

On some properties of linear recurrences

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1. Introduction

A linear recurrence $G = \{G_n\}_{n=0}^{\infty}$ of order k (>1) is defined by rational integers A_1, A_2, \dots, A_k and by recursion $G_n = A_1 G_{n-1} + A_2 G_{n-2} + \dots + A_k G_{n-k}$ ($n \geq k$), where the initial values G_0, G_1, \dots, G_{k-1} are fixed not all zero rational integers and $A_k \neq 0$. We suppose that the roots $\alpha_1, \alpha_2, \dots, \alpha_k$ of the characteristic polynomial

$$g(x) = x^k - A_1 x^{k-1} - \dots - A_k$$

are distinct and so

$$D = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \dots & \alpha_k^{k-1} \end{vmatrix} = \prod_{1 \leq i < j \leq k} (\alpha_j - \alpha_i) \neq 0.$$

Let x_0, x_1, \dots, x_{k-1} be variables. We define a $k \times k$ determinant M_i from D by rejecting the i -th column ($i=1, 2, \dots, k$) and inscribing x_0, x_1, \dots, x_{k-1} as first column. Thus

$$M_i = \begin{vmatrix} x_0 & 1 & \dots & 1 & 1 & \dots & 1 \\ x_1 & \alpha_1 & \dots & \alpha_{i-1} & \alpha_{i+1} & \dots & \alpha_k \\ x_2 & \alpha_1^2 & \dots & \alpha_{i-1}^2 & \alpha_{i+1}^2 & \dots & \alpha_k^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{k-1} & \alpha_1^{k-1} & \dots & \alpha_{i-1}^{k-1} & \alpha_{i+1}^{k-1} & \dots & \alpha_k^{k-1} \end{vmatrix}.$$

Furthermore we define a polynomial $f(x_0, x_1, \dots, x_{k-1})$ by

$$f(x_0, x_1, \dots, x_{k-1}) = D^{2-k} \prod_{i=1}^k M_i.$$

In case $k=2$

$$f(x_0, x_1) = (x_0 \alpha_1 - x_1)(x_0 \alpha_2 - x_1)$$

and if $g(x)$ is irreducible over the rationals and $K = \mathbf{Q}(\alpha_1)$ is a quadratic number field then it can be written in the form

$$f(x_0, x_1) = N_{K/\mathbf{Q}}(\xi) = N(\xi),$$

where $\xi = x_0\alpha_1 - x_1$. On the other hand

$$f(x_0, x_1) = x_1^2 - (\alpha_1 + \alpha_2)x_1x_0 + \alpha_1\alpha_2x_0^2 = x_1^2 - A_1x_1x_0 - A_2x_0^2$$

and one can easily check by induction that

$$f(G_n, G_{n+1}) = C(-A_2)^n$$

for any integer $n(\geq 0)$, where $C = f(G_0, G_1)$. Thus we have

$$N(\xi) = C(-A_2)^n$$

for $\xi = G_n\alpha_1 - G_{n+1}$. It is a well-known identity in the form

$$G_{n+1}^2 - A_1G_{n+1}G_n - A_2G_n^2 = C(-A_2)^n$$

which was proved first by E. LUCAS for case $G_0 = 0, G_1 = 1$ (see L. E. DICKSON [1], p. 396).

In the following we extend this result to arbitrary integer k and give some applications of it.

2. Results

Let $k \geq 2$ be an integer. Using the previous notations we prove:

Theorem 1. *The polynomial*

$$f(x_0, x_1, \dots, x_{k-1}) = D^{2-k} \prod_{i=1}^k M_i$$

in variables x_0, x_1, \dots, x_{k-1} has rational integer coefficients and the coefficient of x_{k-1}^k is one. Furthermore

$$f(G_n, G_{n+1}, \dots, G_{n+k-1}) = F_0[(-1)^{k-1}A_k]^n$$

for all integer $n \geq 0$, where

$$F_0 = f(G_0, G_1, \dots, G_{k-1}).$$

Let the characteristic polynomial $g(x)$ of the sequence G be irreducible over the rationals with roots $\alpha = \alpha_1, \alpha_2, \dots, \alpha_k$. In this case $K = \mathbf{Q}(\alpha)$ is an algebraic number field of degree k and the algebraic numbers $1, \alpha, \alpha^2, \dots, \alpha^{k-1}$ are linearly independent over the field of rational numbers, furthermore the numbers

$$(1) \quad \xi = a_0 + a_1\alpha + \dots + a_{k-1}\alpha^{k-1}$$

with rational integer a_i 's are algebraic integer in K and the set of ξ 's is an order of K . The following statement is true.

Theorem 2. *If the a_i 's ($i=0, 1, \dots, k-1$) in (1) have the form*

$$a_i = G_{n+k-i-1} - \sum_{j=1}^{k-i-1} A_j G_{n+k-i-j-1},$$

where n is an integer, then the norm of ξ is

$$(2) \quad N(\xi) = N(\xi_0) \cdot [(-1)^{k-1} A_k]^n,$$

where ξ_0 is a special case of ξ determining a_i 's by $n=0$.

Norm form equation (2) can be used to express the units of algebraic number fields. A simple case is the cubic field with negative discriminant. In this case, if $K=\mathbf{Q}(\alpha)$, among the conjugates of α there are one real and two complex algebraic number and the number of the fundamental units in K is one. Let ω be one of the fundamental units. ω is an algebraic integer of degree three and so $K=\mathbf{Q}(\alpha)=\mathbf{Q}(\omega)$. Furthermore K does not contain any roots of unity except ± 1 . Thus all units of K are of the form $\varepsilon = \pm \omega^n$. For the units of K we prove:

Theorem 3. *Let $K=\mathbf{Q}(\omega)$ be an algebraic number field of degree three with negative discriminant. Suppose ω is a fundamental unit in K with minimal defining polynomial*

$$h(x) = x^3 - Ax^2 - Bx - C,$$

where $|C|=1$. Let $G = \{G_n\}_{n=-\infty}^{+\infty}$ be a linear recurrence of order three defined by initial terms $G_0=G_1=0$, $G_2=1$ and by recursion

$$G_n = AG_{n-1} + BG_{n-2} + CG_{n-3},$$

which is also defined for negative subscripts since

$$G_n = (G_{n+3} - AG_{n+2} - BG_{n+1})/C$$

is an integer by $|C|=1$. Then the all units in K are the numbers

$$\varepsilon = \pm [G_n \omega^2 + (G_{n+1} - AG_n) \omega + (G_{n+2} - AG_{n+1} - BG_n)] = \pm \varepsilon_n,$$

where n runs over the integers.

We note that C. KLIORYS [5] also has given a connection between the units of some algebraic number fields and linear recurrences. He showed that in algebraic number field $Q(\omega)$ of degree $2k$, where ω is a root of polynomial $x^{2k} - x^k - 1$, the powers of ω can be expressed in the form $\omega^n = a_0 + a_1 \omega + \dots + a_{2k-1} \omega^{2k-1}$, where the a_i 's are Fibonacci numbers.

In [4] we gave a diophantine representation of the terms of generalized Fibonacci sequence which was the generalization of the results of J. P. JONES [2, 3] on Fibonacci and Lucas sequences. These results can be extended to some third order linear recursive sequences, too.

Theorem 4. *Let $K=\mathbf{Q}(\omega)$ and $G = \{G_n\}_{n=-\infty}^{+\infty}$ be an algebraic number field of degree three and a third order linear recurrence respectively defined in Theorem 3 and let $f(x_0, x_1, x_2)$ be the polynomial defined in Theorem 1. Then the set of the nonnegative values of function*

$$F(x, y, z) = [2 - (f(x, y, z))^2] \cdot |x|$$

is the set of the absolute values of the terms of sequence G , as x, y , and z run over the integers.

3. A lemma

In order to prove the theorems we require a lemma.

Let D be the $k \times k$ determinant defined in section 1 and let $D_{i,j}^{(k)}$ be the $(k-1) \times (k-1)$ determinant obtained from D by rejection of the i -th column and the j -th row ($1 \leq i, j \leq k$). Thus

$$D_{i,j}^{(k)} = \begin{vmatrix} 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_{i-1} & \alpha_{i+1} & \dots & \alpha_k \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_1^{j-2} & \alpha_2^{j-2} & \dots & \alpha_{i-1}^{j-2} & \alpha_{i+1}^{j-2} & \dots & \alpha_k^{j-2} \\ \alpha_1^j & \alpha_2^j & \dots & \alpha_{i-1}^j & \alpha_{i+1}^j & \dots & \alpha_k^j \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_1^{k-1} & \alpha_2^{k-1} & \dots & \alpha_{i-1}^{k-1} & \alpha_{i+1}^{k-1} & \dots & \alpha_k^{k-1} \end{vmatrix}.$$

In case $j=k$ we denote $D_{i,j}^{(k)}$ by $D_i^{(k)}$, i.e.

$$D_i^{(k)} = D_{i,k}^{(k)}.$$

Using this notations, we prove:

Lemma. For every indices i, j ($1 \leq i, j \leq k$)

$$(3) \quad D_{i,j}^{(k)} = D_i^{(k)} \cdot S_i^{(k-j)},$$

where $S_i^{(k-j)}$ denotes the elementary symmetrical polynomial of degree $k-j$ of variables $\alpha_1, \alpha_2, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_k$ for $k-j > 0$ and $S_i^{(0)} = 1$.

PROOF. The statement is obviously true in cases $j=1$ and $j=k$, furthermore it can be easily seen for $k=2$ and $k=3$ with arbitrary i and j . Thus we can suppose that $i > 1, 1 < j < k, k > 3$ and it is enough to prove that

$$D_{i,j}^{(k-1)} = D_i^{(k-1)} \cdot S_i^{(k-1-j)}$$

implies equation (3).

Subtracting the first column from the others in determinant $D_{i,j}^{(k)}$, we get $1, 0, 0, \dots, 0$ as first row and so $D_{i,j}^{(k)}$ is equal to a $(k-2) \times (k-2)$ determinant with row-vectors

$$v_t = (\alpha_2^t - \alpha_1^t, \alpha_3^t - \alpha_1^t, \dots, \alpha_{i-1}^t - \alpha_1^t, \alpha_{i+1}^t - \alpha_1^t, \dots, \alpha_k^t - \alpha_1^t)$$

($1 \leq t \leq k-1, t \neq j-1$). The column-vectors of this determinant are divisible by $\alpha_2 - \alpha_1, \alpha_3 - \alpha_1, \dots, \alpha_{i-1} - \alpha_1, \alpha_{i+1} - \alpha_1, \dots$ and $\alpha_k - \alpha_1$ respectively, therefore

$$(4) \quad D_{i,j}^{(k)} = D' \prod_{\substack{2 \leq r \leq k \\ r \neq i}} (\alpha_r - \alpha_1),$$

where D' is a $(k-2) \times (k-2)$ determinant. The coordinates of the row-vectors $v'_t = (c_{t,2}, c_{t,3}, \dots, c_{t,i-1}, c_{t,i+1}, \dots, c_{t,k})$ of determinant D' are

$$c_{t,r} = \alpha_r^{t-1} + \alpha_r^{t-2} \alpha_1 + \dots + \alpha_r \alpha_1^{t-2} + \alpha_1^{t-1}$$

($2 \leq r \leq k, r \neq i$) and so

$$v'_t - \alpha_1 v'_{t-1} = (\alpha_2^{t-1}, \alpha_3^{t-1}, \dots, \alpha_{i-1}^{t-1}, \alpha_{i+1}^{t-1}, \dots, \alpha_k^{t-1})$$

for $1 < t \neq j$ and

$$v'_j - \alpha_1^2 v'_{j-2} = (\alpha_2^{j-1} + \alpha_2^{j-2} \alpha_1, \alpha_3^{j-1} + \alpha_3^{j-2} \alpha_1, \dots, \alpha_{i-1}^{j-1} + \alpha_{i-1}^{j-2} \alpha_1, \alpha_{i+1}^{j-1} + \alpha_{i+1}^{j-2} \alpha_1, \dots, \alpha_k^{j-1} + \alpha_k^{j-2} \alpha_1)$$

if $j \neq 2$. The first row of D' is $v'_1 = (1, 1, \dots, 1)$ in case $j \neq 2$ and $v'_1 = (\alpha_2 + \alpha_1, \alpha_3 + \alpha_1, \dots, \alpha_k + \alpha_1)$ in case $j = 2$. From these follows, using the elementary properties of determinants and our notations, that

$$D' = D_{i-1, j-1}^{(k-1)} + \alpha_1 \cdot D_{i-1, j}^{(k-1)}$$

and the determinants on right are constructed by $\alpha_2, \alpha_3, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_k$. By our supposition we have

$$(5) \quad D' = D_{i-1}^{(k-1)} \cdot S_i^{(k-j)} + \alpha_1 \cdot D_{i-1}^{(k-1)} \cdot S_i^{(k-1-j)}$$

where $S_i^{(t)}$ is the elementary symmetrical polynomial of degree t of variables $\alpha_2, \alpha_3, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_k$.

But

$$D_{i-1}^{(k-1)} \prod_{\substack{2 \leq r \leq k \\ r \neq i}} (\alpha_r - \alpha_1) = D_i^{(k)}$$

and

$$S_i^{(k-j)} + \alpha_1 \cdot S_i^{(k-1-j)} = S_i^{(k-j)}$$

so by (4) and (5) we get

$$D_{i, j}^{(k)} = D_i^{(k)} \cdot S_i^{(k-j)}.$$

Thus the Lemma is true for every integer $k \geq 2$.

4. Proofs of the theorems

PROOF OF THEOREM 1. The discriminant M_i is a linear form of variables x_0, x_1, \dots, x_{k-1} and, by the Lemma, the coefficient of x_{j-1} ($1 \leq j \leq k$) is

$$(-1)^{j-1} \cdot D_{i, j}^{(k)} = (-1)^{j-1} \cdot D_i^{(k)} \cdot S_i^{(k-j)}.$$

From this follows, using the identity

$$\prod_{i=1}^k D_i^{(k)} = D^{k-2},$$

that the coefficient of $x_0^{r_0} \cdot x_1^{r_1} \dots x_{k-1}^{r_{k-1}}$ ($0 \leq r_i \leq k, r_0 + r_1 + \dots + r_{k-1} = k$) in the polynomial $f(x_0, x_1, \dots, x_{k-1})$ is

$$(-1)^q \cdot Q(r_0, r_1, \dots, r_{k-1}),$$

where $q = 0 \cdot r_0 + 1 \cdot r_1 + 2 \cdot r_2 + \dots + (k-1) \cdot r_{k-1}$ and the value of $Q(r_0, r_1, \dots, r_{k-1})$ is

$$\Sigma (S_{i_1}^{(k-1)} \cdot S_{i_2}^{(k-1)} \dots S_{i_{r_0}}^{(k-1)}) \dots (S_{t_1}^{(0)} \cdot S_{t_2}^{(0)} \dots S_{t_{r_{k-1}}}^{(0)}).$$

The summation is extended to such permutations $i_1, i_2, \dots, i_{r_0}, \dots, t_1, t_2, \dots, t_{r_{k-1}}$ of elements $1, 2, \dots, k$ that the elements $(i_1, i_2, \dots, i_{r_0}), \dots$ and $(t_1, t_2, \dots, t_{r_{k-1}})$ respectively are not permuted among themselves. The sum is a symmetrical polynomial

in variables $\alpha_1, \alpha_2, \dots, \alpha_k$, therefore $f(x_0, x_1, \dots, x_{k-1})$ really has integer coefficients and the coefficient of x_{k-1}^k is

$$(-1)^q \cdot Q(0, 0, \dots, 0, k) = S_1^{(0)} \cdot S_2^{(0)} \dots S_k^{(0)} = 1$$

since $S_i^{(0)}=1$ and $q=(k-1) \cdot k$ is an even integer.

We have to prove yet the identity

$$f(G_n, G_{n+1}, \dots, G_{n+k-1}) = F_0[(-1)^{k-1}A_k]^n.$$

It is enough to prove that

$$f(G_{n+1}, G_{n+2}, \dots, G_{n+k}) = (-1)^{k-1}A_k \cdot f(G_n, G_{n+1}, \dots, G_{n+k-1})$$

for any $n \geq 0$, since the statement is trivial for $n=0$.

Let $M_i^{(n)}$ denote the $k \times k$ determinant which we obtain from M_i replacing the element x_0, x_1, \dots, x_{k-1} by $G_t, G_{t+1}, \dots, G_{t+k-1}$ respectively. Let us consider the determinant

$$M_i^{(n+1)} = \begin{vmatrix} G_{n+1} & 1 & \dots & 1 & 1 & \dots & 1 \\ G_{n+2} & \alpha_1 & \dots & \alpha_{i-1} & \alpha_{i+1} & \dots & \alpha_k \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ G_{n+k} & \alpha_1^{k-1} & \dots & \alpha_{i-1}^{k-1} & \alpha_{i+1}^{k-1} & \dots & \alpha_k^{k-1} \end{vmatrix}$$

and let us subtract the first, second, ... and $(k-1)$ -th row multiplied by A_{k-1}, A_{k-2}, \dots and A_1 respectively from the k -th row. Since

$$G_{n+k} - A_1 G_{n+k-1} - \dots - A_{k-2} G_{n+2} - A_{k-1} G_{n+1} = A_k G_n$$

by the definition of sequence G and

$$\begin{aligned} & \alpha_j^{k-1} - A_1 \alpha_j^{k-2} - \dots - A_{k-2} \alpha_j - A_{k-1} = \\ & = \frac{1}{\alpha_j} (\alpha_j^k - A_1 \alpha_j^{k-1} - \dots - A_{k-1} \alpha_j - A_k) + \frac{A_k}{\alpha_j} = \frac{A_k}{\alpha_j} \end{aligned}$$

for $j=1, 2, \dots, k$ by the definition of α_j , furthermore

$$(6) \quad A_k = (-1)^{k+1} \alpha_1 \alpha_2 \dots \alpha_k,$$

we have

$$\begin{aligned} M_i^{(n+1)} &= A_k \begin{vmatrix} G_{n+1} & 1 & \dots & 1 \\ G_{n+2} & \alpha_1 & \dots & \alpha_k \\ \vdots & \vdots & & \vdots \\ G_{n+k-1} & \alpha_1^{k-1} & \dots & \alpha_k^{k-1} \\ G_n & 1/\alpha_1 & \dots & 1/\alpha_k \end{vmatrix} = (-1)^{k+1} \alpha_i \begin{vmatrix} G_{n+1} & \alpha_1 & \dots & \alpha_k \\ G_{n+2} & \alpha_1^2 & \dots & \alpha_k^2 \\ \vdots & \vdots & & \vdots \\ G_{n+k-1} & \alpha_1^{k-1} & \dots & \alpha_k^{k-1} \\ G_n & 1 & \dots & 1 \end{vmatrix} = \\ &= (-1)^{k+1} \cdot (-1)^{k-1} \alpha_i \begin{vmatrix} G_n & 1 & \dots & 1 \\ G_{n+1} & \alpha_1 & \dots & \alpha_k \\ G_{n+2} & \alpha_1^2 & \dots & \alpha_k^2 \\ \vdots & \vdots & & \vdots \\ G_{n+k-1} & \alpha_1^{k-1} & \dots & \alpha_k^{k-1} \end{vmatrix} = \alpha_i M_i^{(n)}. \end{aligned}$$

Thus, using identity (6), we get

$$\begin{aligned} f(G_{n+1}, G_{n+2}, \dots, G_{n+k}) &= D^{2-k} \cdot \prod_{i=1}^k M_i^{(n+1)} = D^{2-k} \cdot \prod_{i=1}^k \alpha_i M_i^{(n)} = \\ &= \alpha_1 \cdot \alpha_2 \dots \alpha_k \cdot D^{2-k} \cdot \prod_{i=1}^k M_i^{(n)} = (-1)^{k-1} A_k \cdot f(G_n, G_{n+1}, \dots, G_{n+k-1}), \end{aligned}$$

which proves the last statement of Theorem 1.

PROOF OF THEOREM 2. First we determine the elementary symmetrical polynomial $S_i^{(t)}$ ($0 \leq t \leq k-1$) by α_i and by the elementary symmetrical polynomials $A_1, -A_2, \dots, (-1)^{k+1} A_k$ of variables $\alpha_1, \alpha_2, \dots, \alpha_k$. On account of symmetry we consider only the case $i=1$. If $t=0$ then, by the definition, $S_1^{(0)}=1$, furthermore obviously

$$S_1^{(1)} = A_1 - \alpha_1 = -(\alpha_1 - A_1)$$

and

$$S_1^{(2)} = -A_2 - \alpha_1 S_1^{(1)} = \alpha_1^2 - A_1 \alpha_1 - A_2.$$

In general we have

$$(7) \quad S_1^{(t)} = (-1)^t \cdot (\alpha_1^t - A_1 \alpha_1^{t-1} - A_2 \alpha_1^{t-2} - \dots - A_{t-1} \alpha_1 - A_t)$$

since if (7) is true for $t=r-1$ (≥ 1), then

$$\begin{aligned} S_1^{(r)} &= (-1)^{r+1} A_r - \alpha_1 S_1^{(r-1)} = (-1)^{r+1} A_r - (-1)^{r-1} \alpha_1 (\alpha_1^{r-1} - A_1 \alpha_1^{r-2} - \dots - A_{r-1}) = \\ &= (-1)^r \cdot (\alpha_1^r - A_1 \alpha_1^{r-1} - \dots - A_{r-1} \alpha_1 - A_r). \end{aligned}$$

Using (7) and the results obtained in the proof of Theorem 1, we get

$$\begin{aligned} M_1 &= \sum_{j=1}^k (-1)^{j-1} D_1^{(k)} S_1^{(k-j)} x_{j-1} = \\ &= (-1)^{k-1} D_1^{(k)} \sum_{j=1}^k (\alpha_1^{k-j} - A_1 \alpha_1^{k-j-1} - \dots - A_{k-j-1} \alpha_1 - A_{k-j}) x_{j-1} = \\ &= (-1)^{k-1} D_1^{(k)} \sum_{i=0}^{k-1} \left(x_{k-i-1} - \sum_{j=1}^{k-i-1} A_j x_{k-i-j-1} \right) \alpha_1^i. \end{aligned}$$

If x_0, x_1, \dots, x_{k-1} are rational integers then

$$\xi = \sum_{i=0}^{k-1} \left(x_{k-i-1} - \sum_{j=1}^{k-i-1} A_j x_{k-i-j-1} \right) \alpha_1^i$$

is an algebraic integer in the number field $K=\mathbf{Q}(\alpha_1)$. Denote the conjugates of ξ by $\xi^{(1)}=\xi, \xi^{(2)}, \dots, \xi^{(k)}$. Now, by the definition of the polynomial f , we get

$$\begin{aligned} f(x_0, x_1, \dots, x_{k-1}) &= D^{2-k} \cdot \prod_{i=1}^k M_i = \\ &= D^{2-k} \cdot \prod_{i=1}^k [(-1)^{k-1} D_i^{(k)} \xi^{(i)}] = D^{2-k} \cdot D^{k-2} \cdot \prod_{i=1}^k \xi^{(i)} = N(\xi) \end{aligned}$$

and from this, by Theorem 1, the statement follows.

PROOF OF THEOREM 3. We can use the previous results taking into our consideration that in this case $k=3, \alpha_1=\omega=\omega_1, \alpha_2=\omega_2, \alpha_3=\omega_3, A_1=A, A_2=B,$ and $A_3=C,$ where $\omega_1, \omega_2, \omega_3$ are the roots of the polynomial $h(x)$.

By the results, obtained in the proof of Theorem 2, we have

$$M_1^{(n)} = D_1^{(3)} \cdot \varepsilon_n,$$

where $\varepsilon_n = G_n \omega^2 + (G_{n+1} - AG_n)\omega + (G_{n+2} - AG_{n+1} - BG_n)$. We have seen in the proof of Theorem 1 that

$$M_1^{(n+1)} = \omega M_1^{(n)}$$

which implies the equality

$$M_1^{(n)} = \omega^n \cdot M_1^{(0)}.$$

But

$$D_1^{(3)} = M_1^{(0)} = \begin{vmatrix} 1 & 1 \\ \omega_2 & \omega_3 \end{vmatrix},$$

and so

$$\varepsilon_n = \omega^n,$$

which proves the statement.

PROOF OF THEOREM 4. First we show that $f(x, y, z)=0$ for rational integers x, y, z if and only if $x=y=z=0$. If $f(x, y, z)=0$, then

$$(8) \quad M_i = D_i^{(3)}[x\omega_i^2 + (y - Ax)\omega_i + (z - Ay - Bx)] = 0$$

for some i ($i=1, 2$ or 3). But $D_i^{(3)} \neq 0$, since the roots of $h(x)$ are distinct, and ω_i is an algebraic integer of degree three, therefore (8) implies the equations

$$x = y - Ax = z - Ay - Bx = 0,$$

thus $x=y=z=0$. If $x=y=z=0$ then obviously $f(x, y, z)=0$.

The polynomial $2 - (f(x, y, z))^2$ for integer x, y, z (not all zero) is non-negative if and only if

$$(9) \quad f(x, y, z) = \pm 1.$$

We have seen in the proof of Theorem 2 that

$$f(x, y, z) = N(x\omega^2 + (y - Ax)\omega + (z - Ay - Bx)) = N(\xi)$$

where $\xi = x\omega^2 + (y - Ax)\omega + (z - Ay - Bx)$ and $N(\xi) = N_{K/Q}(\xi)$ with $K = Q(\omega)$, therefore (9) holds if and only if ξ is a unit in the field K . But, by Theorem 3, ξ is a unit in K if and only if $(x, y, z) = (G_n, G_{n+1}, G_{n+2})$ or $(x, y, z) = (-G_n, -G_{n+1}, -G_{n+2})$ for some integer n and so from $F(x, y, z) \equiv 0$

$$F(x, y, z) = |x| = |\pm G_n| = |G_n|$$

follows. This implies the statement since the function $F(x, y, z)$ takes all values $|G_n|$, namely $F(G_n, G_{n+1}, G_{n+2}) = |G_n|$ for every integer n .

References

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