On quadratic set valued functions

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Let $(X, \|\cdot\|)$ be an arbitrary real normed space and let 2^X denote the family of all subsets of X. For $s, t \in \mathbf{R}$ — the set of all real numbers — and sets $A, B \subset X$ we put $sA+tB := \{x \in X\colon x = sa+tb,\ a \in A,\ b \in B\}$. A set valued function (abbreviated to s. v. function in the sequel) $U\colon \mathbf{R} \to 2^X$ will be called quadratic iff for all $s, t \in \mathbf{R}$

$$U(s+t)+U(s-t) = 2U(s)+2U(t). \tag{1}$$

It is easy to see that if a set $F \subset X$ is convex then the s.v. function U defined by the formula

$$U(t) = t^2 F, \quad t \in \mathbf{R}, \tag{2}$$

is quadratic, but in general the converse implication is not true. In this paper we give some conditions for a quadratic s.v. function whose values are convex and compact subsets of X to be of the form (2). Analogous theorems for quadratic functionals were proved by S. KUREPA [3]. The results given here are a generalization of theorems proved by D. HENNEY [2].

Let $C(X)$ denote the collection of all compact and non-empty subsets of X and $CC(X)$ — the family of all convex members of $C(X)$. $C(X)$ with the Hausdorff distance D is a metric space. We say that the s.v. function $U\colon \mathbf{R} \to 2^X$ is bounded on a set $A \subset \mathbf{R}$ iff there exists a ball $K(\Theta, r) := \{x \in X\colon \|x\| < r\}$ such that $U(t) \subset K(\Theta, r)$ for all $t \in A$.

The main result of this paper is the following

Theorem 1. *Let $A \subset \mathbf{R}$ be a set of positive inner Lebesgue measure or of the second category with the Baire property. If an s.v. function $U\colon \mathbf{R} \to CC(X)$ is quadratic and bounded on A then $U(t) = t^2 U(1)$ for all $t \in \mathbf{R}$.*

To prove this theorem we need the following three lemmas.

Lemma 1 (see [5]). *If $A, B, C \in CC(X)$ and k is a positive number, then $D(A+C, B+C) = D(A, B)$ and $D(kA, kB) = kD(A, B)$. In particular, if $A+C = B+C$, then $A = B$.*

Lemma 2. *If an s.v. function $U\colon \mathbf{R} \to CC(X)$ is quadratic, then for every rational number q and every $t \in \mathbf{R}$ we have $U(qt) = q^2 U(t)$.*

PROOF. Putting in equation (1) $s = t = 0$ we get $U(0)+U(0) = 2U(0)+2U(0)$, whence, because of the convexity of $U(0)$, $2U(0) = 4U(0)$. From here $U(0) = \{\Theta\}$, because $U(0)$ is bounded. Now, fix an arbitrary $n\in\mathbf{N}$ ($\mathbf{N}$ denotes here the set of all positive integers) and assume that $U(kt)=k^2U(t)$ for every $k\in\mathbf{N}$, $k\leqq n$. Then using equation (1), we obtain

$$U((n+1)t)+(n-1)^2U(t) = 2n^2U(t)+2U(t),$$

and so, because of the convexity of $U(t)$,

$$U((n+1)t)+(n-1)^2U(t) = (2n^2+2)U(t) = ((n+1)^2+(n-1)^2)U(t) =$$
$$= (n+1)^2U(t)+(n-1)^2U(t).$$

From Lemma 1, we get $U((n+1)t)=(n+1)^2U(t)$. Thus the equality $U(nt)=n^2U(t)$ holds for every $n\in\mathbf{N}$ and $t\in\mathbf{R}$. Since for $m\in\mathbf{N}$ $U(t)=U\left(m\frac{t}{m}\right)=m^2U\left(\frac{t}{m}\right)$, we have also $U\left(\frac{t}{m}\right)=\frac{1}{m^2}U(t)$. This implies $U(qt)=q^2U(t)$ for every positive rational q and $t\in\mathbf{R}$. To finish the proof it suffices to observe that the s.v. function U is even. Indeed, setting $s=0$ in equation (1), we obtain

$$U(t)+U(-t) = 2\{\theta\}+2U(t) = U(t)+U(t),$$

whence, by Lemma 1, $U(-t)=U(t)$. $\square$

Lemma 3. *Let $A\subset\mathbf{R}$ be a set of positive inner Lebesgue measure or of the second category with the Baire property. If a quadratic s.v. function $U\colon \mathbf{R}\to CC(X)$ is bounded on A, then it is bounded on a neighbourhood of zero.*

Proof. Let $K(\Theta, r)$ be such a ball that $U(t)\subset K(\Theta, r)$ for $t\in A$. Consider the set $H(A):=\{t\in\mathbf{R}\colon A\cap(A+t)\cap(A-t)\neq\emptyset\}$. If $t\in H(A)$, then there exists a point $s\in\mathbf{R}$ such that $s, s-t, s+t\in A$. Hence, using equation (1), we obtain

$$U(s)+U(t) = \frac{1}{2}[U(s+t)+U(s-t)]\subset\frac{1}{2}[K(\Theta, r)+K(\Theta, r)] = K(\Theta, r).$$

Let us fix $x\in U(s)$ arbitrarily. Then $x\in K(\Theta, r)$, and so

$$U(t)\subset U(t)+U(s)-x\subset K(\Theta, r)+K(\Theta, r) = K(\Theta, 2r),$$

which means that U is bounded on the set $H(A)$. Since A has a positive inner Lebesgue measure or it is of the second category with the Baire property, $H(A)$ contains a neighbourhood of zero (see [4]). This concludes the proof. $\square$

Proof of Theorem 1. By Lemma 3, there exists a positive rational number ϱ such that the s.v. function U is bounded on the interval $[0, \varrho]$. Hence and from the fact that the set $U(1)$ is compact, there exists a ball $K(\Theta, r)$ containing all the sets $U(t)$ and $t^2U(1)$ for $t\in[0, \varrho]$. Hence $D(U(t), t^2U(1))\leqq 2r$ for $t\in[0, \varrho]$. Let us put

$$d := \sup\{D(U(t), t^2U(1))\colon t\in[0, \varrho]\}$$

and suppose that $d>0$. Then there exists a number $t_0\in(0, \varrho]$ such that $D(U(t_0),$

$t_0^2U(1)) > \frac{3}{4}d$. Without any loss of generality we may assume that $t_0 \in \left[\frac{\varrho}{2}, \varrho\right]$, for otherwise there exists a rational number q such that $qt_0 \in \left[\frac{\varrho}{2}, \varrho\right]$ whence

$$D\big(U(qt_0), q^2 t_0^2 U(1)\big) = q^2 D\big(U(t_0), t_0^2 U(1)\big) > q^2 \frac{3}{4} d > \frac{3}{4} d.$$

Now, the number $2t_0$ can be written in the form $2t_0 = \varrho + s$, where $s \in [0, \varrho]$. Applying Lemma 2 and the triangle inequality we have

$$D\big(U(2t_0),\ (2t_0)^2 U(1)\big) = D\big(U(2t_0) + U(\varrho - s),\ (2t_0)^2 U(1) + U(\varrho - s)\big) \leqq$$
$$\leqq D\big(U(2t_0) + U(\varrho - s),\ (2t_0)^2 U(1) + (\varrho - s)^2 U(1)\big) +$$
$$+ D\big((2t_0)^2 U(1) + (\varrho - s)^2 U(1),\ (2t_0)^2 U(1) + U(\varrho - s)\big).$$

Since $U(\varrho + s) + U(\varrho - s) = 2U(\varrho) + 2U(s)$, $U(\varrho) = \varrho^2 U(1)$ and

$$(\varrho + s)^2 U(1) + (\varrho - s)^2 U(1) = 2\varrho^2 U(1) + 2s^2 U(1), \quad \text{we get}$$
$$D\big(U(2t_0) + U(\varrho - s),\quad (2t_0)^2 U(1) + (\varrho - s)^2 U(1)\big) =$$
$$= D\big(2\varrho^2 U(1) + 2U(s),\ 2\varrho^2 U(1) + 2s^2 U(1)\big) = 2D\big(U(s),\ s^2 U(1)\big) \leqq 2d$$

and

$$D\big((2t_0)^2 U(1) + (\varrho - s)^2 U(1),\ (2t_0)^2 U(1) + U(\varrho - a)\big) =$$
$$= D\big((\varrho - s)^2 U(1),\ U(\varrho - s)\big) \leqq d.$$

Thus, we have $D\big(U(2t_0), (2t_0)^2 U(1)\big) \leqq 3d$.

On the other hand

$$D\big(U(2t_0), (2t_0)^2 U(1)\big) = D\big(4U(t_0), 4t_0^2 U(1)\big) = 4D\big(U(t_0), t_0^2 U(1)\big) > 3d.$$

The contradiction just obtained implies that $d = 0$ and, consequently, $U(t) = t^2 U(1)$ for every $t \in [0, \varrho]$. Now, fix any $t \in \mathbf{R}$ and take a rational $q \neq 0$ such that $qt \in [0, \varrho]$. By Lemma 2 and by the equality which has been proved previously, it follows that

$$U(t) = \frac{1}{q^2} U(qt) = \frac{1}{q^2} q^2 t^2 U(1) = t^2 U(1),$$

which completes the proof. □

As an immediate consequence of this theorem we obtain the following

Corollary 1. *If a s.v. function $U\colon \mathbf{R} \to CC(X)$ is quadratic and continuous at some point then it is of the form* (2).

Now we shall introduce some further definitions. An s.v. function $U\colon \mathbf{R} \to C(X)$ will be called measurable iff for every Borel set $B \subset C(X)$ (the topology in $C(X)$ is generated by the Hausdorff distance D) the counter-image $U^{-1}(B)$ is a Lebesgue measurable set. We say that an s.v. function $U\colon \mathbf{R}\to C(X)$ has the Baire property iff for every open set $A\subset C(X)$ the set $U^{-1}(A)$ has the Baire property. We say that an s.v. function $V\colon \mathbf{R}\to 2^X$ majorizes a s.v. function $U\colon \mathbf{R}\to 2^X$ on a set $F\subset\mathbf{R}$ iff $U(t)\subset V(t)$ for every $t\in F$.

The next theorem presents another sufficient condition for a quadratic s.v. function to be of the form (2).

Theorem 2. *Let $V\colon \mathbf{R}\to C(X)$ be a measurable s.v. function (or an s.v. function with the Baire property) and let $A\subset\mathbf{R}$ be a set of positive Lebesgue measure (or a second category set having the Baire property). If an s.v. function $U\colon \mathbf{R}\to CC(X)$ is quadratic and if V majorizes U on A, then U is of the form* (2).

PROOF. Let $B_n:=\{F\in C(X)\colon F\subset K(\Theta, n)\}$, $n\in\mathbf{N}$. Since compact sets are bounded, $C(X)=\bigcup_{n=1}^{\infty} B_n$. Hence

$$V^{-1}\big(C(X)\big)=\bigcup_{n=1}^{\infty} V^{-1}(B_n)=\bigcup_{n=1}^{\infty}\{t\in\mathbf{R}\colon V(t)\subset K(\Theta, n)\},$$

and on the other hand, $V^{-1}\big(C(X)\big)=\mathbf{R}$. Since V is measurable and B_n are open sets (see Theorem II—6 in [1]), the sets $\{t\in\mathbf{R}\colon V(t)\subset K(\Theta, n)\}$, $n\in\mathbf{N}$, are Lebesgue measurable sets. Therefore there exists an $n_0\in\mathbf{N}$ such that the sets $\{t\in\mathbf{R}\colon V(t)\subset K(\Theta, n_0)\}\cap A$ has a positive Lebesgue measure. Consequently, V and hence also U is bounded on a set of positive Lebesgue measure, which, by Theorem 1, completes the proof of our theorem in the first case. The proof in the second case is quite analogous. □

Corollary 2. *If a quadratic s.v. function $U\colon \mathbf{R}\to CC(X)$ is measurable or has the Baire property, then it is of the form* (2).

Finally, we give some examples of quadratic s.v. functions which are not of the form (2). Cleary, these functions do not fulfil the assumptions of the theorems proved above.

Examples. Consider the following s.v. functions:

$$U_1(t):=t^2+\mathbf{Q},\quad t\in\mathbf{R},$$

where $\mathbf{Q}$ denotes the set of all rational numbers;

$$U_2(t):=[f(t),\ f(t)+t^2],\quad t\in\mathbf{R},$$

where $f\colon \mathbf{R}\to\mathbf{R}$ is a discontinuous quadratic function (i.e. f is discontinuous and satisfies the equation $f(s+t)+f(s-t)=2f(s)+2f(t)$, $s, t\in\mathbf{R}$);

$$U_3(t):=f(t)F,\quad t\in\mathbf{R},$$

where $f\colon \mathbf{R}\to\mathbf{R}$ is a discontinuous and non-negative quadratic function and F is a convex set.

It is easy to see that these s.v. functions are quadratic but they are not of the form (2).

References

[1] C. Castaing—M. Valadier, Convex analisis and measurable multifunctions, *Lecture Notes in Math.* **580** (1977).
[2] D. Henney, Quadratic set-valued function, *Ark. Mat.* **4** (1962), 377—378.
[3] S. Kurepa, On the quadratic functional, *Publ. Inst. Math. Acad. Serbe Sci. Beograd* **13** (1959), 57—72.
[4] K. Nikodem, Wypukłe i kwadratiwe procesy stochastyczne, *Praca doktorska,* Uniwesytet Śląski, *Katowice* 1980.
[5] H. Radström, An embedding theorem for the space of convex sets, *Proc. Amer. Math. Soc.* **3** (1952), 165—169.

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