The neutrix distribution product $x_+^\lambda \circ x_-^\mu$

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The ordinary summable function x_+^λ is defined for $\lambda > -1$ by

$$x_+^\lambda = \begin{cases} x^\lambda, & \text{for} \quad x > 0, \\ 0, & \text{for} \quad x < 0. \end{cases}$$

The distribution x_+^λ is defined for $-r-1 < \lambda < -r$ and $r = 1, 2, \ldots$ by

$$x_+^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+r+1)} \frac{d^r}{dx^r} x_+^{\lambda+r},$$

where Γ denotes the gamma function.

The ordinary summable function x_-^μ is defined for $\mu > -1$ by

$$x_-^\mu = \begin{cases} |x|^\mu, & \text{for} \quad x < 0, \\ 0, & \text{for} \quad x > 0. \end{cases}$$

The distribution x_-^μ is defined for $-r < \mu < -r$ and $r = 1, 2, \ldots$ by

$$x_-^\mu = \frac{(-1)^r \Gamma(\mu+1)}{\Gamma(\mu+r+1)} \frac{d^r}{dx^r} x_-^{\mu+r}.$$

The following definition for the product fg of two distributions f and g was given in [2].

Definition 1. Let f and g be two distributions for which on the open interval (a, b), f is the r-th derivative of an ordinary summable function F in $L^p(a, b)$ and $g^{(r)}$ is an ordinary summable function in $L^q(a, b)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by

$$fg = gf = \sum_{i=0}^{r} {}_r c_i (-1)^i [F g^{(i)}]^{(r-i)},$$

where

$${}_r c_i = \frac{r!}{i!\,(r-i)!}.$$

The following theorem follows immediately from this definition.

Theorem 1. *The product* $x_+^\lambda x_-^\mu$ *exists and*

$$x_+^\lambda x_-^\mu = 0$$

for $\lambda+\mu>-1$.

Now let ϱ be a fixed infinitely differentiable function having the properties

(i) $\varrho(x)=0$, for $|x|\geqq 1$,
(ii) $\varrho(x)\geqq 0$,
(iii) $\varrho(x)=\varrho(-x)$,
(iv) $\int_{-1}^{1} \varrho(x)\,dx=1$.

Define the function δ_n by

$$\delta_n(x) = n\varrho(nx)$$

for $n=1, 2, \ldots$. It is obvious that $\{\delta_n\}$ is a sequence of infinitely differentiable functions converging to the Dirac deltafunction δ. For an arbitrary distribution f define the function f_n by

$$f_n(x) = f*\delta_n = \int_{-1/n}^{1/n} f(x-t)\delta_n(t)\,dt$$

for $n=1, 2, \ldots$. It is obvious that $\{f_n\}$ is a sequence of infinitely differentiable functions converging to f.

The following definition was given in [3] and extends definition 1 so that a wider class of distribution products could be defined, the resulting product not necessarily being commutative.

Definition 2. Let f and g be arbitrary distributions and let

$$g_n = g*\delta_n.$$

We say that the product $f\circ g$ of f and g exists and is equal to h on the open interval (a, b) if

$$\lim_{n\to\infty} (fg_n, \varphi) = \lim_{n\to\infty} (f, g_n\varphi) = (h, \varphi)$$

for all test functions φ with compact support contained in the interval (a, b).

That definition 2 is an extension of definition 1 is shown by the following theorem which was proved in [3].

Theorem 2. *Let* f *and* g *be distributions. If the product* fg *exists on the open interval* (a, b) *then the product* $f\circ g$ *and* $g\circ f$ *exist and*

$$f\circ g = g\circ f = fg$$

on this interval.

The next definition was given by VAN DER CORPUT, see [1].

Definition 3. A neutrix N is a commutative additive group of functions $\nu(\xi)$ defined on a domain N' with values in an additive group N'', where further if for some ν in N, $\nu(\xi)=\gamma$ for all ξ in N', then $\gamma=0$. The functions in N are called negligible functions. Now let N' be a set contained in a topological space with a limit point b which does not belong to N'. If $f(\xi)$ is a function defined on N' with values in N'' and it is possible to find a constant β such that $f(\xi)-\beta$ is negligible in N, then β is called the neutrix limit or N-limit of f as ξ tends to b and we write

$$N-\lim_{\xi\to b} f(\xi) = \beta,$$

where the limit β must be unique if it exists.

The following definition was given in [4].

Definition 4. Let f and g be arbitrary distributions and let

$$g_n = g * \delta_n.$$

We say that the neutrix product $f\circ g$ of f and g exists and is equal to h on the open interval (a, b) if

$$N-\lim_{n\to\infty} (fg_n, \varphi) = N-\lim_{n\to\infty} (f, g_n\varphi) = (h, \varphi)$$

for all test functions φ with compact support contained in the interval (a, b), where N is the neutrix having domain $N'=\{1, 2, ..., n, ...\}$ and range N'' the real numbers with negligible functions linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n$$

for $\lambda>0$ and $r=1, 2, ...$ and all functions $f(n)$ for which $\lim_{n\to\infty} f(n)=0$.

The following theorems were given in [4].

Theorem 3. *Let f and g be distributions for which the product $f\circ g$ exists by definition 2. Then the neutrix product $f\circ g$ exists and defines the same distribution.*

Theorem 4. *Let f and g be distributions and suppose that the neutrix products $f\circ g$ and $f\circ g'$ exist on the open interval (a, b). Then the neutrix product $f'\circ g$ exists on the interval (a, b) and*

$$(f\circ g)' = f'\circ g + f\circ g'$$

on the interval (a, b).

Theorem 5. *The neutrix products $x_+^\lambda \circ x_-^{-\lambda-r}$ and $x_-^{-\lambda-r}\circ x_+^\lambda$ exist and*

$$x_+^\lambda \circ x_-^{-\lambda-r} = x_-^{-\lambda-r}\circ x_+^\lambda = -\frac{\pi \operatorname{cosec}(\pi\lambda)}{2(r-1)!}\delta^{r-1}(x)$$

for $\lambda\neq 0, \pm 1, \pm 2, ...$ and $r=1, 2, ...$

We now prove the following theorem.

Theorem 6. The neutrix products $x_+^\lambda \circ x_-^\mu$ and $x_-^\lambda \circ x_+^\mu$ exist and

$$x_+^\lambda \circ x_-^\mu = x_-^\lambda \circ x_+^\mu = 0 \tag{1}$$

for $\lambda+\mu\neq -1, -2, ...$

Proof. We first of all note that if $\lambda+\mu>-1$ then equations (1) follow from theorems 1, 2, and 3.

Now suppose that $\lambda > -1$ and $-k < \mu < -k+1$, where k is a positive integer. Then

$$x_-^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+k)} \frac{d^{k-1}}{dx^{k-1}} x_-^{\mu+k-1},$$

$x_-^{\mu+k-1}$ being an ordinary summable function and so

$$(x_-^{\mu})_n = x_-^{\mu} * \delta_n = \frac{\Gamma(\mu+1)}{\Gamma(\mu+k)} \int_x^{1/n} (t-x)^{\mu+k-1} \delta_n^{(k-1)}(t)\, dt.$$

Thus

$$\frac{\Gamma(\mu+k)}{\Gamma(\mu+1)} \int_{-\infty}^{\infty} x_+^{\lambda} (x_-^{\mu})_n x^m \, dx =$$

$$= \int_0^{1/n} x^{\lambda+n} \int_x^{1/n} (t-x)^{\mu+k-1} \delta_n^{(k-1)}(t)\, dt\, dx$$

$$= \int_0^{1/n} \delta_n^{(k-1)}(t) \int_0^t x^{\lambda+m} (t-x)^{\mu+k-1}\, dx\, dt$$

$$= \int_0^{1/n} t^{\lambda+\mu+m+k} \delta_n^{(k-1)}(t) \int_0^1 v^{\lambda+m} (1-v)^{\mu+k-1}\, dv\, dt$$

$$= B(\lambda+m+1, \mu+k) \int_0^{1/n} t^{\lambda+\mu+m+k} \delta_n^{(k-1)}(t)\, dt,$$

where the substitution $x = tv$ has been made and B denotes the beta function.

Making the substitution $nt = s$ we have

$$\int_0^{1/n} t^{\lambda+\mu+m+k} \delta_n^{(k-1)}(t)\, dt = n^{-\lambda-\mu-m-1} \int_0^1 s^{\lambda+\mu+m+k} \varrho^{(k-1)}(s)\, ds$$

and it follows that the functions

$$\int_{-\infty}^{\infty} x_+^{\lambda} (x_-^{\mu})_n x^m \, dx$$

are negligible, or zero, for $m = 0, 1, 2, \ldots$ and $\lambda+\mu \neq -1, -2, \ldots$. Further, choosing a positive integer $p > -\lambda-\mu-1$, we see that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |x_+^{\lambda} (x_-^{\mu})_n x^p|\, dx = 0.$$

Now let φ be an arbitrary test function. Then

$$\varphi(x) = \sum_{m=0}^{p-1} \frac{x^m}{m!} \varphi^{(m)}(0) + \frac{x^p}{p!} \varphi^{(p)}(\xi x) \tag{2}$$

where $0 \leqq \xi \leqq 1$ and so

$$(x_+^\lambda, (x_-^\mu)_n \varphi(x)) = \sum_{m=0}^{p-1} \frac{\varphi^{(m)}(0)}{m!} \int_{-\infty}^{\infty} x_+^\lambda (x_-^\mu)_n x^m \, dx + \frac{1}{p!} \int_{-\infty}^{\infty} x_+^\lambda (x_-^\mu)_n x^p \varphi^{(p)}(\xi x) \, dx.$$

Since

$$\left| \int_{-\infty}^{\infty} x_+^\lambda (x_-^\mu)_n x^p \varphi^{(p)}(\xi x) \, dx \right| \leqq \sup_x \{|\varphi^{(p)}(x)|\} \cdot \int_{-\infty}^{\infty} |x_+^\lambda (x_-^\mu)_n x^p| \, dx$$

it follows from what we have just proved that

$$N - \lim_{n\to\infty} (x_+^\lambda, (x_-^\mu)_n \varphi(x)) = 0 = (0, \varphi).$$

Thus

(3) $$x_+^\lambda \circ x_-^\mu = 0$$

when $\lambda > -1$ and $\mu, \lambda+\mu \neq -1, -2, \ldots$.

Now assume that equation (3) holds when $-k < \lambda < -k+1$ and $\mu, \lambda+\mu \neq -1, -2, \ldots$, where k is a positive integer. It follows from theorem 4 that

$$(x_+^\lambda \circ x_-^\mu)' = 0 = \lambda x_+^{\lambda-1} \circ x_-^\mu - \mu x_+^\lambda \circ x_-^{\mu-1},$$

provided that $\mu \neq 0$ and it follows from our assumption that

(4) $$x_+^{\lambda-1} \circ x_-^\mu = 0$$

when $-k < \lambda < -k+1$ and $\mu - 1, \lambda+\mu \neq -1, -2, \ldots$. To deal with the particular case $\mu = 0$ we notice that the derivative of x_-^0 is $-\delta(x)$. Now it is easily proved that

$$x_+^\lambda \circ \delta(x) = 0$$

for $\lambda \neq 0, -1, -2, \ldots$ and this result was in fact given in [5]. It follows from theorem 4 that

$$(x_+^\lambda \circ x_-^0)' = 0 = \lambda x_+^{\lambda-1} \circ x_-^0 - x_+^\lambda \circ \delta(x)$$

and equation (4) follows from our assumption when $-k < \lambda < -k+1$ and $\mu = 0$. This establishes equation (3) by induction when $\lambda, \mu, \lambda+\mu \neq -1, -2, \ldots$.

We will now consider the neutrix product $x_+^\lambda \circ x_-^{-r}$ for $\lambda, \lambda - r \neq -1, -2, \ldots$ and $r = 1, 2, \ldots$. By definition

$$\frac{d^r}{dx^r} \ln x_- = -(r-1)! \, x_-^{-r},$$

$\ln x_-$ being an ordinary summable function and so

$$(x_-^{-r})_n = x_-^{-r} * \delta_n = -\frac{1}{(r-1)!} \int_x^{1/n} \ln(t-x) \, \delta_n^{(r)}(t) \, dt.$$

Thus if $\lambda > -1$

$$-(r-1)! \int_{-\infty}^{\infty} x_+^{\lambda} (x_-^{-r})_n x^m \, dx = \int_0^{1/n} x^{\lambda+m} \int_x^{1/n} \ln(t-x) \delta_n^{(r)}(t) \, dt \, dx$$

$$= \int_0^{1/n} \delta_n^{(r)}(t) \int_0^t x^{\lambda+m} \ln(t-x) \, dx \, dt$$

$$= \int_0^{1/n} t^{\lambda+m+1} \delta_n^{(r)}(t) \int_0^1 v^{\lambda+m} [\ln(1-v) + \ln t] \, dv \, dt$$

$$= \left\{ \int_0^1 v^{\lambda+m} \ln(1-v) \, dv \right\} \left\{ \int_0^{1/n} t^{\lambda+m+1} \delta_n^{(r)}(t) \, dt \right\} + \frac{1}{\lambda+m+1} \int_0^{1/n} t^{\lambda+m+1} \ln t \delta_n^{(r)}(t) \, dt,$$

where the substitution $x = tv$ has been made.

Making the substitution $nt = s$ we see that the functions

$$\int_{-\infty}^{\infty} x^{\lambda} (x_-^{-r})_n x^m \, dx$$

are negligible, or zero, for $n = 0, 1, 2, \ldots$ and $\lambda - r \neq -1, -2, \ldots$. Further, choosing a positive integer $p > r - \lambda - 1$, we see that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} |x_+^{\lambda} (x_-^{-r})_n x^p| \, dx = 0.$$

Now let φ be an arbitrary test function. Then equation (2) holds and it follows that

$$N-\lim_{n \to \infty} \left(x_+^{\lambda}, (x_-^{-r})_n \varphi(x)\right) = 0 = (0, \varphi).$$

Thus

$$x_+^{\lambda} \circ x_-^{-r} = 0 \tag{5}$$

when $\lambda > -1$, $\lambda - r \neq -1, -2, \ldots$ and $r = 1, 2, \ldots$.

We will now assume that equation (5) holds when $-k < \lambda < -k+1$ and $r = 1, 2, \ldots$, where k is a positive integer. It follows from theorem 4 that

$$(x_+^{\lambda} \circ x_-^{-r})' = 0 = \lambda x_+^{\lambda-1} \circ x_-^{-r} + r x_+^{\lambda} \circ x_-^{-r-1}$$

and it follows from our assumption that

$$x_+^{\lambda-1} \circ x_-^{-r} = 0$$

when $-k < \lambda < -k+1$ and $r = 1, 2, \ldots$. This establishes equation (5) by induction for $\lambda, \lambda - r \neq -1, -2, \ldots$ and $r = 1, 2, \ldots$.

We will now consider the neutrix product $\ln x_+ \circ x_-^{\mu}$ for $\mu \neq -1, -2, \ldots$. It follows from theorems 1, 2 and 3 that this product exists and

$$\ln x_+ \circ x_-^{\mu} = 0 \tag{6}$$

for $\mu > -1$. Let us therefore suppose that $-k < \mu < -k+1$, where k is a positive integer. Then

$$\frac{\Gamma(\mu+k)}{\Gamma(\mu+1)} \int_{-\infty}^{\infty} \ln x_+ (x_-^\mu)_n x^m \, dx = \int_0^{1/n} x^m \ln x \int_x^{1/n} (t-x)^{\mu+k-1} \delta_n^{(k-1)}(t) \, dt \, dx$$

$$= \int_0^{1/n} \delta_n^{(k-1)}(t) \int_0^t x^m (t-x)^{\mu+k-1} \ln x \, dx \, dt$$

$$= \int_0^{1/n} t^{\mu+m+k} \delta_n^{(k-1)}(t) \int_0^1 v^m (1-v)^{\mu+k-1} (\ln t + \ln v) \, dv \, dt$$

$$= B(m+1, \mu+k) \int_0^{1/n} t^{\mu+m+k} \ln t \delta_n^{(k-1)}(t) \, dt +$$

$$+ \Big\{ \int_0^1 v^m (1-v)^{\mu+k-1} \ln v \, dv \Big\} \Big\{ \int_0^{1/n} t^{\mu+m+k} \delta_n^{(k-1)}(t) \, dt \Big\},$$

where the substitution $x = tv$ has been made.

Making the substitution $nt = s$ we see that the functions

$$\int_{-\infty}^{\infty} \ln x_+ (x_-^\mu)_n x^m \, dx$$

are negligible, or zero, for $m = 0, 1, 2, \dots$. Further, choosing a positive integer $p > -\mu - 1$, we see that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |\ln x_+ (x_-^\mu)_n x^p| \, dx = 0.$$

Now let φ be an arbitrary test function. Then equation (2) holds and it follows that

$$N-\lim_{n\to\infty} \big(\ln x_+, (x_-^\mu)_n \varphi(x)\big) = 0 = (0, \varphi),$$

when $-k < \mu < -k+1$. This establishes equation (6) for $\mu \neq -1, -2, \dots$.

It now follows from theorem 4 that

$$(\ln x_+ \circ x_-^\mu)' = 0 = x_+^{-1} \circ x_-^\mu - \mu \ln x_+ \circ x_-^{\mu-1}$$

$$= x_+^{-1} \circ x_-^\mu$$

for $\mu \neq 0, -1, -2, \dots$. Let us therefore assume that the neutrix product $x_+^{-r} \circ x_-^\mu$ exists and

$$x_+^{-r} \circ x_-^\mu = 0 \tag{7}$$

for some positive integer r and $\mu, \mu - r \neq -1, -2, \dots$. Then by theorem 4

$$(x_+^{-r} \circ x_-^\mu)' = 0 = -r x_+^{-r-1} \circ x_-^\mu - \mu x_+^{-r} \circ x_-^{\mu-1}$$

$$= -r x_+^{-r-1} \circ x_-^\mu$$

from our assumption. This establishes equation (7) by induction for $\mu, \mu-r\neq -1, -2, \ldots$.

We have therefore proved the neutrix product $x_+^\lambda \circ x_-^\mu$ exists and

$$x_+^\lambda \circ x_-^\mu = 0$$

for $\lambda+\mu\neq -1, -2, \ldots$. By replacing x by $-x$ in this equation we see that the neutrix product $x_-^\lambda \circ x_+^\mu$ exists and

$$x_-^\lambda \circ x_+^\mu = 0$$

for $\lambda+\mu\neq -1, -2, \ldots$. This completes the proof of the theorem.

We can now define further neutrix products of distributions. For example let $\sin x_+^{\frac{1}{2}}$ be the ordinary summable function defined by

$$\sin x_+^{\frac{1}{2}} = \sum_{m=1}^{\infty} x_+^{m-\frac{1}{2}}/(2m-1)!.$$

Then for a non-negative integer r the function

$$\sum_{m=r+1}^{\infty} x_+^{m-\frac{1}{2}}/(2m-1)!$$

is r times continuously differentiable and it follows from definition 1 and theorem 2 that

$$\left\{\sum_{m=r+1}^{\infty} x_+^{m-\frac{1}{2}}/(2m-1)!\right\}\circ x_-^{-r-\frac{1}{2}} = x_-^{-r-\frac{1}{2}}\circ\left\{\sum_{m=r+1}^{\infty} x_+^{m-\frac{1}{2}}/(2m-1)!\right\} = 0.$$

Since the neutrix product is obviously distributive with respect to addition we see that

$$\sin x_+^{\frac{1}{2}}\circ x_-^{-r-\frac{1}{2}} = \sum_{m=1}^{r} (x_+^{m-\frac{1}{2}}\circ x_-^{-r-\frac{1}{2}})/(2m-1)! + \left\{\sum_{m=r+1}^{\infty} x_+^{m-\frac{1}{2}}/(2m-1)!\right\}\circ x_-^{-r-\frac{1}{2}} =$$

$$= \sum_{m=1}^{r} \frac{(-1)^m \pi}{2(r-m)!\,(2m-1)!}\,\delta^{(r-m)}(x) = x_-^{-r-\frac{1}{2}}\circ \sin x_+^{\frac{1}{2}},$$

where theorems 5 and 6 have been used, for $r=1, 2, \ldots$ and

$$\sin x_+^{\frac{1}{2}}\circ x_-^{-\frac{1}{2}} = x_-^{-\frac{1}{2}}\circ \sin x_+^{\frac{1}{2}} = 0.$$

Now define the function $x_+^{-\frac{1}{2}}\cos x_+^{\frac{1}{2}}$ by

$$x_+^{-\frac{1}{2}}\cos x_+^{\frac{1}{2}} = 2(\sin x_+^{\frac{1}{2}})' = \sum_{m=0}^{\infty} x_+^{m-\frac{1}{2}}/(2m)!.$$

Then it can be proved similarly that

$$(x_+^{-\frac{1}{2}}\cos x_+^{\frac{1}{2}})\circ x_-^{-r-\frac{1}{2}} = x_-^{-r-\frac{1}{2}}\circ(x_+^{-\frac{1}{2}}\cos x_+^{\frac{1}{2}}) = \sum_{m=0}^{r} \frac{(-1)^m \pi}{2(r-m)!\,(2m)!}\,\delta^{(r-m)}(x)$$

for $r=0, 1, 2, \ldots$.

The Bessel function $J_\nu(x)$, see for exampfe SNEDDON [6], is defined by

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{1}{2}x\right)^{\nu+2m}}{m!\,\Gamma(m+\nu+1)}$$

for $\nu \neq -1, -2, \ldots$. We will define the distribution $J_\nu(x_+)$ by

$$J_\nu(x_+) = \sum_{m=0}^{\infty} \frac{(-1)^m \frac{1}{2}^{\nu+2m} x_+^{\nu+2m}}{m!\,\Gamma(m+\nu+1)},$$

for $\nu \neq -1, -2, \ldots$, where the x_+^ν for $\nu < -1$ are interpretted in the distributional sense. It follows as above that

$$J_\nu(x_+) \circ x_-^{-r-\nu} = x_-^{-r-\nu} \circ J_\nu(x_+) =$$

$$= -\frac{1}{2}\pi \operatorname{cosec}(\nu\pi) \sum_{m=0}^{k} \frac{(-1)^m \frac{1}{2}^{\nu+2m}}{m!\,(r-2m-1)!\,\Gamma(m+\nu+1)}\, \delta^{(r-2m-1)}(x),$$

where

$$k = \begin{cases} \frac{1}{2}(r-1), & r \text{ odd}, \\ \frac{1}{2}(r-2) & r \text{ even} \end{cases}$$

for $r = 1, 2, \ldots$ and $\nu \neq 0, \pm 1, \pm 2, \ldots$.

References

[1] J. G. VAN DER CORPUT, Introduction to the neutrix calculus, *J. Analyse Math.* **7** (1959—60), 291—398.

[2] B. FISHER, The product of distributions, *Quart. J. Math. (2)*, **22** (1971), 291—298.

[3] B. FISHER, On defining the product of distributions, *Math. Nachr.* **99** (1980), 239—249.

[4] B. FISHER, A non-commutative neutrix product of distributions, *Math. Nachr.*, **108** (1982), 117—127.

[5] B. FISHER, The neutrix distribution product $x_+^{\lambda} \circ \delta^{(r)}(x)$, *Indian J. Pure Appl. Math., to appear.*

[6] I. N. SNEDDON, Special functions of mathematical physics and chemistry, *Longman Mathematical Texts,* (1980).

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