Local polynomials and functional equations

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1. Introduction. Polynomials on groups play an important role in the theory of functional equations because in most cases the general solution of a linear functional equation can be described by means of polynomials. Results of this type can be found in [2], [7], [8], [9], [10]. However, in some applications the equations do not hold for all values of the variables, but only for values in some domain and the above mentioned results cannot be directly applied. In this paper we extend the main results of the theory to a local direction by introducing the notion of local polynomials on topological Abelian groups. Using this notion we solve a general local functional equation and apply our results to a local mean value property.

In this paper **C** denotes the set of complex numbers. If G is a topological Abelian group and H is an Abelian group, further $U\subset G$ is a neighborhood of zero and n is a positive integer then a function $A\colon U^n\to H$ is called locally n-additive, if

$$A(x_1, \dots, x_{i-1}, x_i+\bar{x}_i, x_{i+1}, \dots, x_n) = A(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n)+$$

$$+A(x_1, \dots, x_{i-1}, \bar{x}_i, x_{i+1}, \dots, x_n)$$

holds for $i=1, \dots, n$ whenever $x_1, \dots, x_{i-1}, x_i, \bar{x}_i, x_i+\bar{x}_i, x_{i+1}, \dots, x_n\in U$.

We use the notations

$$A^{(k)}(x, y) = A(x_1, \dots, x_n)$$

with $x_i=x$ for $i=1, \dots, k$ and $x_i=y$ for $i=k+1, \dots, n$ whenever $k=1, \dots, n-1$ and $x, y\in U$. Further let

$$A^{(0)}(x, y) = A(x_1, \dots, x_n)$$

with $x_i=y$ for $i=1, \dots, n$,

$$A^{(n)}(x, y) = A(x_1, \dots, x_n)$$

with $x_i=x$ for $i=1, \dots, n$ and

$$A^{(n)}(x) = A^{(n)}(x, y)$$

for all $x, y\in U$.

Let $D\subset G$ be an open set and $f\colon D\to H$ a function. Then f is called a local polynomial of degree at most n at the point $x_0\in D$, if there exists a neighborhood $U\subset G$ of zero and there exist $A_k\colon U^k\to H$ locally k-additive symmetric functions

$(k=0, 1, \dots, n)$ such that $U+x_0 \subset D$ and

$$f(x) = \sum_{k=0}^{n} A_k^{(k)}(x-x_0)$$

holds whenever $x \in U+x_0$. (Here $U^0 = U$ and we mean by locally 0-additive function an arbitrary constant in H). If f is a local polynomial of degree at most n at every point of D, then f is called a local polynomial of degree at most n on D.

We shall use the difference operators defined by

$$\Delta_{y_1, \dots, y_n} f(x) = \sum_{0 \leq i_1 < \dots < i_k \leq n} (-1)^{n-k} f(x+y_{i_n}+\dots+y_{i_k})$$

whenever $x+y_{i_1}+\dots+y_{i_k} \in D$. In particular we write

$$\Delta_y^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+ky)$$

whenever $x+ky \in D$ for $k=0, 1, \dots, n$.

2. Characterization of local polynomials. If in the definition of locally n-additive function $U=G$, then we simply call A n-additive. Similarly, if in the definition of local polynomials $D=U=G$, then we call f a polynomial. It is wellknown (see [2]) that polynomials of degree at most n can be characterized by the functional equation:

$$\Delta_{y_1, \dots, y_{n+1}} f(x) = 0$$

for all $x, y_1, \dots, y_{n+1} \in G$ whenever H is divisible and torsion-free. Our first theorem is a similar characterization for local polynomials. In order to prove it we need a simple lemma which can easily be proved by induction on n (see [2]):

Lemma 2.1. *Let G be a topological Abelian group, H an Abelian group, $U \subset G$ a neighborhood of zero, n a positive integer and A: $U^n \to H$ be a locally n-additive function. Then*

i) $$A^{(n)}(x+y) = \sum_{k=0}^{n} \binom{n}{k} A^{(k)}(x, y), \quad \textit{whenever} \quad x, y, x+y \in U;$$

ii) $$\Delta_{y_1, \dots, y_k} A^{(k)}(x) = \begin{cases} 0 & \textit{if} \quad k > n \\ n!\, A(y_1, \dots, y_n) & \textit{if} \quad k = n, \quad \textit{whenever} \end{cases}$$

$$x+y_{i_1}+\dots+y_{i_k} \in U \quad \textit{for} \quad 0 \leq i_1 < \dots < i_k \leq n.$$

Theorem 2.2. *Let G be a topological Abelian group, H a divisible and torsion-free Abelian gorup, n a nonnegative integer, $D \subset G$ an open set and f: $D \to H$ be a function. Then f is a local polynomial of degree at most n on D if and only if for every $x_0 \in D$ there exists a neighborhood $U \subset G$ of zero such that $U+x_0 \subset D$ and*

$$\Delta_{y_1, \dots, y_{n+1}} f(x) = 0$$

whenever $x-x_0, y_1, \dots, y_{n+1} \in U$.

PROOF. (see [2]). The necessity is obvious by lemma 2.1. The sufficiency we prove by induction on n. For $n=0$ the statement is trivial. Supposing that it is proved for $k\leqq n-1$ we prove it for $k=n$.

Hence we fix an $x_0\in D$, a neighborhood $U\subset G$ of zero for which $U+x_0\subset D$ and assume that

$$\Delta_{y_1,\dots,y_{n+1}}f(x)=0$$

whenever $x-x_0, y_1, \dots, y_{n+1}\in U$. It follows that for all $y_1, \dots, y_n$ in U, the function $\Delta_{y_1,\dots,y_n}f$ is constant on $U+x_0$. Let

$$A_n(y_1, \dots, y_n)=\frac{1}{n!}\Delta_{y_1,\dots,y_n}f(x_0).$$

Obviously A_n is symmetric. If $y, \bar{y}_1, y_1+\bar{y}_1, y_2, \dots, y_n\in U$, then

$$A_n(y_1+\bar{y}_1, y_2, \dots, y_n)-A_n(y_1, y_2, \dots, y_n)-A_n(\bar{y}_1, y_2, \dots, y_n)=$$

$$=\frac{1}{n!}(\Delta_{y_1+\bar{y}_1}-\Delta_{y_1}-\Delta_{\bar{y}_1})\Delta_{y_2,\dots,y_n}f(x_0)=\frac{1}{n!}\Delta_{y_1,\bar{y}_1,y_2,\dots,y_n}f(x_0)=0$$

and hence A_n is locally n-additive. By lemma 2.1. we have

$$\Delta_{y_1,\dots,y_n}A_n^{(n)}(x-x_0)=n!\,A_n(y_1, \dots, y_n)$$

whenever $x-x_0, y_1, \dots, y_n\in V$, where $V\subset U$ is an appropriate neighborhood of zero. Let $D_1=V+x_0$ and

$$g(x)=f(x)-A_n^{(n)}(x-x_0)$$

whenever $x\in D_1$. Obviously for every $x_1\in D_1$ there exists a neighborhood $U_1\subset D$ of zero such that $U_1+x_1\subset D_1$ and $x-x_1, y_1, \dots, y_n\in U_1$ implies $x+y_1+\dots+y_n\in D_1$ further

$$\Delta_{y_1,\dots,y_n}g(x)=\Delta_{y_1,\dots,y_n}f(x)-\Delta_{y_1,\dots,y_n}A_n^{(n)}(x-x_0)=$$

$$=n!\,A_n(y_1, \dots, y_n)-n!\,A_n(y_1, \dots, y_n)=0.$$

Hence by induction g is a local polynomial on D_1 of degree at most $n-1$, and our theorem is proved.

3. The local extension property. In this paragraph we deal with the following problem: when has each local polynomial on G a unique extension to a polynomial? We say that the topological Abelian group G has the local extension property with respect to the Abelian group H, if there is a base $\mathcal{U}$ for the neighborhoods of zero in G with the following property: if $U\in\mathcal{U}$ and $A\colon U\to H$ is locally additive, then A has a unique extension $\tilde{A}\colon G\to H$ which is additive. A base $\mathcal{U}$ with this property we call an extension base.

Lemma 3.1. *Let G be a topological Abelian group having the local extension property with respect to the divisible and torsion-free Abelian group H with the extension base $\mathcal{U}$. If n is a positive integer, $U\in\mathcal{U}$ and $A\colon U^n\to H$ is a locally n-additive symmetric function, then A has a unique extension $\tilde{A}\colon G^n\to H$, which is n-additive and symmetric.*

PROOF. We prove by induction on n. Suppose that we have proved the statement for $k \leqq n$ and let $A: U^{n+1} \to H$ be a locally $(n+1)$-additive symmetric function. Let $x \in U$ be fixed and $(y_1, \ldots, y_n) \to B(x, y_1, \ldots, y_n)$ be an n-additive symmetric extension of the locally n-additive symmetric function $(y_1, \ldots, y_n) \to A(x, y_1, \ldots, y_n)$. If $x, y, x+y \in U$ then the functions $(y_1, \ldots, y_n) \to B(x, y_1, \ldots, y_n) + B(y, y_1, \ldots, y_n)$ and $(y_1, \ldots, y_n) \to B(x+y, y_1, \ldots, y_n)$ are n-additive symmetric extensions of the locally n-additive symmetric function $(y_1, \ldots, y_n) \to A(x+y, y_1, \ldots, y_n)$ and hence by the uniqueness it follows that B is locally additive in the first variable. Now fix $y_1, \ldots, y_n \in G$ and let $x \to \tilde{B}(x, y_1, \ldots, y_n)$ be the additive extension of

$$x \to B(x, y_1, \ldots, y_n).$$

Then for every $y_i, \bar{y}_i \in G$ the functions

$$x \to B(x, y_1, \ldots, y_{i-1}, y_i + \bar{y}_i, y_{i+1}, \ldots, y_n)$$

and

$$x \to \tilde{B}(x, y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_n) + \tilde{B}(x, y_1, \ldots, y_{i-1}, \bar{y}_i, y_{i+1}, \ldots, y_n)$$

are additive extensions of the locally additive function

$$x \to B(x, y_1, \ldots, y_{i-1}, y_i + \bar{y}_i, y_{i+1}, \ldots, y_n),$$

hence they are equal. It means that $\tilde{B}$ is additive in each variable and by symmetrization we have

$$\tilde{A}(y_1, y_2, \ldots, y_{n+1}) = \frac{1}{(n+1)!} \sum_{1 \leqq i_j \neq i_k \leqq n+1} \tilde{B}(y_{i_1}, y_{i_2}, \ldots, y_{i_{n+1}})$$

which is obviously a unique $(n+1)$-additive symmetric extension of A.

Lemma 3.2. *Let G be a local Abelian group having the local extension property with respect to the divisible and torsion-free Abelian group H and n be a positive integer. If $D \subset G$ is an open set and $f: D \to H$ is a local polynomial of degree at most n at the point $x_0 \in D$, then there exists a neighborhood $U \subset G$ of zero and a unique polynomial $P: G \to H$ of degree at most n such that $U + x_0 \subset D$ and*

$$f(x) = P(x - x_0)$$

whenever $x \in U + x_0$.

PROOF. The existence of P is a consequence of lemma 3.1., and we have to show only the uniqueness. Let $\mathcal{U}$ be an extension base of G and suppose that $P, Q: G \to H$ are polynomials of degree at most n such that $P(x) = Q(x)$ whenever $x \in U$ for some $U \in \mathcal{U}$. It means that

$$P(x) = \sum_{k=0}^{n} A_k^{(k)}(x) = \sum_{k=0}^{n} B_k^{(k)}(x) = Q(x)$$

holds for $x \in U$, where $A_k, B_k: G^k \to H$ are k-additive symmetric functions $(k = 0, 1, \ldots, n)$. Let $V \subset U$ be a neighborhood of zero for which $V \in \mathcal{U}$ and $x, y_1, \ldots, y_n \in V$ implies $x + y_1 + \ldots + y_n \in U$.

Then by lemma 2.1. we have

$$A_n(y_1, \ldots, y_n) = \frac{1}{n!} \Delta_{y_1, \ldots, y_n} A_n^{(n)}(x) = \frac{1}{n!} \Delta_{y_1, \ldots, y_n} B_n^{(n)}(x) = B_n(y_1, \ldots, y_n)$$

whenever $x, y_1, \ldots, y_n \in V$. That is, A_n and B_n are the same on V^n. By the local extension property of G it follows $A_n = B_n$ and continuing this procedure we have $P = Q$.

Theorem 3.3. *Let G be a topological Abelian group having the local extension property with respect to the divisible and torsion free Abelian group H and let $D \subset G$ be an open and path-connected set. If $f\colon D \to H$ is a local polynomial of degree at most n on D, then f is the restriction of a unique polynomial $P\colon G \to H$ of degree at most n.*

PROOF. The conditions of the theorem and lemma 3.2. imply that for every $x \in D$ there exists a neighborhood U_x of the zero and a polynomial $P_x\colon G \to H$ of degree at most n such that $U_x + x \subset D$ and

$$f(y) = P_x(y-x)$$

whenever $y \in U_x + x$. Let $Q_x(y) = P_x(y-x)$ whenever $x \in D$, $y \in G$. Then $Q_x\colon G \to H$ is a polynomial of degree at most n. First we show that $(U_x+x) \cap (U_y+y) \neq \emptyset$ implies $Q_x = Q_y$. Let $\mathcal{U}$ denote an extension base of G and let $z \in (U_x+x) \cap (U_y+y)$, $W \in \mathcal{U}$ such that $W + z \subset (U_x+x) \cap (U_y+y)$. For all $w \in W$ we have $w + z \in U_x + x$ and $w + z \in U_y + y$, hence

$$Q_x(w+z) = P_x(w+z-x) = f(w+z) = P_y(w+z-y) = Q_y(w+z)$$

that is, the polynomials $w \to Q_x(w+z)$ and $w \to Q_y(w+z)$ are equal on W, and by the local extension property they are identical: $Q_x = Q_y$. Now let $x, y \in D$. We have to show that $Q_x = Q_y$. As D is path-connected, there is a continuous function $\varphi\colon [0, 1] \to D$ with $\varphi(0) = x$, $\varphi(1) = y$. The set $R = \operatorname{range} \varphi$ is a compact connected subset of D and the open sets $U_x + x$ for $x \in R$ cover it. Hence there exist elements $x_1, \ldots, x_k \in D$ such that $(U_{x_i} + x_i) \cap (U_{x_{i+1}} + x_{i+1}) \neq \emptyset$ $(i = 1, \ldots, k-1)$ and $x \in U_{x_1} + x_1$, $y \in U_{x_k} + x_k$. It follows that $Q_x = Q_{x_1} = Q_{x_k} = Q_y$ and the theorem is proved.

Theorem 3.4. *If G is the additive topological Abelian group of a locally convex topological vectorspace then G has the local extension property with respect to any Abelian group.*

PROOF. Let $\mathcal{U}$ be the set of all balanced absorbing neighborhoods of zero in G. Then $\mathcal{U}$ is a base for the neighborhoods of zero. Let H be an Abelian group, $U \in \mathcal{U}$ and $A\colon U \to H$ be a locally additive function. First we extend A to the set $2U$ by the formula

$$A_1(x) = 2A\left(\frac{x}{2}\right)$$

whenever $x \in 2U$. As U is balanced, hence $U \subset 2U$ and obviously A_1 is a locally additive extension of A. Continuing this process we have the locally additive functions $A_n\colon 2^{n+1}U \to H$ with A_{n+1} is an extension of A_n. As U is absorbing, $\bigcup_{n=0}^{\infty} 2^n U = G$ and the function $\tilde{A}$ defined by

$$\tilde{A}(x) = A_n(x)$$

whenever $x \in 2^{n+1}U$ is a well-defined additive extension of A. For the uniqueness let $\tilde{A}, \tilde{B}\colon G \to H$ be additive extensions of A and let $x \in G$. Then $n^{-1}x \in U$ for some positive integer n and hence

$$\tilde{A}(x) = n\tilde{A}(n^{-1}x) = nA(n^{-1}x) = n\tilde{B}(n^{-1}x) = \tilde{B}(x)$$

which proves the theorem.

4. A local functional equation

Theorem 4.1. *Let G be a topological Abelian group, H a divisible and torsion-free Abelian group, φ_i a local isomorphism of G $(i=1, \ldots, n+1)$. Let $D\subset G$ be an open set and f, f_i: $D\to H$ be functions $(i=1, \ldots, n+1)$ for which*

$$f(x)+\sum_{i=1}^{n+1} f_i\big(x+\varphi_i(y)\big) = 0 \tag{1}$$

holds whenever $x, x+\varphi_i(y)\in D$ $(i=1, \ldots, n+1)$. Then f is a local polynomial of degree at most n on D.

Proof. Suppose that $U_0\subset G$ is a symmetric neighborhood of the zero contained in the range of φ_i for which $\varphi_i(x)$, $\varphi_i(y)$ and $\varphi_i(x+y)$ is defined whenever $x, y\in U_0$ $(i=1, \ldots, n+1)$. First we prove the following statement: for every $x_0\in D$ there exists an open set $D_1\subset D$ containing x_0 and there exists a neighborhood $V\subset G$ of zero such that for all $t\in V$ there exists $s\in G$ with the property, that $x, x+\varphi_i(y)\in D_1$ $(i=1, \ldots, n)$ implies $x+t, x+t+\varphi_i(y+s)\in D$ $(i=1, \ldots, n+1)$ and $t+\varphi_{n+1}(s)=0$. Further the sets D_1, V and the element s are independent of f, f_i $(i=1, \ldots, n+1)$.

Let $V_1\subset U_0$ be a symmetric neighborhood of zero for which $V_1+x_0\subset D$. Let $V_2\subset V_1$ be a symmetric neighborhood of zero such that $V_2+V_2\subset V_1$. Then let $V_3\subset V_2$ be a symmetric neighborhood of zero for which $V_3+V_3\subset \varphi_1\circ\varphi_{n+1}^{-1}(V_2)$. Let $D_1=x_0+V_3$, then we have $D_1+V_2\subset D$ and $W=\bigcap_{i=1}^{n}\varphi_i^{-1}(V_3)$ is a symmetric neighborhood of zero. Let $W_1\subset W$ be a symmetric neighborhood of zero such that $W_1+W_1\subset W$ and let $V=\bigcup_{i=1}^{n+1}\varphi_i(W_1)\cap U_0$. For all $t\in V$ we have $t\in U_0$ and hence there is an s such that $t+\varphi_{n+1}(s)=0$, that is $s\in\varphi_{n+1}^{-1}(V)\subset W_1$. Then $\varphi_i(s)\in\varphi_i(W_1)$ and

$$t+\varphi_i(s)\in\varphi_i(W_1)+\varphi_i(W_1)\subset\varphi_i(W_1+W_1)\subset\varphi_i(W)\subset V_3 \quad (i=1, \ldots, n).$$

Let $x, x+\varphi_i(y)\in D_1$ $(i=1, \ldots, n)$, then $x+t\in D_1+V\subset D_1+V_2\subset D$. On the other hand, $x+t+\varphi_i(y+s)=x+\varphi_i(y)+t+\varphi_i(s)\in D_1+V_3\subset D_1+V_2\subset D$ $(i=1, \ldots, n)$. Finally, $x, x+\varphi_1(y)\in D_1$ implies $\varphi_1(y)\in V_3+V_3\subset\varphi_1\circ\varphi_{n+1}^{-1}(V_2)$ and then $y\in\varphi_{n+1}^{-1}(V_2)$, $\varphi_{n+1}(y)\in V_2$, which means $x+\varphi_{n+1}(y)\in D_1+V_2\subset D$.

After proving this statement we can write $x+t$ for x and $y+s$ for y in (1) obing

$$f(x+t)+\sum_{i=1}^{n} f_i\big(x+\varphi_i)\,y(t+\varphi_i(s)\big)+f_{n+1}\big(x+\varphi_{n+1}(y)\big) = 0. \tag{2}$$

By subtraction it follows

$$\Delta_t f(x) + \sum_{i=1}^{n} \Delta_{t+\varphi_i(s)} f_i(x+\varphi_i(y)) = 0$$

for all $x, x+\varphi_i(y) \in D_1$ $(i=1, \ldots, n)$.

Repeating this argument $n+1$ times we get an open set $D_{n+1} \subset D$ containing x_0 and a neighborhood $V_{n+1} \subset G$ of zero such that

$$\Delta_{t_1, \ldots, t_{n+1}} f(x) = 0$$

whenever $x \in D_{n+1}$ and $t_1, \ldots, t_{n+1} \in V_{n+1}$.

Then letting $U = (D_{n+1} - x_0) \cap V_{n+1}$ the conditions of theorem 2.2. are satisfied, hence our theorem is proved.

5. A local mean-value property. Let G be a topological Abelian group, $U \subset G$ a neighborhood of zero and $\varphi\colon U \to U$ a continuous local homomorphism. Suppose that there is a positive integer n for which $\varphi^{n+1}(x) = x$ holds whenever $x \in U$. Obviously in this case φ is a local isomorphism with a continuous inverse, and $\varphi(U) = U$. We call φ a local cyclic isomorphism of degree $n+1$. Suppose, that the elements $x+y$, $x+\varphi(y), \ldots, x+\varphi^n(y)$ are defined for some $x, y \in G$. Then the set $\{x+y, x+\varphi(y), \ldots, x+\varphi^n(y)\}$ is called a regular φ-polygon with center x.

Let $D \subset G$ be an open set and H a divisible and torsion-free Abelian group. The function $f\colon D \to H$ is said to have the φ-mean-value property on D, if the arithmetical mean of the values of f on every regular φ-polygon in D is equal to the value of f at the center of this polygon. More precisely, if then

$$f(x) = \frac{1}{n+1} \sum_{k=0}^{n} f(x+\varphi^k(y)).$$

The following two theorems are consequences of the preceding results.

Theorem 5.1. *Let G be a topological Abelian group, H a divisible and torsion-free Abelian group, n a positive and φ a local cyclic isomorphism of degree $n+1$ on G. Let $D \subset G$ be an open set and $f\colon D \to H$ a function having the φ-mean-value property on D. Then f is a local polynomial of degree at most n on D.*

Theorem 5.2. *In the preceding theorem, if G has the local extension property with respect to H and D is path-connected, then f is the restriction of a polynomial of degree at most n.*

We note that theorems 5.1. and 5.2. are generalizations of well-known results concerning the classical mean-value property in [1], [3], [4], [5], [6], [7], [11].

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