Syntopogenous g-families and their applications in extension theory

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0. Introduction

In [8] the following generalization of the notion of an Hacque's E-mapping ([5]—[6]) was introduced:

For a given single valued mapping $g: E \to E'$, a mapping $\mathfrak{z}$ of $2^{E'}$ into the set of all systems of subsets of E is called a *g-mapping* iff it satisfies the following conditions for any $A', B' \subset E'$ and $X, Y \subset E$:

(M0) $\mathfrak{z}(A') \neq \emptyset$, *and* $Y \supset X \in \mathfrak{z}(A')$ *implies* $Y \in \mathfrak{z}(A')$.

(M1) $\emptyset \in \mathfrak{z}(A')$ *iff* $A' = \emptyset$.

(M2) $X \in \mathfrak{z}(A')$ *implies* $g^{-1}(A') \subset X$.

(M3) $A' \subset B'$ *implies* $\mathfrak{z}(B') \subset \mathfrak{z}(A')$.

If $\mathfrak{z}_1, \mathfrak{z}_2$ are two g-mappings then $\mathfrak{z}_1 \subset \mathfrak{z}_2$ ($\mathfrak{z}_1$ is *coarser* than $\mathfrak{z}_2$, or $\mathfrak{z}_2$ is *finer* than $\mathfrak{z}_1$) means $\mathfrak{z}_1(A') \subset \mathfrak{z}_2(A')$ for any $A' \subset E'$. A g.mapping $\mathfrak{z}$ is said to be *topogenous* iff $X \in \mathfrak{z}(A')$ and $Y \in \mathfrak{z}(B')$ imply $X \cap Y \in \mathfrak{z}(A' \cap B')$ and $X \cup Y \in \mathfrak{z}(A' \cup B')$. $\mathfrak{z}$ is called *perfect* iff $X_i \in \mathfrak{z}(A'_i)$ $(i \in I)$ implies $\bigcup_{i \in I} X_i \in \mathfrak{z}(\bigcup_{i \in I} A'_i)$. It is known that $\mathfrak{z}$ is a perfect topogenous g-mapping iff, for every $x' \in E'$, there exists a filter $\mathfrak{f}(x')$ in E such that $x \in E$, $X \in \mathfrak{f}(g(x))$ imply $x \in X$, $\mathfrak{z}(\emptyset) = 2^E$, and $\mathfrak{z}(A') = \bigcap_{x' \in A'} \mathfrak{f}(x')$ for any $\emptyset \neq A' \subset E'$.

If $<'$ is a semi-topogenous order ([1]) on E', and $x \in E'$, $x <' V' \subset E'$ imply $g(E) \cap V' \neq \emptyset$, then a g-mapping $\mathfrak{z}_{<'}$ can be defined by

$$\mathfrak{z}_{<'}(A') = \{X \subset E\colon A' <' E' - g(E - X)\}.$$

As an extension of a result of S. GACSÁLYI ([4], prop. 1), the following duality theorem was proved ([8], (2.4)): the correspondence $<' \to \mathfrak{z}_{<'}$ is surjective iff g is injective, and $<' \to \mathfrak{z}_{<'}$ is injective iff g is surjective. Consequently $<' \to \mathfrak{z}_{<'}$ is one-to-one iff so is g. The proof of this theorem required the construction of two semi-topogenous orders: If $\mathfrak{z}$ is a g-mapping then the definitions

$$A' <_{\mathfrak{t}_\mathfrak{z}} B' \Leftrightarrow A' \subset B' \quad \text{and} \quad g^{-1}(B') \in \mathfrak{z}(A'),$$

and

$$A <_{\mathfrak{i}_\mathfrak{z}} B \Leftrightarrow B \in \mathfrak{z}(g(A))$$

yield semi-topogenous orders $<_{\mathfrak{t}_\mathfrak{z}}$ and $<_{\mathfrak{i}_\mathfrak{z}}$ on E' and E respectively.

In the present paper the notion of a *syntopogenous g-family* $\mathfrak{Z}$ will be introduced, which consists of topogenous g-mappings with special properties. If $g(E)$ is dense in the syntopogenous space $[E', \mathscr{S}']$ ([1]), then $\mathfrak{Z}_{\mathscr{S}'}=\{\mathfrak{z}_{<'}: <'\in\mathscr{S}'\}$ is a syntopogenous g-family. Conversely, if $\mathfrak{Z}$ is a given syntopogenous g-family then $\mathscr{S}_{\dagger\mathfrak{Z}}=\{<_{\dagger\mathfrak{z}}: \mathfrak{z}\in\mathfrak{Z}\}$ and $\mathscr{S}_{\downarrow\mathfrak{Z}}=\{<_{\downarrow\mathfrak{z}}: \mathfrak{z}\in\mathfrak{Z}\}$ are syntopogenous structures on E' and E respectively. We shall study the correspondence $\mathscr{S}'\to\mathfrak{Z}_{\mathscr{S}'}$. In Chapter 3 some connections between the extensions of syntopogenous structures and the syntopogenous g-families will be examined. E.g. a class of the extensions of a syntopogenous structure $\mathscr{S}$ will be given, in which the extension $h(\mathscr{S})$ (see [7]; [3], th. 3.1) is the coarsest one, and this is a generalizations of a remarkable property of the topological strict extensions ([2], (6.1.2)).

1. Syntopogenous g-families

This chapter will deal with various kinds of families of g-mappings, therefore first of all we need to mention the following lemmas:

(1.1) Lemma. *If* $\{\mathfrak{z}_i: i\in I\neq\emptyset\}$ *is a family of g-mappings then there exists a g-mapping* $\mathfrak{z}$, *which is the coarsest of all g-mappings finer than every* $\mathfrak{z}_i$ $(i\in I)$. $\mathfrak{z}$ *can be defined by* $\mathfrak{z}(A')=\bigcup_{i\in I}\mathfrak{z}_i(A')$ *for* $A'\subset E'$. $\mathfrak{z}$ *will be denoted by* $\bigsqcup_{i\in I}\mathfrak{z}_i$. ∎

(1.2) Lemma.

(1.2.1) *If* $<'_1$ *and* $<'_2$ *are semi-topogenous orders on* E', $g(E)$ *is* $<'_2$*-dense* ([8], ch. 2) *and* $<'_1\sqsubset<'_2$, *then* $g(E)$ *is also* $<'_1$*-dense, and* $\mathfrak{z}_{<'_1}\sqsubset\mathfrak{z}_{<'_2}$.

(1.2.2) *If* $\mathfrak{z}_1$ *and* $\mathfrak{z}_2$ *are g-mappings and* $\mathfrak{z}_1\sqsubset\mathfrak{z}_2$, *then* $<_{\dagger\mathfrak{z}_1}\sqsubset<_{\dagger\mathfrak{z}_2}$ *and* $<_{\downarrow\mathfrak{z}_1}\sqsubset<_{\downarrow\mathfrak{z}_2}$. ∎

(1.3) Proposition. *Let* $\mathfrak{z}$ *be a g-mapping. We have a g-mapping denoted by* $\mathfrak{z}^2$, *for which*

(1.3.1) $X\in\mathfrak{z}^2(A')$ *iff there exists* $Y\in\mathfrak{z}(A')$ *such that* $X\in\mathfrak{z}(g(Y))$.
$\mathfrak{z}^2$ *is coarser than* $\mathfrak{z}$, *and it has the properties listed below:*

(1.3.2) *If* $<'$ *is a semi-topogenous order on* E', *and* $g(E)$ *is* $<'$*-dense, then* $\mathfrak{z}^2_{<'}\sqsubset\mathfrak{z}^2_{<'^2}$.

(1.3.3) *If* g *is injective and* $\mathfrak{z}$ *is topogenous, then* $<_{\dagger\mathfrak{z}^2}\sqsubset<^2_{\dagger\mathfrak{z}}$.

(1.3.4) $<_{\downarrow\mathfrak{z}^2}\sqsubset<^2_{\downarrow\mathfrak{z}}$ *always holds.*

Proof. $\mathfrak{z}^2(A')\subset\mathfrak{z}(A')$ is true, because $X\in\mathfrak{z}^2(A')$ implies the existence of a set $Y\in\mathfrak{z}(A')$ such that $X\in\mathfrak{z}(g(Y))$, and by (M2) $X\supset g^{-1}(g(Y))\supset Y\in\mathfrak{z}(A')$, thus in view of (M0) $X\in\mathfrak{z}(A')$. $\mathfrak{z}^2$ is a g-mapping. (M0): $E\in\mathfrak{z}^2(A')$, since $E\in\mathfrak{z}(A')$ and $E\in\mathfrak{z}(g(E))$. If $Y\supset X\in\mathfrak{z}^2(A')$ then for a suitable $Y_0\in\mathfrak{z}(A')$ we have $Y\supset X\in\mathfrak{z}(g(Y_0))$, therefore $Y\in\mathfrak{z}^2(A')$. (M1): $\emptyset\in\mathfrak{z}(\emptyset)$ and $\emptyset\in\mathfrak{z}(g(\emptyset))$, thus $\emptyset\in\mathfrak{z}^2(\emptyset)$. Conversely, if $\emptyset\in\mathfrak{z}^2(A')$, and $Y\in\mathfrak{z}(A')$ such that $\emptyset\in\mathfrak{z}(g(Y))$, then by (M2) $Y\subset g^{-1}(g(Y))\subset\emptyset$, thus $Y=\emptyset$. Consequently because of property (M1) of $\mathfrak{z}$ we get $A'=\emptyset$. (M2): If $X\in\mathfrak{z}^2(A')$ and $Y\in\mathfrak{z}(A')$ such that $X\in\mathfrak{z}(g(Y))$, then $g^{-1}(A')\subset Y\subset g^{-1}(g(Y))\subset X$. (M3): If $X\in\mathfrak{z}^2(B')$ and $A'\subset B'$, then from $Y\in\mathfrak{z}(B')\subset\mathfrak{z}(A')$ and $X\in\mathfrak{z}(g(Y))$ the relation $X\in\mathfrak{z}^2(A')$ follows.

(1.3.2): Suppose $X \in \mathfrak{z}_{<'^2}(A')$. Then $A' <' C <' E' - g(E-X)$, but this implies $g^{-1}(C') \in \mathfrak{z}_{<'}(A')$, and in view of $g(g^{-1}(C')) \subset C'$, we have $X \in \mathfrak{z}_{<'}(g(g^{-1}(C')))$, that is $X \in \mathfrak{z}^2_{<'}(A')$.

(1.3.3): Let us assume that g is injective and $\mathfrak{z}$ is topogenous. Then $A' <_{\uparrow\mathfrak{z}^2} B'$ implies $A' \subset B'$ and $g^{-1}(B') \in \mathfrak{z}^2(A')$. This means that there exists a set $Y \in \mathfrak{z}(A')$ such that $g^{-1}(B') \in \mathfrak{z}(g(Y))$. Putting $C' = A' \cup g(Y)$, we get $A' \subset C'$ and $g^{-1}(C') = Y \in \mathfrak{z}(A')$, since $g^{-1}(A') \subset Y$. Similarly $Y = g^{-1}(g(Y)) \subset g^{-1}(B')$ gives that $g(Y) \subset g(g^{-1}(B')) \subset B'$ and $g^{-1}(B') \in \mathfrak{z}(A')$. From these $C' \subset B'$ and $g^{-1}(B') \in \mathfrak{z}(A' \cup g(Y)) = \mathfrak{z}(C')$ follows, so that $A' <_{\uparrow\mathfrak{z}} C' <_{\uparrow\mathfrak{z}} B'$.

(1.3.4): If $A <_{\downarrow\mathfrak{z}^2} B$ then $B \in \mathfrak{z}^2(g(A))$, thus there exists a set $Y \in \mathfrak{z}(g(A))$ such that $B \in \mathfrak{z}(g(Y))$. This is equivalent to $A <_{\downarrow\mathfrak{z}} Y <_{\downarrow\mathfrak{z}} B$. ∎

A family $\mathfrak{Z}$ of topogenous g-mappings will be called a *syntopogenous g-family*, if the following conditions are fulfilled:

(F1) *For any* $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{Z}$ *there exists* $\mathfrak{z} \in \mathfrak{Z}$ *such that* $\mathfrak{z}_1 \cup \mathfrak{z}_2 \subset \mathfrak{z}$.

(F2) *If* $\mathfrak{z} \in \mathfrak{Z}$ *then there is* $\mathfrak{z}_1 \in \mathfrak{Z}$ *with* $\mathfrak{z} \subset \mathfrak{z}_1^2$.

If $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ are syntopogenous g-families, then we shall say that $\mathfrak{Z}_1$ is *coarser* than $\mathfrak{Z}_2$, or equivalently $\mathfrak{Z}_2$ is *finer* than $\mathfrak{Z}_1$, iff for any $\mathfrak{z}_1 \in \mathfrak{Z}_1$ there exists $\mathfrak{z}_2 \in \mathfrak{Z}_2$ such that $\mathfrak{z}_1 \subset \mathfrak{z}_2$. This fact will be denoted by $\mathfrak{Z}_1 < \mathfrak{Z}_2$. We shall write $\mathfrak{Z}_1 \sim \mathfrak{Z}_2$ iff $\mathfrak{Z}_1 < \mathfrak{Z}_2$ and $\mathfrak{Z}_2 < \mathfrak{Z}_1$. Such families will be said to be *equivalent*.

A syntopogenous g-family is *topogenous*, if it consists of a single topogenous g-mapping.

(1.4) Proposition. *If* $\mathfrak{Z}$ *is a syntopogenous g-family then a topogenous g-family* $\mathfrak{Z}^t$ *can be defined as follows:*

(1.4.1) $\mathfrak{Z}^t = \{\mathfrak{z}_0\}$, *where* $\mathfrak{z}_0 = \bigcup\{\mathfrak{z} : \mathfrak{z} \in \mathfrak{Z}\}$.

$\mathfrak{Z}^t$ *is the coarset of all topogenous g-families finer than* $\mathfrak{Z}$.

PROOF. Let us prove that $\mathfrak{z}_0$ is topogenous. Suppose $X \in \mathfrak{z}_0(A')$ and $Y \in \mathfrak{z}_0(B')$. Then $X \in \mathfrak{z}_1(A')$ and $Y \in \mathfrak{z}_2(B')$ for some $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{Z}$. If $\mathfrak{z}_1 \cup \mathfrak{z}_2 \subset \mathfrak{z} \in \mathfrak{Z}$, then from the topogenity of $\mathfrak{z}$ we can deduce $X \cap Y \in \mathfrak{z}(A' \cap B') \subset \mathfrak{z}_0(A' \cap B')$ and $X \cup Y \in \mathfrak{z}(A' \cup B') \subset \mathfrak{z}_0(A' \cup B')$. (F1) is obviously satisfied by $\mathfrak{Z}^t$. Finally put $X \in \mathfrak{z}_0(A')$. Then $X \in \mathfrak{z}(A')$ for a suitable $\mathfrak{z} \in \mathfrak{Z}$. By (F2) $X \in \mathfrak{z}_1^2(A')$ for some $\mathfrak{z}_1 \in \mathfrak{Z}$, and from this $X \in \mathfrak{z}_0^2(A')$ follows, hence (F2) is also fulfilled. Clearly $\mathfrak{Z} < \mathfrak{Z}^t$. Let $\mathfrak{Z}_1$ be a topogenous g-family finer than $\mathfrak{Z}$. Then, for $\mathfrak{Z}_1 = \{\mathfrak{z}_1\}$, we have $\mathfrak{z} \subset \mathfrak{z}_1$ for every $\mathfrak{z} \in \mathfrak{Z}$, therefore $\mathfrak{z}_0 \subset \mathfrak{z}_1$ and $\mathfrak{Z}^t < \mathfrak{Z}_1$. ∎

A syntopogenous g-family will be said to be *perfect* if its elements are perfect topogenous g-mappings.

(1.5) Proposition. *Let* $\mathfrak{Z}$ *be a syntopogenous g-family. Then the definition*

(1.5.1) $$\mathfrak{Z}^p = \{\mathfrak{z}^p : \mathfrak{z} \in \mathfrak{Z}\},$$

where

(1.5.2) $$\mathfrak{z}^p(\emptyset) = 2^E \quad \text{and} \quad \mathfrak{z}^p(A') = \bigcap_{x' \in A'} \mathfrak{z}(x') \quad \text{for} \quad \emptyset \neq A' \subset E',$$

yields a perfect syntopogenous g-family which is the coarsest of all perfect syntopogenous g-families finer than $\mathfrak{Z}$. *If* $\mathfrak{Z}$ *is topogenous then so is* $\mathfrak{Z}^p$, *too.*

PROOF. If $\mathfrak{z}$ is a topogenous g-mapping then $\mathfrak{z}^p$ is a perfect topogenous g-mapping by [8], (1.5). If $\mathfrak{z}_1, \mathfrak{z}_2 \in \mathfrak{Z}$ then $\mathfrak{z}_1 \cup \mathfrak{z}_2 \in \mathfrak{Z}$ implies $\mathfrak{z}_1^p \cup \mathfrak{z}_2^p \subset \mathfrak{z}^p$. If $\mathfrak{z} \in \mathfrak{Z}$ and $\mathfrak{z}_1 \in \mathfrak{Z}$ such that $\mathfrak{z} \subset \mathfrak{z}_1^2$, then $\mathfrak{z}^p \subset \mathfrak{z}_1^{p2}$. In fact, suppose $X \in \mathfrak{z}^p(A')$, $A' \neq \emptyset$. Then, for any $x' \in A'$, there is a set $Y_{x'} \in \mathfrak{z}_1(x')$ such that $X \in \mathfrak{z}_1(g(Y_{x'}))$. It is easy to show $Y = \bigcup_{x' \in A'} Y_{x'} \in \mathfrak{z}_1^p(A')$, and $X \in \mathfrak{z}_1^p\big(\bigcup_{x' \in A'} g(Y_{x'})\big) = \mathfrak{z}_1^p(g(Y))$, that is $X \in \mathfrak{z}_1^{p2}(A')$. If $A' = \emptyset$ then $X \in \mathfrak{z}_1^{p2}(A')$ is trivial by (M1) and (M0). From [8], (1.5) $\mathfrak{Z} < \mathfrak{Z}^p$ follows, and similarly, if $\mathfrak{Z} < \mathfrak{Z}_1$ for a perfect syntopogenous g-family $\mathfrak{Z}_1$, then $\mathfrak{Z}^p < \mathfrak{Z}_1$. If $\mathfrak{Z}$ is topogenous then $\mathfrak{Z}^p$ consists of a single g-mapping, too. ∎

(1.6) Corollary. *If* $\mathfrak{Z}$ *is a syntopogenous g-family then* $\mathfrak{Z}^{tp}$ *is the coarsest of all perfect topogenous g-families finer than* $\mathfrak{Z}$. ∎

2. Syntopogenous g-families and syntopogenous structures

Let us consider a syntopogenous structure $\mathscr{S}'$ on E' such that $g(E)$ is $\mathscr{S}'$*-dense* (that is $g(E)$ is $<'$-dense for every $<' \in \mathscr{S}'$). Then a family of topogenous g-mappings is obtained by

(2.1) $$\mathfrak{Z}_{\mathscr{S}'} = \{\mathfrak{z}_{<'}\colon <' \in \mathscr{S}'\}$$

(cf. [8], (2.3.1)).

(2.2) Proposition. *If* $\mathscr{S}'$ *is a syntopogenous structure on* E' *such that* $g(E)$ *is* $\mathscr{S}'$*-dense, then* $\mathfrak{Z}_{\mathscr{S}'}$ *is a syntopogenous g-family having the following properties:*

(2.2.1) $$\mathfrak{Z}^t_{\mathscr{S}'} = \mathfrak{Z}_{\mathscr{S}'^t}.$$

(2.2.2) $$\mathfrak{Z}^p_{\mathscr{S}'} = \mathfrak{Z}_{\mathscr{S}'^p}.$$

(2.2.3) $$\mathfrak{Z}^{tp}_{\mathscr{S}'} = \mathfrak{Z}_{\mathscr{S}'^{tp}}.$$

(2.2.4) *If* $\mathscr{S}'$ *is topogenous or perfect, then* $\mathfrak{Z}_{\mathscr{S}'}$ *also has the corresponding property.*

PROOF. If $<'_1, <'_2 \in \mathscr{S}'$ and $<'_1 \cup <'_2 \subset <' \mathscr{S}'$, then from (1.2.1) $\mathfrak{z}_{<'_1} \cup \mathfrak{z}_{<'_2} \subset \mathfrak{z}_{<'}$ follows. Put $<' \in \mathscr{S}'$ and $<'_1 \in \mathscr{S}'$ so that $<' \subset <'^2_1$, then in view of (1.3.2) we have $\mathfrak{z}_{<'} \subset \mathfrak{z}_{<'^2_1} \subset \mathfrak{z}^2_{<'_1}$, thus $\mathfrak{Z}_{\mathscr{S}'}$ is a syntopogenous g-family. The properties (2.2.1)—(2.2.3) are obvious (see [8], (2.3.2)), and because of (1.4)—(1.5) the statement (2.2.4) is their direct consequence. ∎

Furthei we shall study the correspondence $\mathscr{S}' \to \mathfrak{Z}_{\mathscr{S}'}$.

(2.3) Theorem. *Let* g *be an injection, and* $\mathfrak{Z}$ *be a syntopogenous g-family. Then*

(2.3.1) $$\mathscr{S}_{\dagger\mathfrak{Z}} = \{<_{\dagger\mathfrak{z}}\colon \mathfrak{z} \in \mathfrak{Z}\}$$

is a syntopogenous structure on E', $g(E)$ *is* $\mathscr{S}_{\dagger\mathfrak{Z}}$*-dense, and* $\mathfrak{Z} = \mathfrak{Z}_{\mathscr{S}_{\dagger\mathfrak{Z}}}$. $\mathscr{S}_{\dagger\mathfrak{Z}}$ *is the finest of all syntopogenous structures* $\mathscr{S}'$ *on* E', *for which* $g(E)$ *is* $\mathscr{S}'$*-dense and* $\mathfrak{Z}_{\mathscr{S}'} < \mathfrak{Z}$.

PROOF. By [8], (2.8.1) and statements (1.2.2), (1.3.3) of the present paper $\mathscr{S}_{\uparrow\mathfrak{Z}}$ is in fact a syntopogenous structure on E', and $g(E)$ is $\mathscr{S}_{\uparrow\mathfrak{Z}}$-dense (see [8], (2.7)). From [8], (2.7) $\mathfrak{Z}=\mathfrak{Z}_{\mathscr{S}_{\uparrow\mathfrak{Z}}}$ follows. Let us suppose that $\mathscr{S}'$ is a syntopogenous structure on E', $g(E)$ is $\mathscr{S}'$-dense, and $\mathfrak{Z}_{\mathscr{S}'}\prec\mathfrak{Z}$. Then, for $<'\in\mathscr{S}'$, there is $\mathfrak{z}\in\mathfrak{Z}$ such that $\mathfrak{z}_{<'}\sqsubset\mathfrak{z}$, and because of [8], (2.7) we have $<'\sqsubset<_{\uparrow\mathfrak{z}}$, so that $\mathscr{S}'\prec\mathscr{S}_{\uparrow\mathfrak{Z}}$. ∎

(2.4) Proposition. *If g is an injection then the syntopogenous structure $\mathscr{S}_{\uparrow\mathfrak{Z}}$ has the following properties for any syntopogenous g-family $\mathfrak{Z}$:*

(2.4.1) $$\mathscr{S}^t_{\uparrow\mathfrak{Z}}=\mathscr{S}_{\uparrow\mathfrak{Z}^t}.$$

(2.4.2) $$\mathscr{S}^p_{\uparrow\mathfrak{Z}}=\mathscr{S}_{\uparrow\mathfrak{Z}^p}.$$

(2.4.3) $$\mathscr{S}^{tp}_{\uparrow\mathfrak{Z}}=\mathscr{S}_{\uparrow\mathfrak{Z}^{tp}}.$$

If $\mathfrak{Z}$ is topogenous or perfect, then $\mathscr{S}_{\uparrow\mathfrak{Z}}$ is of this kind, too.

PROOF. It can be put together from (1.4), (1.5), and [8], (2.8). ∎

(2.5) Proposition. *Let $\mathfrak{Z}$ be an arbitrary syntopogenous g-family. Then we have a syntopogenous structure $\mathscr{S}_{\downarrow\mathfrak{Z}}$ on E determined as follows:*

(2.4.1) $$\mathscr{S}_{\downarrow\mathfrak{Z}}=\{<_{\downarrow\mathfrak{z}}:\ \mathfrak{z}\in\mathfrak{Z}\}.$$

The syntopogenous structure $\mathscr{S}_{\downarrow\mathfrak{Z}}$ has the following properties:

(2.5.2) $$\mathscr{S}^t_{\downarrow\mathfrak{Z}}=\mathscr{S}_{\downarrow\mathfrak{Z}^t}.$$

(2.5.3) $$\mathscr{S}^p_{\downarrow\mathfrak{Z}}=\mathscr{S}_{\downarrow\mathfrak{Z}^p}.$$

(2.5.4) $$\mathscr{S}^{tp}_{\downarrow\mathfrak{Z}}=\mathscr{S}_{\downarrow\mathfrak{Z}^{tp}}.$$

If $\mathfrak{Z}$ is topogenous or perfect, then so is $\mathscr{S}_{\downarrow\mathfrak{Z}}$, too.

PROOF. (1.2.2), (1.3.4), [8], (2.11.1) give that $\mathscr{S}_{\downarrow\mathfrak{Z}}$ is a syntopogenous structure on E. The properties listed in (2.5.2)—(2.5.4) can be deduced from (1.4), (1.5) and [8], (2.11.2). ∎

(2.6) Theorem. *Let $\mathscr{S}'$ be a syntopogenous structure on E', and $g(E)$ be $\mathscr{S}'$-dense. Then $\mathscr{S}_{\downarrow\mathfrak{Z}_{\mathscr{S}'}}=g^{-1}(\mathscr{S}')$ holds.*

PROOF. See [8], (2.10). ∎

(2.7) Theorem. *Suppose that g is a surjection, and $\mathfrak{Z}$ is a syntopogenous g-family. Then the mapping g is compatible with the syntopogenous structure $\mathscr{S}_{\downarrow\mathfrak{Z}}$, and $\mathscr{S}'=g(\mathscr{S}_{\downarrow\mathfrak{Z}})$ is the unique syntopogenous structure on E' (up to equivalence) such that $\mathfrak{Z}\sim\mathfrak{Z}_{\mathscr{S}'}$.*

PROOF. If $A<_{\downarrow\mathfrak{z}}B$ for a mapping $\mathfrak{z}\in\mathfrak{Z}$, then $B\in\mathfrak{z}(g(A))$, and from (M2) we deduce $g^{-1}(g(A))\subset B$, therefore g is compatible with $\mathscr{S}_{\downarrow\mathfrak{Z}}$ (see [1], p. 106). This gives that the syntopogenous structure $\mathscr{S}'=g(\mathscr{S}_{\downarrow\mathfrak{Z}})$ on E' is defined. We show that $\mathfrak{Z}_{\mathscr{S}'}\sim\mathfrak{Z}$. In fact, suppose $\mathfrak{z}\in\mathfrak{Z}$ and $<'=g(<_{\downarrow\mathfrak{z}})$. In this case $X\in\mathfrak{z}_{<'}(A')$ implies $A'<'B'$ and $g^{-1}(B')\subset X$ for some set $B'\subset E'$ (see [8], (2.2)). This means that $g^{-1}(A')<_{\downarrow \mathfrak{z}} g^{-1}(B')\subset X$, thus by $A'=g(g^{-1}(A'))$ from the definition of $<_{\downarrow \mathfrak{z}}$ we get $X\in\mathfrak{z}(A')$, so that $\mathfrak{z}_{<'}\sqsubset\mathfrak{z}$. Conversely, let $\mathfrak{z}$ be an element of $\mathfrak{Z}$, further $\mathfrak{z}_1, \mathfrak{z}_2\in\mathfrak{Z}$ such that $\mathfrak{z}\sqsubset\mathfrak{z}_1^2$, $\mathfrak{z}_1\sqsubset\mathfrak{z}_2^2$, and put $<'=g(<_{\downarrow \mathfrak{z}_2})$. If $X\in\mathfrak{z}(A')$ then there exists $Y\in\mathfrak{z}_1(A')$ for which $X\in\mathfrak{z}_1(g(Y))$, and there is a set $Z\in\mathfrak{z}_2(A')$ such that $Y\in\mathfrak{z}_2(g(Z))$. $g^{-1}(A')\subset Z<_{\downarrow \mathfrak{z}_2} Y\subset g^{-1}(g(Y))$, thus $A'<'g(Y)$ and $g^{-1}(g(Y))\subset X$, consequently $X\in\mathfrak{z}_{<'}(A')$. We got $\mathfrak{z}\sqsubset\mathfrak{z}_{<'}$. If $\mathscr{S}''$ is another syntopogenous structure on E' such that $\mathfrak{Z}_{\mathscr{S}''}\sim\mathfrak{Z}$, then from (2.6) we can deduce $g^{-1}(\mathscr{S}'')\sim\mathscr{S}_{\downarrow \mathfrak{Z}}\sim g^{-1}(\mathscr{S}')$, and by [1], (9.30) this implies $\mathscr{S}''\sim\mathscr{S}'$. ∎

The results mentioned above can be summarized as follows:

(2.8) Theorem. *If we do not distinguish equivalent syntopogenous structures and g-families respectively from each other, then*

(2.8.1) *if g is injective, then $\mathscr{S}'\to\mathfrak{Z}_{\mathscr{S}'}$ is surjective.*

(2.8.2) *If g is surjective, then $\mathscr{S}'\to\mathfrak{Z}_{\mathscr{S}'}$ is one-to-one.*

(2.8.3) *If $\mathscr{S}'\to\mathfrak{Z}_{\mathscr{S}'}$ is injective and E' has at least two elements, then g is surjective.*

PROOF. (2.8.1) follows from (2.3). (2.8.2) is a consequence of (2.7). Finally (2.8.3) can be deduced from [8], (2.13), because with the notations of this example $\{<_1'\}$ and $\{<_2'\}$ are perfect topogenous structures on E'. ∎

Let us note that from the perfect topogenous g-mapping $\mathfrak{z}$ of examples [8], (2.5.1), (2.5.2) we cannot make a perfect topogenous g-family $\{\mathfrak{z}\}$, since $\mathfrak{z}\sqsubset\mathfrak{z}^2$ does not hold, consequently in such a direction the converse of (2.8.1) cannot be proved. But in general, we should vainly look for a perfect topogenous g-family, which cannot be deduced from some topology on E'; it will be shown that such a family does not exist (see (2.10)).

In the following statement we shall use the notion of a *round filter*, the definition of which can be found e.g. in [7] or [3].

(2.9) Theorem $\mathfrak{Z}=\{\mathfrak{z}\}$ *is a perfect topogenous g-family iff a topology $\mathscr{T}$ on E, and for any $x'\in E'$, a filter $\mathfrak{f}(x')$ in E can be chosen such that*

(2.9.1) *$\mathfrak{f}(x')$ is $\mathscr{T}$-round for any $x'\in E'$, in particular $\mathfrak{f}(g(x))$ is the $\mathscr{T}$-neighbourhood filter of the point $x\in E$,*

and

(2.9.10) $$\mathfrak{z}(\emptyset)=2^E,\quad \mathfrak{z}(A')=\bigcap_{x'\in A'}\mathfrak{f}(x')\quad \textit{for}\quad \emptyset\neq A'\subset E'.$$

In this case we have $\mathscr{T}=\mathscr{S}_{\downarrow \mathfrak{Z}}$. For a topology $\mathscr{T}'$ on E' the equality $\mathfrak{Z}=\mathfrak{Z}_{\mathscr{T}'}$ holds iff for any $x'\in E'$ the filter $\mathfrak{f}(x')$ agrees with the filter generated by the inverse images $g^{-1}(V')$ of $\mathscr{T}'$-neighbourhoods V' of x'.

PROOF. Supposing that $\mathfrak{Z}$ is a perfect topogenous g-family, with the choice $\mathfrak{f}(x')=\mathfrak{z}(x')$ (2.9.2) is true (see [8], (1.6)). If we put $\mathscr{T}=\mathscr{T}_{\downarrow \mathfrak{Z}}$, then $\mathscr{T}$ is a topology on E (cf. (2.5.4)). $\mathfrak{z}(x')$ is $\mathscr{T}$-round, because $X\in\mathfrak{z}(x')$ and $\mathfrak{z}\sqsubset\mathfrak{z}^2$ imply $Y\in\mathfrak{z}(x')$ and $X\in\mathfrak{z}(g(Y))$ for a set $Y\subset E$, that is $Y<_{\downarrow \mathfrak{z}}X$. Finally $x<_{\downarrow \mathfrak{z}}V$, iff $V\in\mathfrak{z}\,g(x)),)$ thus (2.9.1) is satisfied.

Conversely, (2.9.1)—(2.92) implies [8] (1.6.1)—(1.62), therefore $\mathfrak{z}$ is a perfect topogenous g-mapping. We have to prove $\mathfrak{z}\subset\mathfrak{z}^2$ in accordance with axiom (F2). If $X\in\mathfrak{z}(A')$ for a set $A'\neq\emptyset$, then $x'\in A'$ implies the existence of a set $Y_{x'}\in\mathfrak{f}(x')$ such that $Y_{x'}<X$, where $\mathcal{T}=\{<\}$. From the perfectness of $<$ we get $Y=\bigcup_{x'\in A'} Y_{x'}<X$, and $Y\supset Y_{x'}\in\mathfrak{f}(x')$ for every $x'\in A'$, thus $Y\in\mathfrak{z}(A')$. At the same time, for $x\in Y$, the set X is a $\mathcal{T}$-neighbourhood of x, consequently by (2.9.1) $X\in\mathfrak{f}(g(x))$, so that $X\in\mathfrak{z}(g(Y))$. The first part of the theorem is proved.

By (2.9.2) $\mathfrak{z}(x')=\mathfrak{f}(x')$ for any $x'\in E'$, thus owing to (2.9.1) $\mathcal{S}_{\downarrow\mathfrak{Z}}$ is obviously identical with $\mathcal{T}$. Finally let $\mathcal{T}'=\{<'\}$ be a topology on E' such that $g(E)$ is $\mathcal{T}'$-dense. Then we have $\mathfrak{z}_{<'}(x')=\{X\subset E\colon g^{-1}(V')\subset X,\ x'<'V'\}$. If $\mathfrak{z}=\mathfrak{z}_{<'}$, then $\mathfrak{f}(x')=\mathfrak{z}(x')=\mathfrak{z}_{<'}(x')$ for every $x'\in E'$. Conversely, $\mathfrak{z}_{<'}$ is perfect (see [8], (2.3.2)), hence if $\mathfrak{z}_{<'}(x')=\mathfrak{f}(x')$ for any $x\in E'$, then $\mathfrak{z}_{<'}(A')=\bigcap_{x'\in A'}\mathfrak{z}_{<'}(x')=\bigcap_{x'\in A'}\mathfrak{f}(x')=$ $=\mathfrak{z}(A')$. ∎

For the formulation of the following corollary of (2.9) we need to use the notion of a *monotone mapping* h belonging to a given family of filters in E, and the syntopogenous structure $h(\mathcal{S})$, which was defined e.g. in [7] (and also in [3], (0.4), th. 3.1)

(2.10) Corollary. *Let $\mathfrak{Z}=\{\mathfrak{z}\}$ be a perfect topogenous g-family, $\mathcal{T}=\mathcal{S}_{\downarrow\mathfrak{Z}}$, and let h denote the monotone mapping belonging to the filters $\mathfrak{z}(x')$ $(x'\in E')$. Then $\mathfrak{Z}=\mathfrak{Z}_{h(\mathcal{T})^p}$.*

Proof. By (2.9) the conditions of [7], (2.3) are fulfilled, thus the filter generated by the inverse images of $h(\mathcal{T})^p$-neighbourhoods of x' agrees with the filter $\mathfrak{z}(x')$ for every $x'\in E'$. ∎

3. Applications in extension theory

In the first part of this chapter a generalization of th. (6.1.2) of [2] will be given with a determination of a class of syntopogenous structures on E', in which the structure $h(\mathcal{S})$ studied in ch. 1—2. of [7] is the coarsest one.

First of all let us consider the following construction:

(3.1) Theorem. *Let $\mathcal{S}$ be a syntopogenous structure on E and $\mathfrak{Z}_0=\{\mathfrak{z}_0\}$ be a topogenous g-family such that $\mathcal{S}^t<\mathcal{S}_{\downarrow\mathfrak{Z}_0}$. Then we have a syntopogenous g-family $\mathfrak{Z}^*=\mathfrak{Z}(\mathcal{S},\mathfrak{Z}_0)$ consisting of the topogenous g-mappings $\mathfrak{z}(<,\mathfrak{z}_0)$ defined for every $<\in\mathcal{S}$ as follows:*

(3.1.1) *$X\in\mathfrak{z}(<,\mathfrak{z}_0)(A')$ iff there is $V\in\mathfrak{z}_0(A')$ with $V<X$.*
The syntopogenous g-family $\mathfrak{Z}^*$ has the following properties:

(3.1.2) $\mathcal{S}_{\downarrow\mathfrak{Z}^*}\sim\mathcal{S}$.

(3.1.3) $\mathfrak{Z}^{*t}<\mathfrak{Z}_0$.

(3.1.4) $\mathfrak{Z}^{*t}=\mathfrak{Z}_0$ *provided $\mathcal{S}^t=\mathcal{S}_{\downarrow\mathfrak{Z}_0}$.*

(3.1.5) $\mathfrak{Z}^*$ *is finer than every syntopogenous g-family $\mathfrak{Z}_1$ such that $\mathcal{S}_{\downarrow\mathfrak{Z}_1}<\mathcal{S}$ and $\mathfrak{Z}_1^t<\mathfrak{Z}_0$.*
In view of the last property of $\mathfrak{Z}^*$, it will be called the fine syntopogenous g-family corresponding to $\mathcal{S}$ and $\mathfrak{Z}_0$.

PROOF. We prove that $\mathfrak{z}=\mathfrak{z}(<, \mathfrak{z}_0)$ is a topogenous g-mapping for every $<\in\mathcal{S}$. (M0): $E<E$ and $E\in\mathfrak{z}_0(A')$ implies $E\in\mathfrak{z}(A')$, i.e. $\mathfrak{z}(A')\neq\emptyset$. If $Y\supset X\in\mathfrak{z}(A')$, then we have a set $V\in\mathfrak{z}_0(A')$ such that $V<X$, thus $V<Y$ and $Y\in\mathfrak{z}(A')$. (M1): $\emptyset<\emptyset$ and $\emptyset\in\mathfrak{z}_0(\emptyset)$, therefore $\emptyset\in\mathfrak{z}(\emptyset)$. Conversely, if $\emptyset\in\mathfrak{z}(A')$, then $V<\emptyset$ for some $V\in\mathfrak{z}_0(A')$, but this implies $V=\emptyset$, and owing to property (M1) of $\mathfrak{z}_0$ we get $A'=\emptyset$. (M2): If $X\in\mathfrak{z}(A')$, then $g^{-1}(A')\subset V<X$ with a suitable $V\in\mathfrak{z}_0(A')$, hence $g^{-1}(A')\subset\subset X$. (M3): If $A'\subset B'$ and $X\in\mathfrak{z}(B')$, then there is a set $V\in\mathfrak{z}_0(B')\subset\mathfrak{z}_0(A')$ such that $V<X$, from this $X\in\mathfrak{z}(A')$ follows.

Suppose $X\in\mathfrak{z}(A')$ and $Y\in\mathfrak{z}(B')$. Then $V<X$ and $W<Y$ for some $V\in\mathfrak{z}_0(A')$ and $W\in\mathfrak{z}_0(B')$. $<$ and $\mathfrak{z}_0$ are topogenous, therefore $V\cap W<X\cap Y$ and $V\cup W<<X\cup Y$, further $V\cap W\in\mathfrak{z}_0(A'\cap B')$ and $V\cup W\in\mathfrak{z}_0(A'\cup B')$. This gives the topogenity of $\mathfrak{z}$.

Obviously, if $<_1, <_2\in\mathcal{S}$ and $<_1\cup<_2\subset<\in\mathcal{S}$, then $\mathfrak{z}(<_1, \mathfrak{z}_0)\cup\mathfrak{z}(<_2, \mathfrak{z}_0)\subset\subset\mathfrak{z}(<, \mathfrak{z}_0)$, thus $\mathfrak{Z}^*$ satisfies axiom (F1). Finally suppose that $<$ is an arbitrary member of $\mathcal{S}$, and let us choose $<_1\in\mathcal{S}$ so that $<\subset<_1^3$, after this let $\mathfrak{z}$ and $\mathfrak{z}_1$ denote $\mathfrak{z}(<, \mathfrak{z}_0)$ and $\mathfrak{z}(<_1, \mathfrak{z}_0)$ respectively. If $X\in\mathfrak{z}(A')$ then $V<X$ for some $V\in\mathfrak{z}_0(A')$. Assume $V<_1Y<_1W<_1X$. In this case we have $Y\in\mathfrak{z}_1(A')$, and from $\mathcal{S}^t<\mathcal{S}_{\downarrow\mathfrak{Z}_0}$ the relation $W\in\mathfrak{z}_0(g(Y))$ follows, hence $X\in\mathfrak{z}_1(g(Y))$. We got $\mathfrak{z}\subset\mathfrak{z}_1^2$, i.e. $\mathfrak{Z}^*$ satisfies axiom (F2), too.

Let us show that $\mathfrak{Z}^*$ has the properties listed in (3.1.2)—(3.1.5).

(3.1.2): Put $\mathfrak{z}=\mathfrak{z}(<, \mathfrak{z}_0)$, where $<$ is an arbitrary element of $\mathcal{S}$. If $A<_{\downarrow\mathfrak{z}}B$ then $B\in\mathfrak{z}(g(A))$, thus there is a set $V\in\mathfrak{z}_0(g(A))$ such that $V<X$. But because of (M2) $A\subset g^{-1}(g(A))\subset V$, so that $A<B$, i.e. $<_{\downarrow\mathfrak{z}}\subset<$. Conversely, suppose that $<\in\mathcal{S}$ and $<_1\in\mathcal{S}$ for which $<\subset<_1^2$. If $A<B$ then $A<_1V<_1B$ for some $V\subset E$. By $\mathcal{S}^t<\mathcal{S}_{\downarrow\mathfrak{Z}_0}$ we have $V\in\mathfrak{z}_0(g(A))$, therefore $B\in\mathfrak{z}_1(g(A))$, where $\mathfrak{z}_1=\mathfrak{z}(<_1, \mathfrak{z}_0)$. This shows $<\subset<_{\downarrow\mathfrak{z}_1}$.

(3.1.3): If $X\in\mathfrak{z}(A')$ for some $\mathfrak{z}=\mathfrak{z}(<, \mathfrak{z}_0)$, $<\in\mathcal{S}$, then $V<X$ holds for a set $V\in\mathfrak{z}_0(A')$, and since $V\subset X$, from (M0) we get $X\in\mathfrak{z}_0(A')$.

(3.1.4): Assume that $\mathcal{S}^t=\mathcal{S}_{\downarrow\mathfrak{Z}_0}$ and $X\in\mathfrak{z}_0(A')$. In view of $\mathfrak{z}_0\subset\mathfrak{z}_0^2$ we can choose a set $V\in\mathfrak{z}_0(A')$ such that $X\in\mathfrak{z}_0(g(V))$. But this means $V<_{\downarrow\mathfrak{z}_0}X$, and because of our condition we have $V<X$ for some $<\in\mathcal{S}$. If $\mathfrak{z}(<, \mathfrak{z}_0)$ is denoted by $\mathfrak{z}$, then we can write $X\in\mathfrak{z}(A')$, and obviously $\mathfrak{Z}_0<\mathfrak{Z}^{*t}$.

(3.1.5): Let $\mathfrak{Z}_1$ denote a syntopogenous g-family such that $\mathcal{S}_{\downarrow\mathfrak{Z}}<\mathcal{S}$ and $\mathfrak{Z}_1^t<\mathfrak{Z}_0$. Suppose that $\mathfrak{z}_1\in\mathfrak{Z}_1$ is arbitrary, $\mathfrak{z}_2\in\mathfrak{Z}_1$, for which $\mathfrak{z}_1\subset\mathfrak{z}_2^2$, further $<\in\mathcal{S}$ such that $<_{\downarrow\mathfrak{z}_2}\subset<$, finally let us consider $\mathfrak{z}=\mathfrak{z}(<, \mathfrak{z}_0)$. If $X\in\mathfrak{z}_1(A')$ then there exists $V\in\mathfrak{z}_2(A')$ such that $X\in\mathfrak{z}_2(g(V))$. This means $V<_{\downarrow\mathfrak{z}_2}X$, consequently $V<X$. From $\mathfrak{Z}_1^t<\mathfrak{Z}_0$ we can deduce $V\in\mathfrak{z}_0(A')$, thus in accordance with our notations, this gives $X\in\mathfrak{z}(A')$, accordingly $\mathfrak{z}_1\subset\mathfrak{z}$, and in general $\mathfrak{Z}_1<\mathfrak{Z}^*$. ∎

(3.2) Theorem. *Let $\mathcal{S}$ be a syntopogenous structure on E. Suppose that $\mathfrak{f}(x')$ is an $\mathcal{S}$-round filter in E for any $x'\in E'$ such that $\mathfrak{f}(g(x))$ is the filter of $\mathcal{S}$-neighbourhoods of every point $x\in E$. Then we have a perfect topogenous g-family $\mathfrak{Z}_0=\{\mathfrak{z}_0\}$ determined by*

$$(3.2.1.)\qquad \mathfrak{z}_0(\emptyset)=2^E \quad \textit{and} \quad \mathfrak{z}_0(A')=\bigcap_{x'\in A'}\mathfrak{f}(x') \quad \textit{for} \quad \emptyset\neq A'\subset E',$$

for which $\mathscr{S}^{tp}=\mathscr{S}_{\downarrow\mathfrak{z}_0}$. *If* $\mathfrak{Z}^*$ *is the fine syntopogenous g-family corresponding to* $\mathscr{S}$ *and* $\mathfrak{Z}_0$, *then denoting by* h *the monotone mapping belonging to the filters* $\mathfrak{f}(x')$

(3.2.2) $\mathfrak{Z}^*\sim\mathfrak{Z}_{h(\mathscr{S})}$ *and* $\mathfrak{Z}_{h(\mathscr{S})^{tp}}=\mathfrak{Z}_0$.

(3.2.3) $h(\mathscr{S})$ *is the coarsest of all syntopogenous structures* $\mathscr{S}'$ *on* E' *such that* $g(E)$ *is* $\mathscr{S}'$*-dense and* $\mathfrak{Z}^*<\mathfrak{Z}_{\mathscr{S}'}<\mathfrak{Z}_0$.

PROOF. Putting $\mathscr{T}=\mathscr{S}^{tp}$, the first part of the theorem can be read from (2.9).

(3.2.2): Supposing $<\in\mathscr{S}$ and $\mathfrak{z}=\mathfrak{z}(<,\mathfrak{z}_0)$, we have $\mathfrak{z}\subset\mathfrak{z}_{<'}$, where $<'=$ $=h(<)^q\in h(\mathscr{S})$, and at the same time $\mathfrak{Z}^*<\mathfrak{Z}_{h(\mathscr{S})}$. In fact, if $X\in\mathfrak{z}(A')$, then $V<X$ for some $V\in\mathfrak{z}_0(A')$. In this case $A'\subset h(V)h(<)^q h(X)$. $g^{-1}(h(X)\subset X$, since $x\in E$, $X\in\mathfrak{f}(g(x))$ gives that X is an $\mathscr{S}$-neighbourhood of x, consequently $x\in X$. This means $X\in\mathfrak{z}_{<'}(A')$. Conversely, from [7], (2.3) we can deduce that the filter generated by the inverse images of $h(\mathscr{S})$-neighbourhoods of any point $x'\in E'$ is exactly $\mathfrak{f}(x')$ hence by (2.9) we have $\mathfrak{Z}_{h(\mathscr{S})^{tp}}=\mathfrak{Z}_0$. After this the inequality $\mathfrak{Z}_{h(\mathscr{S})}<\mathfrak{Z}^*$ will be verified. On the basis of (1.5) and (2.2.3) we can state $\mathfrak{Z}^t_{h(\mathscr{S})}<\mathfrak{Z}^{tp}_{h(\mathscr{S})}=\mathfrak{Z}_{h(\mathscr{S})^{tp}}=\mathfrak{Z}_0$. Simultaneously $\mathscr{S}_{\downarrow\mathfrak{Z}_{h(\mathscr{S})}}=g^{-1}(h(\mathscr{S}))\sim\mathscr{S}$ (see (2.6) and [7], (2.3), (1.2)). In view (3.1.5), from these we get $\mathfrak{Z}_{h(\mathscr{S})}<\mathfrak{Z}^*$.

(3.2.3): Assume that $\mathscr{S}'$ is a syntopogenous structure on E' such that $\mathfrak{Z}^*<\mathfrak{Z}_{\mathscr{S}'}<\mathfrak{Z}_0$. We shall show $h(\mathscr{S})<\mathscr{S}'$. Indeed, suppose that $<^*=h(<)^q$ for some $<\in\mathscr{S}$, and let us choose an element $<'$ of $\mathscr{S}'$ such that $\mathfrak{z}\subset\mathfrak{z}_{<'}$ for $\mathfrak{z}=\mathfrak{z}(<,\mathfrak{z}_0)$. Putting $<'_1\in\mathscr{S}'$, for which $<'\subset<'^2_1$, we can state $<^*\subset<'_1$. In fact, if $A'\subset h(A)$, $h(B)\subset B'$ and $A<B$, then $A\in\mathfrak{f}(x')$ for $x'\in A'$, thus $A\in\mathfrak{z}_0(A')$, and consequently $B\in\mathfrak{z}(A')$. We have a set $X'\subset E'$ such that $A'<'X'$ and $g^{-1}(X')\subset B$. If $A'<'_1Y'<'_1X'$, then for any $x'\in Y'$ the inequality $x'<'_1X'$ holds, therefore $B\supset g^{-1}(X')\in\mathfrak{z}_{<'_1}(x')\subset\mathfrak{z}_0(x')=\mathfrak{f}(x')$. This gives $Y'\subset h(B)$, accordingly $A'<'_1B'$. We got $h(<)\subset<'_1$, consequently $h(<)^q\subset<'^q_1=<'_1$, i.e. $h(\mathscr{S})<\mathscr{S}'$. ∎

(3.3) Corollary. *Under the conditions of* (3.2) *let* g *be an injection. Then* $h(\mathscr{S})$ *is the coarsest of all syntopogenous structures* $\mathscr{S}'$ *on* E' *such that* $(E',\mathscr{S},g)$ *is an extension of* $[E,\mathscr{S}]$ *giving the filters* $\mathfrak{f}(x')$, *as trace filters, and* $\mathscr{S}'$ *induces the fine syntopogenous g-family corresponding to* $\mathscr{S}$ *and* $\mathfrak{Z}_0$.

PROOF. For such an extension $(E',\mathscr{S}',g)$ by (2.9) we have $\mathfrak{Z}^*\sim\mathfrak{Z}_{\mathscr{S}'}<\mathfrak{Z}_{\mathscr{S}'}\,\mathrm{tp}=$ $=\mathfrak{Z}_0$, thus from (3.1.5) $h(\mathscr{S})<\mathscr{S}'$ follows. ∎

(3.4) Corollary. ([2], (6.1.2)). *Under the conditions of* (3.2) *let* g *be an injection, and* $\mathscr{S}=\mathscr{T}$ *be a topology. Then* $h(\mathscr{T})^p$ *is the coarsest of those topologies* $\mathscr{T}'$ *on* E' *for which* $(E',\mathscr{T}',g)$ *is an extension of* $[E,\mathscr{T}]$ *with the trace filters* $\mathfrak{f}(x')$.

PROOF. Denoting by $\mathfrak{Z}^*$ the fine syntopogenous g-family corresponding to $\mathscr{T}$ and $\mathfrak{Z}_0$, from (3.1.3) $\mathfrak{Z}^*<\mathfrak{Z}_0$ follows. But simultaneously $\mathscr{S}_{\downarrow\mathfrak{Z}_0}=\mathscr{T}$ and (3.1.5) implies $\mathfrak{Z}_0<\mathfrak{Z}^*$, thus $\mathfrak{Z}_0=\mathfrak{Z}^*$ (let us observe that by (3.1.1) $\mathfrak{Z}^*$ consists of a single g-mapping). Consequently, if $\mathscr{T}'$ is a topology as in the theorem, then $\mathfrak{Z}^*=\mathfrak{Z}_0=\mathfrak{Z}_{\mathscr{T}'}$, therefore $h(\mathscr{T})<\mathscr{T}'$. From this $h(\mathscr{T})^p<\mathscr{T}'^p=\mathscr{T}'$ can be deduced. ∎

(3.5) Theorem. *Let* g *be an injection,* $\mathscr{S}$ *be a syntopogenous structure on*

E and $\mathfrak{Z}$ be an arbitrary syntopogenous g-family. In order that there exist a syntopogenous structure $\mathcal{S}'$ on E' such that $(E', \mathcal{S}', g)$ is an extension of $[E, \mathcal{S}]$ and $\mathfrak{Z} \sim \mathfrak{Z}_{\mathcal{S}'}$, it is necessary and sufficient that $\mathcal{S} \sim \mathcal{S}_{\downarrow\mathfrak{Z}}$. In this case $\mathcal{S}_{\uparrow\mathfrak{Z}}$ is the finest of all syntopogenous structures on E' with these properties.

PROOF. If $(E', \mathcal{S}', g)$ is an extension of $[E, \mathcal{S}]$ such that $\mathfrak{Z} \sim \mathfrak{Z}_{\mathcal{S}'}$, then $\mathcal{S} \sim g^{-1}(\mathcal{S}') = \mathcal{S}_{\downarrow\mathfrak{Z}_{\mathcal{S}'}} \sim \mathcal{S}_{\downarrow\mathfrak{Z}}$ (see (2.6)). Conversely, suppose $\mathcal{S} \sim \mathcal{S}_{\downarrow\mathfrak{Z}}$, and let $\mathcal{S}^*$ denote the syntopogenous srtructure $\mathcal{S}_{\uparrow\mathfrak{Z}}$ on E'. Then $g(E)$ is $\mathcal{S}^*$-dense, $\mathfrak{Z} = \mathfrak{Z}_{\mathcal{S}^*}$, and $\mathcal{S} \sim \mathcal{S}_{\downarrow\mathfrak{Z}} = \mathcal{S}_{\downarrow\mathfrak{Z}_{\mathcal{S}^*}} = g^{-1}(\mathcal{S}^*)$, finally $\mathcal{S}^*$ is the finest of all syntopogenous structures on E' inducing $\mathfrak{Z}$ (see (2.3) and (2.6)). ∎

The knowledges concerning extensions providing a prescribed family of trace filters can be completed as follows:

(3.6) Theorem. *Under the conditions of* (3.2) *let g be an injection. Then $\mathcal{S}_{\uparrow\mathfrak{Z}^*}$ is the finest of all syntopogenous structures $\mathcal{S}'$ on E' such that $(E', \mathcal{S}', g)$ is an extension of $[E, \mathcal{S}]$ with the filters $\mathfrak{f}(x')$, as trace filters.*

PROOF. Put $\mathcal{S}^* = \mathcal{S}_{\uparrow\mathfrak{Z}^*}$. Then $(E', \mathcal{S}^*, g)$ is in fact an extension of $[E, \mathcal{S}]$ (see (3.1.2) and (3.5)). $\mathfrak{Z}_{\mathcal{S}^*} = \mathfrak{Z}^*$, therefore by (3.1.3) $\mathfrak{Z}_{\mathcal{S}^{*tp}} = \mathfrak{Z}^{*tp} < \mathfrak{Z}_0$, at the same time (3.2.3) implies $h(\mathcal{S}) < \mathcal{S}^*$. From this and form (3.2.2) we can deduce $\mathfrak{Z}_0 = \mathfrak{Z}_{h(\mathcal{S})^{tp}} < \mathfrak{Z}_{\mathcal{S}^{*tp}}$, so that $\mathfrak{Z}_{\mathcal{S}^{*tp}} = \mathfrak{Z}_0$. In view of (2.9) this means that the extension $(E', \mathcal{S}^*, g)$ gives the filters $\mathfrak{f}(x')$, as trace filters. Suppose that the extension $(E', \mathcal{S}', g)$ of $[E, \mathcal{S}]$ has the same trace filters $\mathfrak{f}(x')$ (i.e. $\mathfrak{Z}_{\mathcal{S}'^{tp}} = \mathfrak{Z}_0$). Then putting $\mathfrak{Z} = \mathfrak{Z}_{\mathcal{S}'}$, we have $\mathcal{S} \sim \mathcal{S}_{\downarrow\mathfrak{Z}}$ and $\mathfrak{Z}^t = \mathfrak{Z}_{\mathcal{S}'^t} < \mathfrak{Z}_0$ (cf. (3.5)). From the fineness of $\mathfrak{Z}^*$ (see (3.1.5)) the inequality $\mathfrak{Z} < \mathfrak{Z}^*$ follows, but owing to (2.3) this gives $\mathcal{S}' < \mathcal{S}^*$. ∎

Remark. This finest extension described in the theorem can be constructed in another way (see [3], ch. 0 (p. 60), (0.6) and th. 2.2).

On the basis of the statements of [7], (2.1) and (2.4) we can verify that for the existence of an extension $(E', \mathcal{S}', g)$ of the syntopogenous space $[E, \mathcal{S}]$ having a prescribed family $\{\mathfrak{f}(x') : x' \in E'\}$ of trace filters, it is necessary and sufficient that any filter $\mathfrak{f}(x')$ be $\mathcal{S}$-round, and $\mathfrak{f}(g(x))$ be the filter of $\mathcal{S}$-neighbourhoods of every point $x \in E$. In the last part of the chapter this result will be generalized.

For an arbitrary syntopogenous space $[X, \mathcal{S}]$, we shall say that $\mathcal{S}(A) = \{V \subset X : A < V$ for some $< \in \mathcal{S}\}$ is the *$\mathcal{S}$-neighbourhood filter* of $\emptyset \neq A \subset X$. If $(E', \mathcal{S}', g)$ is an extension of the syntopogenous space $[E, \mathcal{S}]$, the system $\{g^{-1}(\mathcal{S}'(A')) : \emptyset \neq A' \subset E'\}$ will be called the *full system of trace filters* of this extension, where $g^{-1}(\mathcal{S}'(A'))$ consists of the inverse image $g^{-1}(V')$ of members V' of $\mathcal{S}'(A')$.

(3.7) Lemma. *Let $(E', \mathcal{S}', g)$ be an extension of the syntopogenous space $[E, \mathcal{S}]$. Then the full system of trace filters of this extension is identical with $\{\mathfrak{z}_{<'}(A') : \emptyset \neq A' \subset E'\}$, where $\mathcal{S}'^t = \{<'\}$.*

PROOF. It is clear by the definition and [8], (2.1)—(2.2). ∎

(3.8) Theorem. *Let g be an injection, and $\mathcal{S}$ be a syntopogenous structure on E. Let us consider a filter $\mathfrak{f}(A')$ in E for any $\emptyset \neq A' \subset E'$. Then the following statements are equivalent:*

(3.8.1) *There exists an extension* $(E', \mathcal{S}', g)$ *of* $[E, \mathcal{S}]$ *such that* $\{\mathfrak{f}(A'): \emptyset \neq A' \subset E'\}$ *is the full system of trace filters of this extension.*

(3.8.2) $\mathfrak{f}(A')$ *is an* $\mathcal{S}$*-round filter in* E *for any* $\emptyset \neq A' \subset E'$, *in particular* $\mathfrak{f}(g(A)) = \mathcal{S}(A)$ *for every* $\emptyset \neq A \subset E$, *finally, if* A', B', *are non empty subsets of* E', *then* $\mathfrak{f}(A') \cap \mathfrak{f}(B') = \mathfrak{f}(A' \cup B')$.

PROOF. (3.8.1)⇒(3.8.2): Suppose $X \in \mathfrak{f}(A')$. Then for some $<' \in \mathcal{S}'$ we have $A' <' V'$ and $g^{-1}(V') = X$. If $<'_1 \in \mathcal{S}'$, $<' \subset <'^2_2$ and $< \in \mathcal{S}$ such that $g^{-1}(<'_1) \subset <$, then $A' <'_1 W' <'_1 V'$ for a suitable W', and with the notation $g^{-1}(W') = Y$ we get $Y \in \mathfrak{f}(A')$ and $Y < X$, therefore $\mathfrak{f}(A')$ is $\mathcal{S}$-round. We know $g^{-1}(\mathcal{S}') \sim \mathcal{S}$, thus $X \in \mathcal{S}(A)$ iff $E' - g(E-X) \in \mathcal{S}'(g(A))$. g is an injection, thus from $X = g^{-1}(E' - g(E-X))$ the equality $\mathfrak{f}(g(A)) = \mathcal{S}(A)$ follows. Obviously $\mathfrak{f}(A' \cup B') \subset \mathfrak{f}(A') \cap \mathfrak{f}(B')$, and if $X = g^{-1}(V') = g^{-1}(W')$, where $A' <'_1 V'$, $B' <'_2 W'$ ($<'_1, <'_2 \in \mathcal{S}'$), then, for $<'_1 \cup <'_2 \subset <' \in \mathcal{S}'$, we have $A' \cup B' <' V' \cup W'$, thus $g^{-1}(V' \cup W') = g^{-1}(V') \cup g^{-1}(W') = X$ shows that $\mathfrak{f}(A') \cap \mathfrak{f}(B') \subset \mathfrak{f}(A' \cup B')$ is also true.

(3.8.2)⇒(3.8.1): First of all we prove that putting $\mathfrak{z}_0(\emptyset) = 2^E$, $\mathfrak{z}_0(A') = \mathfrak{f}(A')$ $(\emptyset \neq A' \subset E')$, $\mathfrak{Z}_0 = \{\mathfrak{z}_0\}$ is a topogenous g-family, for which $\mathcal{S}^t = \mathcal{S}_{\downarrow \mathfrak{Z}_0}$ holds. In fact, $\mathfrak{z}_0$ clearly satisfies axioms (M0) and (M1). If $A' \subset B'$, then $\mathfrak{z}_0(B') = \mathfrak{z}_0(A' \cup B') = \mathfrak{z}_0(A') \cap \mathfrak{z}_0(B') \subset \mathfrak{z}_0(A')$, thus (M3) is also fulfilled. For the verification of (M2) let us suppose $X \in \mathfrak{z}_0(A')$. Then $x \in E$, $g(x) \in A'$ implies $X \in \mathfrak{z}_0(g(x)) = \mathcal{S}(x)$ (see (M3)), hence $x \in X$. This means $g^{-1}(A') \subset X$. The topogenity of $\mathfrak{z}_0$ can be deduced from [8], (1.3). One can see that $\mathfrak{z}_0 \subset \mathfrak{z}_0^2$. In fact, if $X \in \mathfrak{z}_0(A')$, then from the roundness of $\mathfrak{z}_0(A')$ we get a set $Y \in \mathfrak{z}_0(A')$ such that $Y < X$, where $< \in \mathcal{S}$. But this implies $X \in \mathfrak{z}_0^2(A')$, since $X \in \mathcal{S}(Y) = \mathfrak{z}_0(g(Y))$. Finally suppose $\mathcal{S}^t = \{<_0\}$. Then $A <_0 B$ iff $B \in \mathcal{S}(A) = \mathfrak{z}_0(g(A))$, and this is equivalent to $A <_{\downarrow \mathfrak{Z}_0} B$. This shows $\mathcal{S}^t = \mathcal{S}_{\downarrow \mathfrak{Z}_0}$.

From here we shall have an easy job, namely assume that $\mathfrak{Z}^*$ is the fine syntopogenous g-family corresponding to $\mathcal{S}$ and $\mathfrak{Z}_0$. Then from $\mathcal{S}^t = \mathcal{S}_{\downarrow \mathfrak{Z}_0}$ we get $\mathfrak{Z}^{*t} = \mathfrak{Z}_0$ (see (3.1.4)). (3.1.2) and (3.5) show that there is an extension $(E', \mathcal{S}', g)$ of $[E, \mathcal{S}]$ such that $\mathfrak{Z}^* \sim \mathfrak{Z}_{\mathcal{S}'}$. This implies $\mathfrak{Z}_{\mathcal{S}'^t} = \mathfrak{Z}^t_{\mathcal{S}'} = \mathfrak{Z}^{*t} = \mathfrak{Z}_0$, which means that $\{\mathfrak{z}_0(A'): \emptyset \neq A' \subset E'\} = \{\mathfrak{f}(A'): \emptyset \neq A' \subset E'\}$ is the full system of trace filters of the extension in question (see lemma (3.7)). ∎

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