsilon_l' \rangle),$$

$$\theta_*(\langle \varepsilon_{j+1}, \ldots, \varepsilon_k \rangle) < \theta_*(\langle \varepsilon_{j+1}', \ldots, \varepsilon_l' \rangle).$$

If, in each case, x and y are the left and right sides, then $\theta_*(r)=\eta_1\ldots\eta_j(x)$ and $\theta_*(r')=\eta_1\ldots\eta_j(y)$. Hence in all cases $\theta_*(r)<\theta_*(r')$.

To obtain the analogues of Theorems 3.3 and 3.4, we need one more fact: if f is a solution of $(\mathrm{AS})_*$ with $f(1)=1$ and $f(2)=2$, then

$$f(\theta_*(r))=r \tag{7}$$

for each r in R. First, direct substitutions into $(\mathrm{AS})_*$ immediately give (7) when $r=\frac{3}{2}$, $r=\langle 1\rangle=\frac{5}{4}$, and $r=\langle 2\rangle=\frac{7}{4}$. To establish (7) for all $r=\langle\varepsilon_1, \ldots, \varepsilon_k\rangle$, we proceed by induction on the length of the finite sequences $\varepsilon_1, \ldots, \varepsilon_k$. Fix $k>1$ and assume that (7) holds for those r represented by sequences of length $k-1$. From (6) and the definition of θ_*, $(\mathrm{AS})_*$ becomes

$$2f(\theta_*(r))=f(\eta\theta_*(r)-\varepsilon_1)+\varepsilon_1.$$

But $\eta\theta_*(r)-\varepsilon_1=\theta_*(\langle\varepsilon_2, \ldots, \varepsilon_k\rangle)$, whence by the induction hypothesis,

$$2f(\theta_*(r))=\langle\varepsilon_2, \ldots, \varepsilon_k\rangle+\varepsilon_1.$$

A simple calculation gives $\langle\varepsilon_2, \ldots, \varepsilon_k\rangle+\varepsilon_1=2r$, and (7) thus holds.

Now, the arguments employed in proving Theorems 3.3 and 3.4 carry over to yield:

Theorem 3.7. *When K_* is dense in* $[1, 2]$ *there is a unique continuous solution φ to (A_2) and (S_2) on $[1, +\infty]$, which of necessity is strictly increassing.*

Theorem 3.8. *There is a unique continuous and increasing solution φ to (A_2) and (S_2) on $[1, +\infty]$. It is strictly increasing if and only if K_* is dense in* $[1, 2]$.

There is also a counterpart to Theorem 3.5. Consider the equation

$$v=\frac{3}{2}+\frac{(-1)^{\varepsilon_1}}{2^2}+\ldots+\frac{(-1)^{\varepsilon_k}}{2^{k+1}}+\ldots$$

which establishes a one-to-one correspondence between the real numbers v in $(1, 2)$ that are not in R and the infinite sequences $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots$ of integers 1 and 2 that are not eventually constant. Note, for $k>1$, that

$$\langle\varepsilon_1, \ldots, \varepsilon_k\rangle<v<\langle\varepsilon_1, \ldots, \varepsilon_{k-1}\rangle \quad \text{if} \quad \varepsilon_k<\varepsilon_{k+1},$$

and

$$\langle\varepsilon_1, \ldots, \varepsilon_k\rangle>v>\langle\varepsilon_1, \ldots, \varepsilon_{k-1}\rangle \quad \text{if} \quad \varepsilon_k>\varepsilon_{k+1}.$$

Let I_k for $k>1$ be the interval whose endpoints are $\langle\varepsilon_1, \ldots, \varepsilon_k\rangle$ and $\langle\varepsilon_1, \ldots, \varepsilon_{k-1}\rangle$. The I_k are closed nested intervals such that v is interior to I_k and I_{k+1} is interior to I_k for all k. Moreover, $\bigcap_{k=2}^{\infty} I_k=\{v\}$. Since θ_* is one-to-one and order preserving, the proof of Theorem 3.5 can be duplicated to prove

Theorem 3.9. *K_* is dense in* $[1, 2]$ *if and only if, for each infinite sequence* $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k, \ldots$ *of integers* 1 *and* 2 *not eventually constant,*

$$\eta_{\varepsilon_1}\eta_{\varepsilon_2}\cdots\eta_{\varepsilon_{k_n}}(\eta^{-1}(3))-\eta_{\varepsilon_1}\eta_{\varepsilon_2}\cdots\eta_{\varepsilon_{k_n}-1}(\eta^{-1}(3))\to 0$$

as $n\to\infty$, *where* $\varepsilon_{k_1}, \ldots, \varepsilon_{k_n}, \ldots$ *is the subsequence of* $\varepsilon_1, \ldots, \varepsilon_k, \ldots$ *obtained by deleting all occurrences of* 1.

Some important comments about the results of this section are presented in Section 6.

4. Sets of Uniqueness

Consider Archimedean T_0, T with diagonals δ_0, δ and s-sections σ_0, σ, respectively. Throughout this section assume that d_0, d, c_0, c as defined in Section 2 are such that $d_0=d$ and $c_0=c$. Let g: $[0, 1]\to[0, a]$ be the generator of T for which $g(s)=1$. Then there is a strictly increasing φ from $[0, a]$ onto $[0, a_0]$ with $\varphi(1)=1$ such that the generator g_0: $[0, 1]\to[0, a_0]$ with $g_0(s)=1$ of T_0 is given by $g_0=\varphi g$. Consider subsets D of $[d, 1]$ and S of $[c, 1]$. According to Theorems 2.3 and 2.5, if the restrictions of δ, δ_0 to D coincide and the restrictions of σ, σ_0 to S coincide, then

(S) $$\varphi(2u) = 2\varphi(u) \quad \text{when} \quad u\in g(D)$$

(A) $$\varphi(u+1) = \varphi(u)+1 \quad \text{when} \quad u\in g(S).$$

For certain D and S the only common solution φ to (S) and (A) is the identify function. When, in such cases, it can be shown that φ must be the identity on all of $[0, a]$, then necessarily $T_0=T$. Archimedean T are thus uniquely determined by their restrictions to these sets $D\cup S$.

Of special interest are the cases when $\delta_0=\delta$ or when $\sigma_0=\sigma$.

Theorem 4.1. *Suppose* $\delta_0=\delta$. *Then* $T_0=T$ *in each of the following cases:*

(a) $c\leqq s$ *and* $\sigma_0(x)=\sigma(x)$ *when* $s\leqq x\leqq 1$;
(b) T *is strict and there is an* $\varepsilon>0$ *for which* $\sigma_0(x)=\sigma(x)$ *when* $0\leqq x\leqq\varepsilon$;
(c) $c\leqq\sigma^2(s)$ *and* $\sigma_0(x)=\sigma(x)$ *when* $\sigma^2(s)\leqq x\leqq s$.

To prove part (a), note first that φ must satisfy (S) and (A) with $D=[d, 1]$ and $S=[s, 1]$. By Theorem 2.6, $c\leqq d\leqq s$, whence $1=g(s)\leqq g(d)=\frac{a}{2}$. Since $g(D)=\left[0, \frac{a}{2}\right]$ and $g(S)=[0, 1]$, it then follows that φ satisfies (S) and (A) at least for all u in $[0, 1]$. By virtue of Theorem 3.4, with $\xi(x)=x+1$, there is only one common solution to (S) and (A) on $[0, 1]$. And since the identity function is a solution, we have that $\varphi(u)=u$ when $0\leqq u\leqq 1$. Finally, since (S) holds for $0\leqq u\leqq\frac{a}{2}$, it is clear that $\varphi(u)=u$ when $0\leqq u\leqq a$.

For part (b), $a=+\infty$ and $c=d=0$, so φ satisfies (S) for all $u\geqq 0$ and (A) for all $u\geqq g(\varepsilon)$. Let k be a positive integer for which $2^k\geqq g(\varepsilon)$. Note that, from (S), $\varphi(2^n u)=2^n\varphi(u)$ for all $u\geqq 0$ and all integers n. And because $\varphi(1)=1$, it then follows, from (A), that $\varphi(m)=m$ for all integers $m\geqq 2^k$. Let u be any positive dyadic rational. There is an integer $n \geqq k$ such that $2^n u$ is an integer exceeding 2^k. Hence, $\varphi(u)=\frac{1}{2^n}\varphi(2^n u)=\frac{1}{2^n}(2^n u)=u$. The continuity by of φ ensures that $\varphi(u)=u$ for all $u \geqq 0$, as required.

Finally, consider part (c), in which $D=[d, 1]$ and $S=[\sigma^2(s), s]$ — that is, $g(D)=\left[0, \frac{a}{2}\right]$ and $g(S)=[1, 3]$. By assumption $a-1=g(c) \geqq g(\sigma^2(s))=3$, so $a \geqq 4$. Both (S) and (A) thus hold at least for $1 \leqq u \leqq 2$. By virtue of the continuity of φ and the fact that (S) holds for $0 \leqq u \leqq \frac{a}{2}$, it suffices to show that $\varphi(u)=u$ for all dyadic rationals $u=\frac{m}{2^n}$ in $(1, 2)$ where m is odd. We proceed by induction on n. If $n=1$, then $m=3$ and $\varphi\left(\frac{3}{2}\right)=\frac{1}{2}\varphi(3)=\frac{1}{2}[\varphi(1)+2]=\frac{3}{2}$. Suppose $n>1$. Then either $u<\frac{3}{2}$ or $\frac{3}{2}<u$. In the first case, $1<2u-1<2$ and $2u-1=\frac{m-2^{n-1}}{2^{n-1}}$ so that $\varphi(u)=\frac{1}{2}\varphi(2u)=\frac{1}{2}[\varphi(2u-1)+1]=\frac{1}{2}[2u-1+1]=u$. In the second case, $1<2u-2<2$ and $2u-2=\frac{m-2^n}{2^{n-1}}$ so that

$$\varphi(u)=\frac{1}{2}\varphi(2u)=\frac{1}{2}[\varphi(2u-2)+2]=\frac{1}{2}[2u-2+2]=u.$$

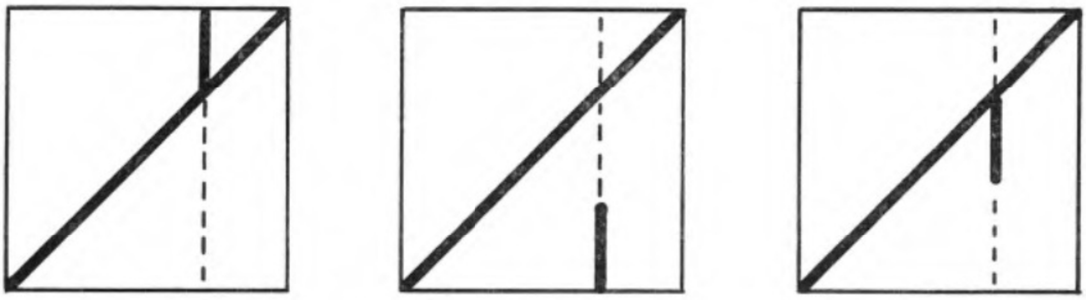
The proof is complete.

Theorem 4.2. *Suppose* $\sigma_0=\sigma$. *Then* $T_0=T$ *in each of the following cases:*

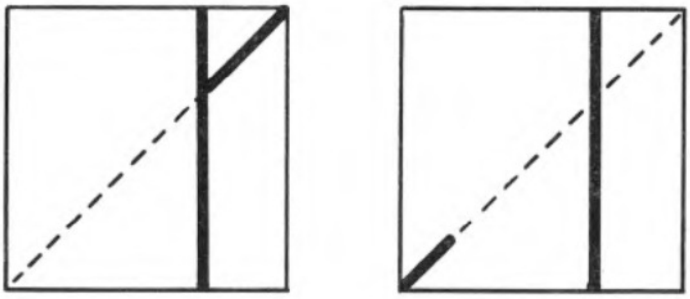
(a) $d \leqq s$ *and* $\delta_0(x)=\delta(x)$ *when* $s \leqq x \leqq 1$;
(b) T *is strict and there is an* $\varepsilon>0$ *for which* $\delta_0(x)=\delta(x)$ *when* $0 \leqq x \leqq \varepsilon$.

The proofs of parts (a) and (b) are analogous to the corresponding parts of Theorem 4.1. For (a), $c \leqq d \leqq s$ so that $1=g(s) \leqq g(c)=a-1$ and $a \geqq 2$. Then φ satisfies (S) on $[0, 1]$ and (A) on $[0, a-1]$. Again we can invoke Theorem 3.4 to conclude that φ is the identify on $[0, 1]$ and, by (A), on $[0, a]$ as well, as required. Now for part (b), we have $a=+\infty$ and $c=d=0$ so that φ must satisfy (S) on $[g(\varepsilon), +\infty]$ and (A) on $[0, +\infty]$. Choose an integer $k \geqq g(\varepsilon)$. For any positive dyadic rational $u=\frac{m}{2^n}$, we have $\varphi(u)=\varphi(u+k)-k=\frac{1}{2^n}\varphi(2^n u+2^n k)-k=\frac{1}{2^n}\varphi(m+2^n k)-$ $-k=u$, and the conclusion follows by continuity of φ.

The parts of Theorems 3.1 and 3.2 in terms of the restriction of T to pieces of the diagonal and vertical s-section over the unit square are pictured below. The solid lines inside the squares indicate sets of uniqueness.



1 a 1 b 1 c



2 a 2 b

5. Extension

Let D and S be closed subintervals of $[0, 1]$, and assume that $0<s<1$. Consider continuous and strictly increasing functions $\delta\colon D\to[0, 1]$ and $\sigma\colon S\to[0, 1]$. Observe the abuse of notation; here δ and σ need not be diagonal and s-section of any Archimedean T. In the preceding section, upon the assumption that δ and σ are the restrictions of the diagonal and s-section of some Archimedean T, we gave examples of D and S for which T is unique. In this section we shall be concerned with the circumstances under which δ and σ are *compatible* in the sense that there exists an Archimedean T whose diagonal and s-section when restricted to D and S are δ and σ.

Theorems 2.2, 2.4, and 2.6 impose several necessary conditions for compatibility. We shall in the sequel, then, assume the following:

(a) $$\delta(0)=0 \quad \text{if} \quad 0\in D; \quad \sigma(0)=0 \quad \text{if} \quad 0\in S;$$

(b) $$\delta(1)=1 \quad \text{if} \quad 1\in D; \quad \sigma(1)=s \quad \text{if} \quad 1\in S;$$

(c) $$\delta(x)<x \quad \text{if} \quad 0<x<1 \quad \text{and} \quad x\in D;$$
$$\sigma(x)<x \quad \text{if} \quad 0<x\leqq 1 \quad \text{and} \quad x\in S;$$

(d) $$\delta(x)<\sigma(x) \quad \text{if} \quad 0<x<s \quad \text{and} \quad x\in D\cap S;$$

(e) $$\delta(x)=\sigma(s) \quad \text{if} \quad s\in D\cap S;$$

(f) $$\sigma(x)<\delta(x) \quad \text{if} \quad s<x<1 \quad \text{and} \quad x\in D\cap S.$$

By extension of δ from D to $[0, 1]$ (if necessary) a continuous and strictly decreasing $g: D\cup\delta(D)\to[0, +\infty]$ can be constructed, as in the proof Theorem 2.2, so that

$$g\delta(x) = 2g(x) \quad \text{when} \quad x\in D.$$

Note that $2u\in g\delta(D)$ when $u\in g(D)$. Now suppose that φ is strictly increasing from $g(D\cup\delta(D))$ onto $g(D\cup\delta(D))$ and

$$\varphi(2u) = 2\varphi(u) \quad \text{when} \quad u\in g(D).$$

Then, as in Section 2, δ is the restriction to D of the diagonal of some Archimedean T_0 if and only if the restriction to $D\cup\delta(D)$ of a generator of T_0 has the form φg.

Similarly, σ is the restriction to S of the s-section of some Archimedean T_0 if and only if the restriction to $S\cup\sigma(S)$ of a generator of T_0 has the form ψh, where h: $S\cup\sigma(S)\to[0, +\infty]$ is continuous and strictly decreasing with

$$h\sigma(x) = h(x)+1 \quad \text{when} \quad x\in S$$

and ψ is strictly increasing from $h(S\cup\sigma(S))$ onto $h(S\cup\sigma(S))$ with

$$\psi(u+1) = \psi(u)+1 \quad \text{when} \quad u\in h(S).$$

When $D\cup\delta(D)$ and $S\cup\sigma(S)$ are non-overlapping intervals, φg and ψh can be extended and abutted (after possibly multiplying one of them by a constant) to create a generator of some Archimedean T_0 whose diagonal and s-section are extensions of δ and σ. We have thus established

Theorem 5.1. *If $D\cup\delta(D)$ and $S\cup\sigma(S)$ are non-overlapping intervals, then δ and σ are compatible.*

On the other hand, it is easy to find incompatible δ and σ in several cases where the intervals D, S, and $D\cap S$ are large. Consider any Archimedean T and any one of the sets of uniqueness that appear in the preceding section, say, Ia. Let σ_0 be a function whose restriction to $[s, 1]$ coincides with that of σ, but $\sigma_0\neq\sigma$. It is evident that δ and σ_0 cannot possibly be compatible.

The problem of compatibility is, in general, very difficult. We have been able to find conditions for the compatibility of δ and σ in two special cases of interest:

(I) $$D = S = [s, 1]$$

(II) $$D = S = [0, s]$$

To begin case (I), observe that $D\cup\delta(D)=[\sigma(s), 1]$ since $\sigma(s)=\delta(s)<s$. We can construct g on $[\sigma(s), 1]$, as described above, so that

(1) $$g\delta(x) = 2g(x) \quad \text{when} \quad s\leqq x\leqq 1,$$

and $g(s)=1$. Then $g(\sigma(s))=2$, and g is strictly decreasing from $[\sigma(s), 1]$ onto $[0, 2]$. Consequently, any φ, as described above, is strictly increasing from $[0, 2]$ onto $[0, 2]$ and

$$\varphi(2u) = 2\varphi(u) \quad \text{when} \quad 0\leqq u\leqq 1.$$

For compatibility of δ and σ, one such φ must satisfy

$$\varphi g(\sigma(x)) = \varphi g(x)+1 \quad \text{when} \quad s \leqq x \leqq 1,$$

or, equivalently,

$$\varphi g\sigma g^{-1}(u) = \varphi(u)+1 \quad \text{when} \quad 0 \leqq u \leqq 1.$$

Now upon letting

$$\xi = g\sigma g^{-1},$$

this equation becomes

$$\varphi\xi(u) = \varphi(u)+1 \quad \text{when} \quad 0 \leqq u \leqq 1.$$

Observe that ξ is strictly increasing from [0, 1] onto [1, 2].

To summarize, δ and σ are compatible if and only if there is a strictly increasing function φ from [0, 2] onto [0, 2] such that

$$\varphi(2u) = 2\varphi(u)$$

$$\varphi\xi(u) = 1+\varphi(u)$$

when $0 \leqq u \leqq 1$, where g is a solution of (1) and $\xi = g\sigma g^{-1}$; if φ exists, any extension of φg to [0, 1] generates an Archimedean extension of δ and σ.

The problem of the existence of such a φ is just case 1 of Section 3. Thus, with K as defined there, δ and σ are compatible if and only if K is dense in [1, 2] (Theorem 3.4).

To express elements of K in terms of δ and σ directly, observe that (1) can be written as

$$g\delta^{-1}(u) = \frac{1}{2}\, g(u) \quad \text{for} \quad \delta(s) \leqq u \leqq 1.$$

Since $u \leqq \delta^{-1}(u) \leqq 1$, we have

$$g\delta^{-n}(u) = \frac{1}{2^n}\, g(u)$$

for $\delta(s) \leqq u \leqq 1$ and all integers $n>0$, whence

$$\frac{y}{2^n} = g\delta^{-n}g^{-1}(y) \quad \text{when} \quad 0 \leqq y \leqq 2.$$

Also, $\xi_n(y) = \xi\left(\frac{y}{2^n}\right) = g\sigma g^{-1}\left(\frac{y}{2^n}\right)$ for $1 \leqq y \leqq 2$, which means that

$$\xi_n(y) = g\sigma\delta^{-n}g^{-1}(y) \quad \text{when} \quad 1 \leqq y \leqq 2.$$

Consequently, for integers $n_i > 0$,

$$\xi_{n_1}\xi_{n_2}\dots\xi_{n_k}(1) = g\sigma\delta^{-n_1}\sigma\delta^{-n_2}\dots\sigma\delta^{-n_k}(s).$$

Let $G = g^{-1}(K)$, that is, the subset of $[\delta(s), s]$ consisting of all

$$\sigma\delta^{-n_1}\sigma\delta^{-n_2}\dots\sigma\delta^{-n_k}(s).$$

We have, at once,

Theorem 5.2. *In case* (I), δ *and* σ *are compatible if and only if* G *is dense in* $[\delta(s), s]$.

Observe that Theorem 3.5 translates to

Theorem 5.3. *In case* (I), δ *and* σ *are compatible if and only if, for each admissible sequence* $n_1, n_2, \ldots, n_k, \ldots$ *of positive integers,*

$$\sigma\delta^{-n_1}\sigma\delta^{-n_2}\ldots\sigma\delta^{-n_{k_j}}(s)-\sigma\delta^{-n_1}\sigma\delta^{-n_2}\ldots\sigma^{-n_{k_j}}(\delta(s))\to 0$$

as $j\to\infty$.

Given compatible δ and σ defined on $[s, 1]$, one can easily extend φg so as to generate a T whose diagonal or s-section is any specified extension of δ or σ, respecttively, to $[0, 1]$. For a diagonal δ_* whose restriction is δ, let $x_n=\delta_*^n(s)$ for all $n>0$, and define the extension of φg by $2^n\varphi g\delta_*^{-n}(x)$ when $x_{n+1}\leqq x\leqq x_n$. Similarly, for an s-section σ_* whose restriction is σ, let $y_n=\sigma_*^n(s)$ for all $n>0$, and define the extension by $n+\varphi g\sigma_*^{-n}(x)$ when $y_{n-1}\leqq x<y$. Notice that T is uniquely determined in either case. This follows from Theorem 3.4 or, alternatively, from parts (a) of Theorems 4.1 and 4.2.

Now, consider case (II) where, since $D=S=[0, s]$ and $\sigma(s)\leqq s$, $s\cup\sigma(S)=[0, s]$. Here we can construct a strictly decreasing h from $[0, s]$ onto $[1, +\infty]$, as described above, so that

$$h\sigma(x) = h(x)+1 \quad \text{when} \quad 0\leqq x\leqq s. \tag{2}$$

And, continuing the analogy with the previous case, any ψ is strictly increasing from $[1, +\infty]$ onto $[1, +\infty]$ and

$$\psi(u+1) = \psi(u)+1 \quad \text{when} \quad u\geqq 1.$$

For compatibility of δ and σ, one such ψ must satisfy

$$\psi h(\delta(x)) = 2\psi h(x) \quad \text{when} \quad 0\leqq x\leqq s,$$

that is,

$$\psi h\delta h^{-1}(u) = 2\psi(u) \quad \text{when} \quad u\geqq 1,$$

or, with $\eta=h\delta h^{-1}$,

$$\psi\eta(u) = 2\psi(u).$$

Consequently, δ and σ are compatible if and only if there is a strictly increasing ψ from $[1, +\infty]$ onto $[1, +\infty]$ such that

$$\psi(u+1) = \psi(u)+1$$
$$\psi\eta(u) = 2\psi(u)$$

when $u\geqq 1$, where h is a solution of (2) and $\eta=h\delta h^{-1}$; if ψ exists, any extension of ψh to $[0, 1]$ generates an Archimedean extension of δ and σ.

Now we have case 2 of Section 3. Thus, with K_* as defined there, we shall have δ and σ compatible exactly when K_* is dense in $[1, 2]$ (Theorem 3.8).

From (2), it is obvious that

$$x+1 = h\sigma h^{-1}(x)$$
$$x+2 = h\sigma^2 h^{-1}(x)$$

when $x \geqq 1$, and since $\eta^{-1} = h\delta^{-1}h^{-1}$, we have

$$\eta^{-1}(x+1) = h\delta^{-1}\sigma h^{-1}(x)$$
$$\eta^{-1}(x+2) = h\delta^{-1}\sigma^2 h^{-1}(x)$$

when $1 \leqq x \leqq 2$. Thus K_* consists of $h\delta^{-1}h^{-1}(3)$ and all

$$h\delta^{-1}\sigma^{\varepsilon_1}\delta^{-1} \dots \delta^{-1}\sigma^{\varepsilon_{k-1}}\delta^{-1}\sigma^{\varepsilon_k}\delta^{-1}h^{-1}(3).$$

Note that $h^{-1}(3) = \sigma^2(s)$.

Let $H = g^{-1}(K_*)$, that is, the subset of $(\sigma(s), s)$ consisting of

$$\delta^{-1}\sigma^2(s)$$

together with all

$$\delta^{-1}\sigma^{\varepsilon_1}\delta^{-1} \dots \delta^{-1}\sigma^{\varepsilon_{k-1}}\delta^{-1}\sigma^{k}\delta^{-1}(\sigma^2(s)).$$

Theorem 5.4. *In case* (II), δ *and* σ *are compatible if and only if* H *is dense in* $[\delta(s), s]$.

And in translation, Theorem 3.9 becomes

Theorem 5.5. *In case* (II), δ *and* σ *are compatible if and only if, for each infinite sequence* $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k, \dots$ *of integers* 1 *and* 2 *not eventually constant,*

$$\delta^{-1}\sigma^{\varepsilon_1}\delta^{-1}\sigma^{\varepsilon_2} \dots \delta^{-1}\sigma^{(\varepsilon_{k_n}-1)}\big(\delta^{-1}\sigma^2(s)\big) -$$
$$-\delta^{-1}\sigma^{\varepsilon_1}\delta^{-1}\sigma^{\varepsilon_2} \dots \delta^{-1}\sigma^{(\varepsilon_{k_n}-1)}\big(\sigma\delta^{-1}\sigma^2(s)\big) \to 0$$

as $n \to \infty$, *where* $\varepsilon_{k_1}, \dots, \varepsilon_{k_j}, \dots$ *is the subsequence of* $\varepsilon_1, \dots, \varepsilon_k, \dots$ *obtained by deleting all occurrences of* 1.

As was true in case (I), if δ and σ are compatible on $[0, s]$, there is a unique Archimedean T whose diagonal δ_* or s-section σ_* is any prescribed extension of δ or σ. To generate T, we extend ψh as follows. Given δ_*, let $x_n = \delta_*^n(s)$ for all $n < 0$, and extend via $2^n\psi h\delta_*^{-n}(x)$ when $x_{n+1}x < x_n$; given σ_*, by $\psi h\sigma_*(x) - 1$ when $s < x \leqq 1$. Uniqueness follows either from Theorem 3.8 or from parts (b) of Theorems 4.1. and 4.2.

6. Remarks

(1) To each of the theorems on Archimedean T in Sections 4 and 5 there corresponds a dual result for the class of associative functions T' on $[0, 1]^2$ that are continuous, are non-decreasing in each place, satisfy the boundary conditions

$$T'(0, x) = T'(x, 0) = x, \quad T'(1, x) = T'(x, 1) = 1$$

and satisfy

$$T'(x, x) > x \quad \text{when} \quad 0 < x < 1.$$

Each T' admits the representation (R) of Section 2 for a strictly increasing generator g from $[0, 1]$ to $[0, a]$. Therefore, with the obvious changes in notation and assumptions, all of the results in Section 2 have counterparts for the T'. In particular, the conclusions of Theorems 2.3 and 2.5 are valid verbatim.

To get the dual sets of uniqueness for the T', reflect each of the sets in Section 4 about the point (1/2, 1/2). And, upon reversing (I) and (II) of Section 5, one is led to identical conditions for compatibility.

(2) The representation theorem is valid for Archimedean functions defined on any interval. Thus, while we have stated our results for T on [0, 1] in order to simplify the exposition, everything in Sections 4 and 5 is, with transparent modifications, true in the more general setting.

(3) Each of the sets of uniqueness $D \cup S$ obtained in Section 4 is *minimal* in the sense that the removal of any interval yields a set on which T is never uniquely determined. (Observe the exception that must be made in cases 1b and 2b.) For, in fact, should an interval be removed from either $g(D)$ or $g(S)$, it is always possible to construct common solutions φ, other than the identity, to equations (S) and (A) of that section.

(4) The existence proofs presented in Section 3 can be used, at least in principle, to construct solutions of the Abel—Schröder systems. In case 1, for instance, the instructions given in Theorem 3.2, together with equation (1), yield the solution φ. For all but trivial ξ, however, the limit function φ is difficult to construct. On the other hand, one can always find the unique u for which $\varphi(u)$ is any given dyadic rational by applying equation (3) with $y=1$.

(5) It would be desirable to find simple, neat conditions on ξ, say, of Section 3 to replace those occurring in Theorems 3.4 and 3.5. Part of the difficulty in finding any such conditions is that when any composite $\xi_{n_1} \dots \xi_{n_k}$ has two fixed points x and y, then equation (3) of that section implies that $f(x)=f(y)$, and, thus, that the continuous solution to the Abel—Schröder system can not be strictly increasing.

The best we have been able to get is the following sufficient condition, a corollary to Theorem 3.5:

K is dense in [1, 2] if

$$\frac{\xi(y)-\xi(x)}{y-x} \leqq 2$$

when $\frac{1}{2} \leqq x \leqq y \leqq 1$ and there exists an $M<1$ such that for all $n>1$.

$$\frac{\xi(y)-\xi(x)}{y-x} \leqq M\, 2^n$$

when $\frac{1}{2^n} \leqq x<y \leqq \frac{1}{2^{n-1}}$.

From this it can be shown that K is dense in [1, 2] whenever ξ is concave or when ξ is convex and its graph lies above the line $y=2x$ on [0, 1).

For case (I) of Section 5, there is a very weak counterpart to the bounded slope condition above: δ and σ are compatible if there exists an $N \geqq 1$ and an $M<N$ for which

$$\frac{\delta(y)-\delta(x)}{y-x} \geqq N, \quad \frac{\sigma(y)-\sigma(x)}{y-x} \leqq M$$

when $s \leqq x<y \leqq 1$. This follows directly from Theorem 5.3. Case (II) of Section 5 admits no similar slope conditions.

(6) We illustrate some of our results on extension in case (I) of Section 5 with an example. Let $s=\frac{2}{3}$ and

$$\delta(x) = 2x-1$$

$$\sigma(x) = \frac{2}{3}-3(1-x)^2$$

for $\frac{2}{3}\leqq x\leqq 1$. (Note that δ and σ just fail to meet the above slope requirement.) For this δ there is a simple and natural choice of g, namely

$$g(x) = 3(1-x) \quad \text{for} \quad \frac{1}{3} \leqq x \leqq 1,$$

and so $\xi(u)=g\sigma g^{-1}(u)=u^2+1$ for $0\leqq u\leqq 1$.

By virtue of Remark (5), the solution φ of the system

$$\left.\begin{aligned} \varphi(u^2+1) &= \varphi(u)+1 \\ \varphi(2u) &= 2\varphi(u) \end{aligned}\right\} \quad 0 \leqq u \leqq 1$$

is strictly increasing. Hence, δ and σ are compatible, and their extensions T are generated by extensions of φg to [0, 1]. The construction of any such T thus rests on that of φ (but see Remark (4) above). Various aspects of the structure of extensions T in this and similar examples remain to be studied.

(7) The methods used in Section 3 can be extended to solve a wider class of Abel—Schröder systems; however, the general problem is unsettled.

References

[1] J. Aczél, Lectures on Functional Equations and Their Applications, *Academic Press, New York* (1966).
[2] C. Kimberling, On a class of associative functions, *Publ. Math. (Debrecen)* **20** (1973), 21—39.
[3] G. Krause, A strengthened form of Ling's theorem on associative functions, *Doctoral thesis, Illinois Institute of Technology* (1981).
[4] M. Kuczma, On the Schröder equation, *Rozprawy Math.* **34** *Panstowowe Wydawnictwo Naukowe, Warsaw* (1963).
[5] C.-H. Ling, Representation of associative functions, *Publ. Math. (Debrecen)* **12** (1965), 189—212.
[6] B. Schweizer and A. Sklar, Associative functions and statistical triangle inequalities, *Publ. Math. (Debrecen)* **8** (1961), 169—186.

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(Received October 2, 1981.)