# On characterizations of the normal law in Hilbert space

By ISTVÁN FAZEKAS (Debrecen)

## 1. Introduction

KHATRI [5] gives some characterizations of multivariate normality through linear regressions and remarks that these results are valid on Hilbert space too, if the original conditions are replaced by conditions similar to those given by KUMAR and PATHAK [7]. But it is not necessary to change the conditions of Theorem 2 of [5]; it is true in Hilbert space in its original form.

The aim of this paper is to point out that most of the results and methods of [5] are "dimension free" (see [3] and [7]). Those lemmas of [5] that are valid only in finite dimensional spaces are replaced by lemmas which are valid in finite dimensional spaces as well as in Hilbert spaces.

## 2. Notations and preliminary lemmas

Let $H$, $H_1$, $H_2$ etc. denote real separable complete inner product spaces, that is separable Hilbert spaces or finite dimensional Euclidean spaces. For the sake of brevity these spaces are called Hilbert spaces. The inner product is denoted by $\langle .,. \rangle$ and the associated norm by $|.|=\langle .,. \rangle^{\frac{1}{2}}$. The adjoint of a linear bounded operator $A\colon H_1 \to H_2$ is denoted by $A'$. If we consider $h \in H$ as the following map from $\mathbf{R}$ into $H$: $h(x)=xh$ ($x \in \mathbf{R}$, where $\mathbf{R}$ denotes the set of real numbers), then $h'$ is the following map from $H$ into $\mathbf{R}$: $h'(a)=\langle h, a\rangle$ ($a \in H$). In particular for fixed $h_1, h_2 \in H$; $h_1' h_2 = \langle h_1, h_2 \rangle$ and $h_1 h_2'$ is the following operator: $h_1 h_2'(a) = \langle h_2, a \rangle h_1 = h_1 h_2' a$ ($a \in H$).

If $H = \prod_{i=1}^{n} H_i$, where $H_1, \dots, H_n$ are Hilbert spaces, then $H$ is a Hilbert space with the inner product:

$$\langle a, b\rangle = \sum_{i=1}^{n} \langle a_i, b_i \rangle, \quad a = (a_i)_{i=1}^{n}, \quad b = (b_i)_{i=1}^{n} \in H.$$

There is an isomorphism between the algebra of bounded linear operators of $H$ and the algebra of matrices $(A_{ij})_{i,j=1}^{n}$, where $A_{ij}\colon H_j \to H_i$ is a bounded linear operator ($i, j=1, \dots, n$). Through this paper $I_{ii}$ denotes an appropriate identical operator.

Let $\xi$ be a Hilbert space valued random variable (r.v.). $E\xi$, the expectation of $\xi$ is defined by the Bochner integral of $\xi$. The covariance operator of $\xi$ is $D^2\xi=$

$=E[(\xi-E\xi)(\xi-E\xi)']$. If $E|\xi|^2<\infty$, then $D^2\xi$ exists. The conditional expectation $E(\xi|.)$ is defined in Scalora [8]. $\xi$ is said to have a normal distribution if for each contsant vector $h$, the real-valued r.v. $\langle h, \xi\rangle$ has the normal distribution.

**Lemma 1.** *Let $\xi$ and $\eta$ be $H$-valued r.v'.s, $E|\xi|<\infty$. $E(\xi|\eta)$ is constant if and only if*

$$E\{\xi e^{i\langle t,\eta\rangle}\} = E\xi\, Ee^{i\langle t,\eta\rangle}$$

*for each $t\in H$.*

PROOF. See [4] and [2].

**Lemma 2.** (See [5].) *Let $\xi$ and $\eta$ be independent Hilbert space valued r.v'.s, $E|\xi|<\infty$, $E|\eta|<\infty$.*

(a) *If $E(\xi+\eta|\eta)$ is constant, then $\eta$ is degenerate.*

(b) *If $E(\eta|\xi+\eta)$ is constant and the characteristic functional of $\xi$ nowhere vanishes, then $\eta$ is degenerate.*

PROOF. (a) is trivial; (b) follows from Lemma 1.

**Lemma 3.** (See [6].) *Let $\xi$ and $(\eta, \zeta)$ be independent Hilbert space valued r.v'.s. Assume that the characteristic functional of $\xi$ nowhere vanishes. If $\xi+\eta$ and $\zeta$ are independent, then $\eta$ and $\zeta$ are independent.*

PROOF. The proof is based on elementary properties of the characteristic functional.

**Lemma 4.** (See [5].) *Let $H=\prod_{i=1}^{n} H_i$ and let $A=A_{(n)}=(A_{ij})_{i,j=1}^{n}$ be a bounded linear operator of $H$. Assume that $A_{(r)}=(A_{ij})_{i,j=1}^{r}$ is an invertible linear operator for $r=1, 2, \ldots, n-1$. Then there exists a decomposition $A=TS$, where $T=(T_{ij})_{i,j=1}^{n}$ is lower triangular with*

$$T_{ii} = A_{ii}-(A_{i1}, \ldots, A_{i,i-1})[A_{(i-1)}]^{-1}(A'_{1i}, \ldots, A'_{i-1,i})' = A_{ii}-\sum_{k=1}^{i-1} T_{ik}S_{ki}$$

*$(i=2, 3, \ldots, n)$. $T_{ii}$ is invertible for $i=1, 2, \ldots, n-1$. (If $A_{(n)}$ is invertible, then $T_{nn}$ is also invertible.) $S=(S_{ij})_{i,j=1}^{n}$ is upper triangular and $S_{ii}$: $H_i\to H_i$ is identical $(i=1, \ldots, n)$.*

PROOF. For $k=2$

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22}-A_{21}A_{11}^{-1}A_{12} \end{pmatrix} \begin{pmatrix} I_{11} & A_{11}^{-1}A_{12} \\ 0 & I_{22} \end{pmatrix}$$

is the required decomposition. Then one can proceed by induction.

**Lemma 5.** (See [5].) *Let $\zeta$ and $(\xi, \eta_1, \eta_2)$ be independent Hilbert space valued r.v'.s and let $\zeta$ have normal distribution. Let $\zeta_1=\zeta+A_1\xi+\eta_1$, $E|\zeta_1|<\infty$. Assume that $E(\zeta_1|\zeta-B_1\xi, \eta_1, \eta_2)=0$, where $A_1$ and $B_1$ are bounded linear operators satisfying the relation $AB_1=BA_1$ for some operators $A$ and $B$. Then $(\zeta, B\zeta_1)$ is normally distributed and $(\zeta, B\zeta_1)$ and $(\eta_1, \eta_2)$ are independent.*

PROOF. Let $\mu=A_1\xi+\eta_1$. From the conditions of the lemma

$$E(\zeta+\mu|A\zeta-B\mu,\ \eta_1,\eta_2)=0.$$

By Lemma 1

$$E\{(\zeta+\mu)\exp[i\langle A\zeta-B\mu,t\rangle+i\langle\eta_1,t_1\rangle+i\langle\eta_2,t_2\rangle]\}=0.$$

Let $f_1(s)$ and $f_2(s,t_1,t_2)$ denote the characteristic functional of $\zeta$ and $(\mu,\eta_1,\eta_2)$ respectively. Then

$$\left.\frac{df_1(s)}{ds}\right|_{s=A't} f_2(-B't,t_1,t_2)+\left.\frac{\partial f_2(s,t_1,t_2)}{\partial s}\right|_{s=-B't} f_1(A't)=0,$$

where $\frac{df_1(s)}{ds}$ and $\frac{\partial f_2(s,t_1,t_2)}{\partial s}$ denote the Fréchet derivative and the Fréchet partial derivative (cf. [1]) resp. Since $\zeta$ is normally distributed we have (see [9])

$$f_1(s)=\exp\left\{i\langle m,s\rangle-\frac{1}{2}\langle\Sigma s,s\rangle\right\}.$$

Therefore from the above equation

$$B(im-\Sigma A't)f_2(-B't,t_1,t_2)=\frac{\partial f_2(-B't,t_1,t_2)}{\partial t}.$$

This shows that

$$f_2(-B't,t_1,t_2)=\exp\left\{i\langle Bm,t\rangle-\frac{1}{2}\langle B\Sigma A't,\ t\rangle\right\}g(t_1,t_2).$$

Hence $B\mu$ is normally distributed and $B\mu$ and $(\eta_1,\eta_2)$ are independent. This proves the lemma.

**Lemma 6.** (See [5].) *Let $\zeta$ and $(\mu,\eta)$ be independent Hilbert space valued r.v'.s. Let*

$$\zeta_0=A\zeta+(A-I)\mu,$$

*where $A$ is an arbitrary bounded linear operator and $I$ is the identical operator. Assume that $E|\zeta_0|^2<\infty$ and*

(1) $$E(\zeta_0|\zeta+\mu,\eta)=0;$$

(2) $$E(\zeta_0\zeta_0'|\zeta+\mu,\eta)=\Sigma,$$

*where $\Sigma$ is a constant operator.*

*Then $A\zeta$ and $(A-I)\mu$ are normally distributed. If $A$ is invertible, then $\zeta$, $(A-I)\mu$ and $\eta$ are independent.*

PROOF. Let $\varphi_1(t)$ and $\varphi_2(t,t_1)$ be the characteristic functional of $\zeta$ and $(\mu,\eta)$ respectively. With the help of Lemma 1 from (1) and (2) one can get

(3) $$A\frac{d\varphi_1(t)}{dt}\varphi_2(t,t_1)+(A-I)\frac{\partial\varphi_2(t,t_1)}{\partial t}\varphi_1(t)=0$$

and

$$A\frac{d^2\varphi_1(t)}{dt^2}A'\varphi_2(t,t_1)+A\frac{d\varphi_1(t)}{dt}\left[\frac{\partial\varphi_2(t,t_1)}{\partial t}\right]'(A-I)'+ \tag{4}$$

$$+(A-I)\frac{\partial\varphi_2(t,t_1)}{\partial t}\left[\frac{d\varphi_1(t)}{dt}\right]'A'+(A-I)\frac{\partial^2\varphi_2(t,t_1)}{\partial t^2}(A-I)'\varphi_1(t)=$$

$$=-\Sigma\varphi_1(t)\varphi_2(t,t_1).$$

From (3) and (4) we obtain for $t$ and $t_1$ in some neighborhood of the origin:

$$A\frac{d^2\varphi_1(t)}{dt^2}\frac{1}{\varphi_1(t)}-A\frac{d\varphi_1(t)}{dt}\left[\frac{d\varphi_1(t)}{dt}\right]'\frac{1}{\varphi_1^2(t)}=-\Sigma \tag{5}$$

and

$$(A-I)\frac{\partial^2\varphi_2(t,t_1)}{\partial t^2}\frac{1}{\varphi_2(t,t_1)}-(A-I)\frac{\partial\varphi_2(t,t_1)}{\partial t}\left[\frac{\partial\varphi_2(t,t_1)}{\partial t}\right]'\frac{1}{\varphi_2^2(t,t_1)}=\Sigma.$$

From these equations we have

$$\frac{d^2\ln[\varphi_1(A't)]}{dt^2}=-\Sigma A'$$

and

$$\frac{\partial^2\ln\{\varphi_2[(I-A)'t,t_1]\}}{\partial^2 t}=-\Sigma(I-A)'.$$

By the theorem of Marcinkiewicz (see Lemma 2.4.3 of [4]) $A\zeta$ and $(I-A)\mu$ are normally distributed.

If $A$ is invertible, then from (5) we have

$$\varphi_1(t)=\exp\left\{i\langle m,t\rangle-\frac{1}{2}\langle A^{-1}\Sigma t,t\rangle\right\}.$$

In this case (3) can be rewritten in the form

$$(iAm-\Sigma t)\varphi_2(t,t_1)+(A-I)\frac{\partial\varphi_2(t,t_1)}{\partial t}=0.$$

Let $t=(I-A)'s$, then

$$[iAm-\Sigma(I-A)'s]\bar\varphi_2(s,t_1)-\frac{\partial\bar\varphi_2(s,t_1)}{\partial s}=0,$$

where $\bar\varphi_2(s,t_1)=\varphi_2[(I-A)'s,t_1]$ is the characteristic functional of $[(I-A)\mu,\eta]$. Its general solution is

$$\bar\varphi_2(s,t_1)=\exp\left\{i\langle Am,s\rangle-\frac{1}{2}\langle\Sigma(I-A')s,s\rangle\right\}\psi(t_1),$$

where $\psi$ is a function of $t_1$ only. This shows us that if $A$ is invertible, then $(I-A)\mu$ and $\eta$ are independent. Thus the lemma is proved.

In Theorem 2 we shall need the following facts on conditional distributions. Let $H_1$ and $H_2$ be Hilbert spaces and let $(\xi,\eta)$ be a $H_1\times H_2$-valued r.v. Assume that

$(\xi, \eta)$ has canonical form. Denote by $Q_{\xi,\eta}$, $Q_\xi$ and $Q_\eta$ the distribution of $(\xi, \eta)$, $\xi$ and $\eta$ respectively. Let $P_{\xi|\eta=y}(C; y)$ denote a regular conditional distribution of $\xi$ given $\eta=y$, where $y\in H_2$ and $C\in\mathscr{B}(H_1)$, the Borel $\sigma$-algebra of $H_1$. If

$$P_{\xi|\eta=y}(C+E(\xi|\eta = y); y) = P(C), \quad C\in\mathscr{B}(H_1), \quad y\in H_2,$$

where the distribution $P(.)$ does not depend on $y$, we shall say, that the conditional distribution of $\xi$ given $\eta$ depends on $\eta$ only through the conditional expectation.

**Lemma 7.** *If the above conditions are satisfied, then* $\xi-E(\xi|\eta)$ *and* $\eta$ *are independent.*

Proof. Let $C\in\mathscr{B}(H_1)$, $D\in\mathscr{B}(H_2)$.

$$\begin{aligned}Q_{\xi-E(\xi|\eta),\eta}(C\times D) &= P([\xi-E(\xi|\eta)]\in C,\ \eta\in D)\\ &= P(\xi\in C+E(\xi|\eta),\ \eta\in D)\\ &= \int_D P_{\xi|\eta=y}(C+E(\xi|\eta = y); y) \quad Q_\eta(dy)\\ &= \int_D P(C)Q_\eta(dy).\end{aligned}$$

Hence the conditional distribution of $\xi-E(\xi|\eta)$ given $\eta=y$ does not depend on $y$, that is $\xi-E(\xi|\eta)$ and $\eta$ are independent.

## 3. Characterization theorems

**Proposition 1.** (See [5].) *Let* $\xi_1, \xi_2, \eta_1, \eta_2$ *be Hilbert space valued r.v'.s. Let*

$$\zeta_1 = \xi_1+A_{12}\xi_2+\eta_1$$

*and*

$$\zeta_2 = A_{21}\xi_1+\xi_2+\eta_2.$$

*Let* $E|\zeta_2|^2<\infty$. *Assume that* $A=A_{12}A_{21}$ *is invertible and the range of* $B=I-A$ *is closed. If*

(i) $\zeta_1$ *and* $(\xi_2, \eta_1, \eta_2)$ *are independent;*
(ii) $E(\zeta_2|\xi_1, \eta_1, \eta_2)=0$;

*and*

(iii) $E(\zeta_2\zeta_2'|\xi_1, \eta_1, \eta_2)=\Sigma$,

*where* $\Sigma$ *is constant, then* $(A\zeta_1, A_{12}\zeta_2)$ *is normally distributed and* $(\zeta_1, A_{12}\zeta_2)$ *and* $(\eta_1, \eta_2)$ *are independent.*

Proof. Let $\xi_{(2)}=A_{12}\xi_2$ and $\eta_{(2)}=\eta_2-A_{21}\eta_1$, then

$$\zeta_0 = A_{12}\zeta_2 = A\zeta_1+(I-A)\xi_{(2)}+A_{12}\eta_{(2)}.$$

From (ii) and (iii) we obtain

(6) $$E(\zeta_0|\zeta_1-\xi_{(2)},\ \eta_{(2)},\ \eta_1) = 0,$$

(7) $$E(\zeta_0\zeta_0'|\zeta_1-\xi_{(2)},\ \eta_{(2)},\ \eta_1) = A_{12}\Sigma A_{12}' = \Sigma_1.$$

There exists a linear bounded operator $B^-$ such that $BB^-B=B$. From (6) we then find

$$E\{(I-BB^-)A\zeta_1+(I-BB^-)A_{12}\eta_{(2)}|\eta_{(2)}\} = 0.$$

By Lemma 2 $(I-BB^-)A_{12}\eta_{(2)}$ is degenerate.

We can assume that $EA_{12}\eta_{(2)}=0$, $E(I-A)\xi_{(2)}=0$ and $EA\zeta_1=0$. It follows that $(I-BB^-)A_{12}\eta_{(2)}=0$. Let $\mu=\xi_{(2)}+B^-A_{12}\eta_{(2)}$, then

$$(I-A)\mu = (I-A)\xi_{(2)}+A_{12}\eta_{(2)}.$$

Hence $\zeta_0=A\zeta_1+(I-A)\mu$. Then (i), (6) and (7) give

(8) $\zeta_1$ and $(\mu, \eta)$ are independent;

(9) $E(\zeta_0|\zeta_1-\mu, \eta)=0$;

(10) $E(\zeta_0\zeta_2'|\zeta_1-\mu, \eta)=\Sigma_1$,

where $\eta=(\eta_1, \eta_2)$. The required result follows from Lemma 6.

**Proposition 2.** (See [6].) *Let $\xi_1, \xi_2, \eta_1, \eta_2$ be Hilbert space valued r.v.'s. Let*

$$\zeta_1 = \xi_1+A_{12}\xi_2+\eta_1$$

*and*

$$\zeta_2 = A_{21}\xi_1+\xi_2+\eta_2.$$

*If $\zeta_1$ and $(\xi_2, \eta_1, \eta_2)$ are independent and $\zeta_2$ and $(\xi_1, \eta_1, \eta_2)$ are independent, then $(A_{21}\zeta_1, A_{12}\zeta_2)$ is normally distributed and $(\zeta_1, \zeta_2)$ and $(\eta_1, \eta_2)$ are independent.*

Proof. Let $T_{22}=I-A_{21}A_{12}$ and $\eta=\eta_2-A_{21}\eta_1$. Then

$$\xi_1 = \zeta_1-A_{12}\xi_2-\eta_1$$

and

(11) $$\zeta_2 = A_{21}\zeta_1+T_{22}\xi_2+\eta.$$

From the assumptions of the proposition $A_{21}\zeta_1+T_{22}\xi_2+\eta$ and $(\zeta_1-A_{12}\xi_2, \eta)$ are independent. Hence

(12) $$\varphi(A_{21}'t_1+A_{21}'t_2)\psi(T_{22}'t_1-A_{12}'A_{21}'t_2,\ t_1+t_3) =$$
$$= \varphi(A_{21}'t_1)\psi(T_{22}'t_1,\ t_1)\varphi(A_{21}'t_2)\psi(-A_{12}'A_{21}'t_2,\ t_3),$$

where $\varphi$ and $\psi$ denotes the characteristic functional of $\zeta_1$ and $(\xi_2, \eta)$ respectively. Let $\bar\varphi$ and $\bar\psi$ be the characteristic functional of $(\zeta_1, 0, 0)$ and $(\xi_2, \eta, 0)$ respectively.

Since $\bar\varphi$ and $\bar\psi$ do not vanish in a neighborhood of the origin, we can take their logarithm in this neighborhood. Then from (12) we have

$$f(B_1t)+g(B_2t) = f(B_3t+g(B_4t)+f(B_5t)+g(B_6t)$$

in a neighborhood of 0, where $t'=(t_1', t_2', t_3')$, $f(t)=\log\bar{\varphi}(t)$, $g(t)=\log\bar{\psi}(t)$ and

$$B_1=\begin{pmatrix} A_{21}' & A_{21}' & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad B_2=\begin{pmatrix} T_{22}' & -A_{12}'A_{21}' & 0\\ I & 0 & I\\ 0 & 0 & 0\end{pmatrix},\quad B_3=\begin{pmatrix} A_{21}' & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$B_4=\begin{pmatrix} T_{22}' & 0 & 0\\ I & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad B_5=\begin{pmatrix} 0 & A_{21}' & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad B_6=\begin{pmatrix} 0 & -A_{12}'A_{21}' & 0\\ 0 & 0 & I\\ 0 & 0 & 0\end{pmatrix}.$$

It is easy to see that the conditions of Lemma 2.3 of [7] are satisfied. It follows that $f(B_1 t)$ is a polynomial in a neighborhood of $t=0$. Marcinkiewicz's theorem implies that $A_{21}\zeta_1$ is normal. By similar arguments we get that $A_{12}\zeta_2$ has also a normal distribution. By equation (11)

$$A_{12}\zeta_2 = A_{12}(A_{21}\zeta_1)+A_{12}(T_{22}\xi_2+\eta).$$

Cramèr's theorem implies that $A_{12}(T_{22}\xi_2+\eta)$ is normal. Therefore it follows that $(A_{21}\zeta_1, A_{12}\zeta_2)$ is normally distributed.

Finally Lemma 3 shows that $(\zeta_1, \zeta_2)$ and $(\eta_1, \eta_2)$ are independent.

**Theorem 1.** (See Theorem 2 of [5].) *Let $X_1, \dots, X_n, U_1, \dots, U_n$ be Hilbert space valued random variables and let*

$$Z_j = \sum_{i=1}^{n} A_{ji}X_i+U_j,$$

*where $A_{jj}=I_{jj}$ $(j=1, 2, \dots, n)$. Suppose that the following conditions are satisfied:*

(i) $Z_1$ *and* $(X_2, X_3, \dots, X_n, U_1, \dots, U_n)$ *are independent;*
(ii) $E(Z_j|X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_n, U_1, \dots, U_n)=0$ *for* $j=2, 3, \dots, n;$
(iii) $E(Z_2Z_2'|X_1, X_3, \dots, X_n, U_1, \dots, U_n)=\Sigma,$

*where $\Sigma$ is constant.*

*Let $A_{(i)}=(A_{j,l})_{j,l=1}^{i}$ be invertible $i=1, 2, \dots, n$. Assume that $A_{12}$, $A_{21}$ and $I-T_{ii}$ $(i=3, \dots, n)$ are invertible, where $T_{ii}$ $(i=1, \dots, n)$ are defined as in Lemma 4.*

*Then $(Z_1, \dots, Z_n)$ is normally distributed and $(Z_1, \dots, Z_n)$ and $(U_1, \dots, U_n)$ are independent.*

Proof. According to the decomposition given in Lemma 4 we can write $A= =A_{(n)}=TS$. Let

$$U=\begin{pmatrix} U_1\\ \vdots\\ U_n\end{pmatrix},\quad X=\begin{pmatrix} X_1\\ \vdots\\ X_n\end{pmatrix},\quad Z=\begin{pmatrix} Z_1\\ \vdots\\ Z_n\end{pmatrix},\quad W=SX+T^{-1}U,$$

then $Z=TW$. With these notations $Z_j=\sum_{k=1}^{j} T_{jk}W_k$ and

$$\sigma\{X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_n, U\} \supset$$
$$\supset \sigma\{W_1-S_{1j}W_j, \dots, W_{j-1}-S_{j-1,j}W_j, W_{j+1}, \dots, W_n, U\},$$

where $\sigma$ denotes the generated $\sigma$-algebra. Then (i), (ii) and (iii) give:

(i)' $Z_1$ and $(W_2, \ldots, W_n, U)$ are independent;
(ii)' $E(Z_j|W_1-S_{1j}W_j, \ldots, W_{j-1}-S_{j-1,j}W_j, W_{j+1}, \ldots, W_n, U)=0$ for $j=2, 3, \ldots, n$;
(iii)' $E(Z_2 Z_2'|W_1-S_{12}W_2, W_3, \ldots, W_n, U)=\Sigma$.

Now we can apply Proposition 1 with $\xi_1=W_1-S_{21}W_2$, $\xi_2=W_2$, $\eta_1=0$, $\eta_2=0$. It follows that $(\zeta_1, \zeta_2)=(Z_1, Z_2)$ has a normal distribution.

We proceed by induction. Assume that

(13) $(Z_1, \ldots, Z_{l-1})$ is normally distributed for $1<l\leqq j$;

and

(14) $(Z_1, \ldots, Z_{l-1})$ and $(W_l, \ldots, W_n, U)$ are independent for $1<l\leqq j$, where $1<j\leqq n$. (It follows that $W_1, \ldots, W_{j-1}$ are independent normal r.v'.s. and $(W_1, \ldots \ldots, W_{j-1})$ and $(W_j, \ldots, W_n, U)$ are independent.)

Let

$$\zeta = (T_{j1}, \ldots, T_{j,j-1})[T_{(j-1)}]^{-1}(Z_1', \ldots, Z_{j-1}')',$$

$$\xi = W_j, \quad \eta_1 = 0, \quad \eta_2 = (W_{j+1}, \ldots, W_n, U),$$

where $T_{(j-1)}=(T_{il})_{i,l=1}^{j-1}$. Then from (13) and (14)

(15) $\zeta$ and $(\xi, \eta_1, \eta_2)$ are independent;

(16) $\zeta$ is normally distributed;

and from (ii)

$$E\big(\zeta_1|\zeta-(I-T_{jj})\xi, \eta_1, \eta_2\big) = 0, \tag{17}$$

where $\zeta_1=\zeta+T_{jj}\xi=\sum_{k=1}^{j} T_{jk}W_k=Z_j$. Then Lemma 5 gives that $Z_j$ is normally distributed. Hence $W_j$ is normally distributed and $W_1, \ldots, W_j$ are independent normal r.v'.s, whence $(Z_1, \ldots, Z_j)$ is normal. It follows from Lemma 5 that $(Z_1, \ldots, Z_j)$ and $(W_{j+1}, \ldots, W_n, U)$ are independent. This completes the proof.

**Corollary 1.** *In Theorem 1, condition (iii) can be replaced by the following condition:*

*"$Z_2$ and $(X_1, X_3, \ldots, X_n, U_1, \ldots, U_n)$ are independent".*

PROOF. By Proposition 2 $(Z_1, Z_2)$ is normally distributed. Then one can proceed as in the proof of Theorem 1.

**Theorem 2.** (See Note 5 of [5].) *Let $X_1, \ldots, X_n$ be Hilbert space valued r.v'.s. Assume that*

(a) *the regression of $X_i$ on the rest of the variables is linear, that is*

$$E(X_i|X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) =$$

$$= -[A_{i1}X_1+\ldots+A_{i,i-1}X_{i-1}+A_{i,i+1}X_{i+1}+\ldots+A_{in}X_n] \quad (i = 1, \ldots, n);$$

(b) *the conditional distribution of* $X_1$ *depends on the rest of the variables only through the conditional expectation;*

(c) $D^2(X_2|X_1, X_3, \ldots, X_n)=\Sigma$, *where* $\Sigma$ *is constant.*

*Let* $A=(A_{ij})_{i,j=1}^{n}$ *with* $A_{ii}=I_{ii}$ $(i=1, \ldots, n)$. *Suppose that* $A$ *satisfies the conditions given in Theorem 1.*

*Then* $(X_1, \ldots, X_n)$ *is normal.*

PROOF. Let $X'=(X_1', \ldots, X_n')$ and let $Z=AX$. (a) and (c) imply that the conditions (ii) and (iii) of Theorem 1 are satisfied. Condition (i) of Theorem 1 follows from (a) and (b) with the help of Lemma 7. Therefore $X$ is normal.

**Corollary 2.** *In Theorem 2, condition (c) can be replaced by the condition*

(c)′ *the conditional distribution of* $X_2$ *depends on the rest of the variables only through its conditional expectation.*

PROOF. It is an easy consequence of Corollary 1.

## References

[1] J. DIEUDONNÉ, Foundations of modern analysis, *Academic Press, New York—London,* 1968.
[2] I. FAZEKAS, Quadratic statistics in Hilbert space, to be published.
[3] V. GOODMAN—P. K. PATHAK, A dimension free approach to certain characterizations of the normal law, *Sankhyā, Series A,* **35** (1973), 479—488.
[4] A. KAGAN—YU. V. LINNIK—C. R. RAO, Characterization problems of mathematical statistics, *Wiley, New York,* 1973.
[5] C. G. KHATRI, Characterizations of multivariate normality. II. Through linear regressions, *J. of Multivariate Analysis,* **9** (1979), 589—598.
[6] C. G. KHATRI—C. R. RAO, Characterizations of multivariate normality. I. Through independence of some statistics, *J. Multivariate Anal.* **6** (1976), 81—94.
[7] A. KUMAR—P. K. PATHAK, Some recent dimension free characterizations of the normal law, *Colloq. Math.* **35** (1976), 119—140.
[8] F. SCALORA, Abstract martingale convergence theorems, *Pacific J. Math.* **11** (1961), 347—374.
[9] A. V. SKOROHOD, Integration in Hilbert space, *Springer, Berlin—Heidelberg—New York,* 1974.

*(Received March 12, 1982.)*