Attractivity theorems for second order nonlinear differential equations

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1. Introduction

In this paper we give conditions which ensure that all solutions of the equation

(E1) $$\big(p(t)\dot{x}\big)^{\cdot}+q(t)f(x)=e(t)$$

tend to zero as $r\to\infty$.

This equation appears in applications very often. For instance, motions of certain forced mechanical systems of one degree of freedom are described by an equation of such form. In this case $\lim\limits_{t\to\infty} x(t)=0$ means that the system asymptotically approaches to its equilibrium position. A great number of papers have been devoted to establish this property. First the linear case was investigated. One of the best of these results from a theoretical viewpoint is due to G. SANSONE [2]. As the most of oscillations are nonlinear, the examinations were extended to the nonlinear equations [1, 2, 4, 5, 7, 8].

In [7] and [8] F. J. SCOTT gave two sufficient conditions of this property for equation (E1) by different methods. He proposed the question combining these two methods preserving advantages of both of them. In this paper we are going to present such a method.

Throughout the paper the following assumptions are made for functions in [E1]:

(I) $$p,\, q\in C^1[t_0,\infty);\quad p(t),\, q(t)>0;$$

(II) $$f\in C(-\infty,\infty),\quad xf(x)>0\quad (x\neq 0);$$

(III) $$F(x):=2\int_0^x f\to\infty\quad (x\to\infty);$$

(IV) $$e\in L_{\mathrm{loc}}[t_0,\infty);$$

(V) $$\int^{\infty}\frac{[(pq)^{\cdot}]^-}{pq}<\infty;$$

(VI) $$\int^{\infty}\frac{|e|}{(pq)^{1/2}}<\infty,$$

where $[a]^+=\max(a,0)$, $[a]^-=\max(-a,0)$.

The energy

$$E(t) = \frac{p(t)}{q(t)}\dot{x}^2(t) + F(x(t))$$

is used as a Ljapunov function. For its derivative the following estimate holds:

$$\dot{E}(t) = 2\left(\frac{p}{q}\right)^{1/2}(t)\dot{x}(t)\frac{e}{(pq)^{1/2}}(t) - \frac{p(t)}{q(t)}\dot{x}^2(t)\frac{(pq)^{\cdot}}{pq}(t) \leqq \tag{1}$$

$$\leqq 2\frac{|e|}{(pq)^{1/2}}(t)E^{1/2}(t) + \frac{[(pq)^{\cdot}]^{-}}{pq}(t)E(t).$$

Using this inequality, it can be proved that under assumptions (I)—(VI) all solutions of (E1) exist on $[t_0, \infty)$, moreover for every solution the finite limit $\lim_{t\to\infty} E(t) = \lambda$ exists. Hence by virtue of (II), in order to show $\lim_{t\to\infty} x(t) = 0$ it is sufficient to prove $\lambda = 0$.

2. Results

The fundamental theorem is the following:

Theorem 1. *Suppose that there is a nonnegative absolutely continuous function* α *such that* $\int^{\infty} \alpha = \infty$ *and it satisfies the following conditions:*

$$\int_{t_0}^{\infty} \left[\frac{(pq)^{\cdot}}{pq}(\tau) - k\frac{\alpha(\tau)}{\int_{t_0}^{\tau}\alpha}\right]^{-} d\tau < \infty \quad \textit{for every} \quad k > 1; \tag{1.1}$$

$$\alpha(t)\left(\frac{p}{q}\right)^{1/2}(t) = o\left(\int_{t_0}^{t}\alpha\right) \quad (t\to\infty); \tag{1.2}$$

$$\int_{t_0}^{t}\left|\left(\frac{\alpha}{q}\right)^{\cdot}\right|(pq)^{1/2} = o\left(\int_{t_0}^{t}\alpha\right) \quad (t\to\infty). \tag{1.3}$$

Then for all solutions of equation (E1) $\lim_{t\to\infty} E(t) = 0$.

Proof. Let $x(t)$ be an arbitary solution of (E1). First we prove that for every $\varepsilon > 0$ there is a $v > 0$ such that

$$F(x(t)) - vx(t)f(x(t)) < \varepsilon \quad (t \geqq t_0). \tag{2}$$

Let $[a, b]$ be a bounded closed interval $(a > 0)$. It follows from the continuity of f and F that there exists a constant $v = v(a, b)$ such that $F(x) < vxf(x)$ whenever $x \in [a, b]$. Let $b = \sup_{[t_0, \infty]} |x(t)|$ and choose a so that $F(y) < \varepsilon$ if $|y| < a$. Choosing v to this $[a, b]$ we get (2).

Function $E(t)$ can be rewritten and estimated as follows:

(3)
$$E(t) = \frac{p(t)}{q(t)}\dot{x}^2(t)+F(x(t)) = (1+\nu)\frac{p(t)}{q(t)}\dot{x}^2(t)-$$
$$-\nu\frac{1}{q(t)}(px\dot{x})^{\cdot}(t)+F(x(t))-\nu x(t)f(x(t))+\nu x(t)\frac{e(t)}{q(t)}<$$
$$<(1+\nu)\frac{p(t)}{q(t)}\dot{x}^2(t)-\nu\frac{1}{q(t)}(px\dot{x})^{\cdot}(t)+\nu\frac{e(t)}{q(t)}x(t)+\varepsilon.$$

Therefore, if $W(t)=E(t)\int_{t_0}^{t}\alpha$, then

(4)
$$\dot{W}(t)<-\nu(px\dot{x})^{\cdot}(t)\frac{\alpha(t)}{q(t)}+2\left(\frac{p}{q}\right)^{1/2}(t)|\dot{x}(t)|\frac{|e|}{(pq)^{1/2}}(t)\int_{t_0}^{t}\alpha+$$
$$+\nu\frac{|e|\,\alpha}{q}(t)\,|x(t)|+\varepsilon\alpha(t)+\varphi(t)W(t)\leqq-\nu(px\dot{x})^{\cdot}(t)\frac{\alpha(t)}{q(t)}+$$
$$+2K\frac{|e|}{(pq)^{1/2}}(t)\int_{t_0}^{t}\alpha+\nu M\frac{|e|\,\alpha}{q}(t)+\varepsilon\alpha(t)+\varphi(t)W(t),$$

where $M=\sup\limits_{[t_0,\infty)}(|x(t)|)$, $K^2=\sup\limits_{[t_0,\infty)}\left(\frac{p(t)}{q(t)}\dot{x}^2(t)\right)$,
$$\varphi(t)=\left[\frac{(pq)^{\cdot}}{pq}(t)-(1+\nu)\frac{\alpha(t)}{\int_{t_0}^{t}\alpha}\right]^{-}.$$

By virtue of (VI) there is a number T such that $\int_T^t |e|(pq)^{-1/2}<\varepsilon/2K$ whenever $t>T$. Integrating (4) over the interval $[T,t]$ we get

(5)
$$W(t)<W(T)-\nu\frac{\alpha}{q}(px\dot{x})\Big|_T^t+\nu\int_T^t\left(\frac{\alpha}{q}\right)^{\cdot}px\dot{x}+\varepsilon\int_T^t\alpha+$$
$$+2K\int_T^t\Big(\int_{t_0}^{\tau}\alpha\Big)\frac{|e|}{(pq)^{1/2}}(\tau)\,d\tau+\nu M\int_T^t\frac{|e|\,\alpha}{q}+\int_T^t\varphi W<$$
$$<\left\{W(T)+\nu\left(\frac{\alpha}{q}px\dot{x}\right)(T)\right\}+\nu MK\alpha(t)\left(\frac{p}{q}\right)^{1/2}(t)+\varepsilon\int_T^t\alpha+$$
$$+\nu MK\int_T^t\left|\left(\frac{\alpha}{q}\right)^{\cdot}\right|(pq)^{1/2}+2K\int_T^t\Big(\int_{t_0}^{\tau}\alpha\Big)\frac{|e|}{(pq)^{1/2}}(\tau)\,d\tau+$$
$$+\nu M\int_T^t\frac{|e|\,\alpha}{q}+\int_T^t\varphi W=\psi(t)+\int_T^t\varphi W.$$

Observe that in consequence of conditions (VI), (1.1), (1.2), (1.3) the following estimates hold:

$$\overline{\lim_{t\to\infty}}\, \psi(t)\Big(\int_{t_0}^{t} \alpha\Big)^{-1} < 2\varepsilon, \quad \int_{T}^{\infty} \varphi = L < \infty.$$

Applying the fundamental theorem of differential inequalities [3] we obtain

$$W(t) = \psi(t) + \exp\Big\{\int_T^t \varphi\Big\} \int_T^t \psi(\tau)\varphi(\tau) \exp\Big\{-\int_T^\tau \varphi\Big\}\, d\tau < \psi(t) + Le^L \sup_{[T,t]} (\psi),$$

and $\lim_{t\to\infty} E(t) < 2\varepsilon + 2Le^L\varepsilon$. Since ε was arbitary, $\lim_{t\to\infty} E(t) = 0$. The proof iscomplete.

Applying different methods F. J. Scott has proved two theorems ensuring $\lim_{t\to\infty} E(t) = 0$ for all solutions of (E1). In [7] he used the condition

$$\underline{\lim}_{t\to\infty} \frac{(pq)^{\cdot}}{pq}(t)\frac{1}{\alpha(t)} > 0$$

requiring much more than (1.1). In [8] he supposed the boundedness of the functions

$$\alpha(t)\Big(\frac{p}{q}\Big)^{1/2}(t), \quad \int_{t_0}^{t} \Big|\Big(\frac{\alpha}{q}\Big)^{\cdot}\Big| (pq)^{1/2}$$

on $[t_0, \infty)$ instead of conditions (1.2) and (1.3). Thus Th. 1 is a common improvement of these results.

Remark 1. If there is a $\nu > 0$ so that $F(x) \leqq \nu x f(x)$ $(-\infty < x < \infty)$, then condition (1.1) has to be satisfied for $k = 1 + \nu$ only.

Remark 2. If further differentiability conditions hold for $\alpha(t)$ and $q(t)$, then the proof of Theorem 1 can be refined. Integrating by parts the member $\nu(\alpha/q)^{\cdot}(t)(p x\dot{x})(t)$ in (4) we obtain the following modification:

If the conditions

$$\int_{t_0}^{t} \Big[\Big(\Big(\frac{\alpha}{q}\Big)^{\cdot} p\Big)^{\cdot}\Big]^{-} = o\Big(\int_{t_0}^{t} \alpha\Big) \quad (t\to\infty), \tag{1.3'}$$

$$\Big|\Big(\frac{\alpha}{q}\Big)^{\cdot} p\Big|(t) = o\Big(\int_{t_0}^{t} \alpha\Big) \quad (t\to\infty) \tag{1.3''}$$

hold instead of (1.3) then the statement of Theorem 1 remains true.

Remark 3. If the assertion of Th. 1 concerns only the oscillatory solutions of (E1), then condition (1.3″) and, in the case $e(t) = 0$, condition (1.2) can be omitted. Indeed, for oscillatory solutions integrate in (5) from t_1 to t_n, where $\{t_n\}_1^\infty$ is the sequence of the zeros of the solution $x(t)$. Then

$$\nu\Big(\frac{\alpha}{q}\Big)^{\cdot} p\frac{x^2}{2}\Big|_{t_1}^{t_n} = 0 \quad \text{and} \quad \nu\frac{\alpha}{q}\, px\dot{x}\Big|_{t_1}^{t_n} = 0$$

respectively.

Another variant of this statement has been proved by L. HATVANI [4].

In the remain part of this paper we give some applications of Th. 1 to the equations

(E2) $$\ddot{x}+q(t)f(x)=e(t)$$

and

(E2′) $$\ddot{x}+q(t)f(x)=0.$$

It is known [2] that the condition $q(t)\nearrow\infty$ as $t\to\infty$ is necessary but not sufficient for $\lim_{t\to\infty}E(t)=0$. Hence, to ensure this property further conditions must hold for regularity of growth of $q(t)$ as $t\to\infty$. G. SANSONE [2] proved that if $\dot{q}(t)\geqq c>0$ on $[t_0,\infty)$ and $\int^{\infty} 1/q=\infty$ then each solution of the linear equation

(LE) $$\ddot{x}+q(t)x=0$$

tends to zero as $t\to\infty$. I. T. KIGURADZE [5] obtained the following result: If the condition $\dot{q}(t)/q(t)>\sigma/(t\log t)$ is satisfied for all $t\in[t_0,\infty]$ and some $\sigma>0$, then all solutions of (E2′) approach to zero as $t\to\infty$. Using Th. 1 we prove a generalization of these results.

Corollary 1. *Suppose that there is an absolutely continuous nonnegative function α for which* (1.1) *and* $\int^{\infty}\alpha=\infty$ *hold. If either*

a) *there exists a constant* $\beta<\frac{1}{2}$ *such that*

$$\int_{t_0}^{t}\left|\left(\frac{\alpha}{q^{\beta}}\right)^{\cdot}\right|=o\left(\int_{t_0}^{t}\alpha\right)\quad(t\to\infty),$$

or b)

$$\frac{\alpha(t)}{\int_{t_0}^{t}\alpha}\to 0,\quad \int_{t_0}^{t}\left|\left(\frac{\alpha}{q^{1/2}}\right)^{\cdot}\right|=o\left(\int_{t_0}^{t}\alpha\right)\quad(t\to\infty)$$

then for all solutions of (E2) $\lim_{t\to\infty}E(t)=0$.

PROOF. First we prove that from (V)

(6) $$\lim_{t\to\infty}q(t)>c>0$$

follows. Suppose that this statement is false. Then there exists a sequence $\{t_n\}_1^{\infty}$ such that $t_n\to\infty$, $q(t_n)\to 0$, and

$$\int_{t_0}^{t_n}\frac{[\dot{q}]^-}{q}\geqq-\frac{1}{2}\int_{t_0}^{t_n}\frac{\dot{q}}{q}=\frac{1}{2}\left(\log q(t_0)-\log q(t_n)\right)$$

as $n\to\infty$ which contradicts (V).

We show that all conditions of Th. 1 are satisfied. Because of (6) we have (1.2). In order to prove (1.3) in case a), by

$$\int_{t_0}^{t}\left|\left(\frac{\alpha}{q}\right)^{\cdot}\right| q^{1/2} \leqq \int_{t_0}^{t}\left|\left(\frac{\alpha}{q^{\beta}}\right)^{\cdot}\right| q^{\beta-1/2}+(1-\beta)\int_{t_0}^{t}\left(\frac{\alpha}{q^{\beta}}\right)|\dot{q}|\, q^{\beta-3/2}$$

it is sufficient to present that the members on the right-hand side are of $o\Big(\int\limits_{t_0}^{t}\alpha\Big)$ as $t\to\infty$. In view of (6) it holds for the first one. For the second one the following estimate results the desired property:

$$\int_{t_0}^{t}\frac{\alpha}{q^{\beta}}|\dot{q}|\, q^{\beta-3/2} < \sup_{[t_0,\,t]}\left(\frac{\alpha}{q^{\beta}}\right)\left\{2c^{\beta-1/2}\int_{t_0}^{t}\frac{[\dot{q}]^-}{q}+\int_{t_0}^{t}\dot{q}q^{\beta-3/2}\right\} <$$

$$< \sup_{[t_0,\,t]}\left(\frac{\alpha}{q^{\beta}}\right)\left\{2c^{\beta-1/2}\int_{t_0}^{\infty}\frac{[\dot{q}]^-}{q}+\frac{2}{1-2\beta}c^{\beta-1/2}\right\}.$$

In case b) (1.3) follows immediately from

$$\int_{t_0}^{t}\left|\left(\frac{\alpha}{q}\right)^{\cdot}\right| q^{1/2} \leqq \int_{t_0}^{t}\left|\left(\frac{\alpha}{q^{1/2}}\right)^{\cdot}\right|+\frac{1}{2}\int_{t_0}^{t}\alpha\frac{|\dot{q}|}{q^{3/2}}.$$

The corollary is proved.

From case a) of Corollary 1 one gets Sansone's and Kiguradze's results with $\beta=0$, $\alpha(t)=1/q(t)$ and $\alpha(t)=1/(t\log t)$ respectively.

In 1978 R. J. Ballieu and K. Peiffer [1] obtained the following statement. Suppose that $\dot{q}(t)\geqq 0$, $\dot{q}(t)/q^{3/2}(t)$ is bounded on $[t_0,\infty)$ and $\dot{q}(t)/q^{3/2}(t)\geqq a(t)$, where $a(t)$ is nonincreasing, $\int\limits_{t_0}^{\infty} aq^{1/2}=\infty$. Then all solutions of (LE) tend to zero as $t\to\infty$. This theorem is generalized by Corollary 1 (case b), $\alpha(t)=a(t)q^{1/2}(t)$. Instead of monotonicity of $a(t)$ only

$$\int_{t_0}^{t}|\dot{a}| = o\Big(\int_{t_0}^{t} aq^{1/2}\Big) \quad (t\to\infty)$$

is required. It is easy to see that Corollary 1 can be applied to equation

$$\ddot{x}+\exp\left\{\int_{1}^{t}\frac{1}{\tau}\sin^2\tau\, d\tau\right\}xe^{x} = 0$$

but the conditions of the result of Ballieu and Peiffer are not satisfied.

Next we formulate some statements for equation (E2′). As is known [3] conditions (I), (II), (III), and (V) imply that all solutions of (E2′) oscillate on $[t_0,\infty)$.

Choosing $\alpha(t)=\dot{q}(t)/q(t)$ in remarks 2 and 3 we obtain

Corollary 2. *Assume that $\ddot{q}(t)$ is absolutely continuous and $q(t)\nearrow\infty$ as $t\to\infty$. If*

$$\int_{t_0}^{t}[(q^{-1})^{\cdot\cdot}]^{+}=o(\log q(t))\quad(t\to\infty)$$

then for all solution of (E2′) $\lim_{t\to\infty}E(t)=0$.

A. C. LAZER proved [6] that if $q(t)\nearrow\infty(t\to\infty)$ and the function $\int_{t_0}^{t}|(q^{-1/2})^{\cdot\cdot\cdot}|$ is bounded on $[t_0,\infty)$ then all solutions of the linear equation (LE) tend to zero as $t\to\infty$. Applying Remark 1, with $\alpha(t)=\dot{q}(t)q^{-\nu/(\nu+1)}(t)$ we obtain a generalization of Lazer's theorem to the nonlinear equation (E2′).

Corollary 3. *Assume that $\ddot{q}(t)$ is absolutely continuous, $q(t)\nearrow\infty$ as $t\to\infty$ and $F(x)<\nu xf(x)$ for all $-\infty<x<\infty$ and for some constant $\nu>0$. If*

$$\int_{t_0}^{t}[(q^{-\nu/(\nu+1)})^{\cdot\cdot\cdot}]^{+}=o(q^{1/(\nu+1)}(t))\quad(t\to\infty)$$

then for all solutions of (E2′) $\lim_{t\to\infty}E(t)=0$.

Remark 4. Our results can be sharpened if $f(x)$ is "nearer" to a linear function in the following sense: there exist positive numbers μ, ν such that

(VII) $$\mu^2x^2\leqq F(x)\leqq \nu xf(x).$$

If $f(x)=x$ then $\mu=\nu=1$.

By refinements of the proofs of our theorems it can be shown that in this "near-linear" case conditions (1.3) and (1.3′) may be replaced by

(1.3)′ $$\overline{\lim_{t\to\infty}}\left\{\Big(\int_{t_0}^{t}\alpha\Big)^{-1}\int_{t_0}^{t}\left|\Big(\frac{\alpha}{q}\Big)^{\cdot}\right|(pq)^{1/2}\right\}<\frac{\mu}{\nu};$$

(1.3′)′ $$\overline{\lim_{t\to\infty}}\left\{\Big(\int_{t_0}^{t}\alpha\Big)^{-1}\int_{t_0}^{t}\left[\Big(\Big(\frac{\alpha}{q}\Big)^{\cdot}p\Big)^{\cdot}\right]^{-}\right\}<\frac{2\mu^2}{\nu}$$

respectively.

Statements involving these conditions are real sharpening of the corresponding theorems. For example, for the equation

$$((t^{5/2}\log t)\dot{x})^{\cdot}+3t^{1/2}x=0$$

condition (1.3′)′ is satisfied with $\alpha(t)=1/t$ while (1.3′) does not hold.

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References

[1] R. J. Ballieu and K. Peiffer, Attractivity of the origin for the equation $\ddot{x}+f(t, x, \dot{x})|\dot{x}|^{\alpha}\dot{x}+g(x)=0$, *J. Math. Anal. Appl.*, **65** (1978), 321—332.

[2] L. Cesari, Asymptotic behavior and stability problems in ordinary differential equations, *Springer-Verlag, Berlin*, 1959.

[3] A. Halanay, Differential equations, *Academic Press, New York, San Francisco, London*, 1966.

[4] L. Hatvani, On the asymptotic behavior of the solutions of $(p(t)x')'+q(t)f(x)=0$, *Publ. Math. (Debrecen)* **19** (1972), 225—237.

[5] I. T. Kiguradze, On the asymptotic properties of solutions of the equation $u''+a(t)u^n=0$, *Soobsc. Akad. Nauk Gruzin. SSR*, **30** (1963), 129—136.

[6] A. C. Lazer, A stability condition for the differential equation $y''+p(x)y=0$, *Michigan Math. J.*, **12** (1965), 193—196.

[7] F. J. Scott, On a partial asymptotic stability theorem of Willett and Wong, *J. Math. Anal. Appl.*, **63** (1978), 416—420.

[8] F. J. Scott, New partial asymptotic stability results for nonlinear ordinary differential equations, *Pacific J. Math.* **72** (1977), 523—535.

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