Publ. Math. Debrecen
50 / 3-4 (1997), 265–271

On some partial differential inequalities for mappings of $\mathcal{C}^1$ class in $\mathbb{C}^n$

By GABRIELA KOHR (Cluj-Napoca) and PIOTR LICZBERSKI (Łódź)

Abstract. In this paper the authors obtain some partial differential inequalities involving mappings of $\mathcal{C}^1$ class defined on the unit ball of $\mathbb{C}^n$. Some interesting applications are also presented.

1. Introduction

Let $\mathbb{C}^n$ denote the n-dimensional space of complex variables $z = (z_1,\dots,z_n)' = (x_1+iy_1,\dots,x_n+iy_n)'$ with the Euclidean inner product $\langle z,w\rangle = \sum_{k=1}^n z_k\bar{w}_k$ and the Euclidean norm $\|z\| = \sqrt{\langle z,z\rangle}$. A' means the transpose of the matrix A. The open Euclidean ball $\{z\in\mathbb{C}^n : \|z\| < r\}$ is denoted by B_r and the open unit ball is abbreviated by B_1. If G is a domain in $\mathbb{C}^n$ and $f=(f_1,\dots,f_n)' : G\to\mathbb{C}^n$, then we say that f belongs to the class $\mathcal{C}^1(G)$ if for each $j,k=1,\dots,n$ the functions $u_j = \Re f_j,\ v_j = \Im f_j$, have all first order partial derivatives in respect to the real variables x_k, y_k and they are continuous in G. For $f\in\mathcal{C}^1(G)$

$$D_z f(a) = \left[\frac{\partial f_j}{\partial z_k}(a)\right]_{1\le j,k\le n},$$
$$D_{\bar z} f(a) = \left[\frac{\partial f_j}{\partial \bar z_k}(a)\right]_{1\le j,k\le n},$$

Mathematics Subject Classification: 32H99, 30C45.
Key words and phrases: mappings from $\mathcal{C}^1$ class, partial differential inequalities, fixed point.

where, as usually

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}\right),$$
$$\frac{\partial}{\partial \bar{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right).$$

Of course, if f is a holomorphic mapping on G, then $D_{\bar{z}}f = 0$ and $D_z f(a)$ is equal to the Frechet derivative $Df(a)$ of f at the point a. For our purpose we use the following one-dimensional lemma included in the paper [3] of P.T. Mocanu:

Lemma 1. *Let U be the open unit disc $\{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and f a complex function from the class $\mathcal{C}^1(U - \{0\})$. If there exists $\zeta_0 \in U - \{0\}$ such that*

$$\Re f(\zeta_0) = 0 = \min\{\Re f(\zeta) : 0 < |\zeta| \le |\zeta_0|\},$$

then the following relation holds

$$\Im\left[\zeta_0 \frac{\partial f}{\partial \zeta}(\zeta_0) - \bar{\zeta}_0 \frac{\partial f}{\partial \bar{\zeta}}(\zeta_0)\right] = 0. \tag{1}$$

2. Main results

Theorem 1. *Let $p \in \mathcal{C}^1(B - \{0\})$. If there exists an $a \in B - \{0\}$ such that*

$$\Re\langle p(a), a\rangle = 0 = \min\{\Re\langle p(z), z\rangle : 0 < \|z\| \le \|a\|\}, \tag{2}$$

then

$$\left[\overline{D_z p(a)}\right]' a + [D_{\bar{z}} p(a)]' \bar{a} + p(a) = ma \tag{3}$$

and

$$\Im\langle [D_z p(a)]a - [D_{\bar{z}} p(a)]\bar{a} - p(a), a\rangle = 0, \tag{4}$$

where m is a real number.

Proof. Let $r = \|a\| \in (0,1)$ and v be an arbitrary vector of the real tangent space $T_a(\partial B_r)$ at the point a. Since ∂B_r is a real surface at least of $\mathcal{C}^1$ class, it is well known that there exists an $\varepsilon > 0$ and a once differentiable function $\gamma : (-\varepsilon, \varepsilon) \to \partial B_r$ such that $\gamma(0) = a$ and $\frac{d\gamma}{dt}(0) = v$. Let

$$\alpha(t) \quad = \quad \Re\langle p(\gamma(t)), \gamma(t)\rangle \quad , \quad t \in (-\varepsilon, \varepsilon).$$

Then α is the differentiable function on $(-\varepsilon, \varepsilon)$ and it satisfies the following relation

$$\alpha(0) = \Re \langle p(a), a\rangle = \min\{\alpha(t) : t \in (-\varepsilon, \varepsilon)\},$$

hence $\frac{d\alpha}{dt}(0) = 0$. On the other hand

$$\frac{d\alpha}{dt}(0) = \frac{d}{dt}\Re \langle p(\gamma(t)), \gamma(t)\rangle_{|t=0}$$

$$= \Re \left\langle [D_z p(a)]\frac{d\gamma}{dt}(0) + [D_{\bar z} p(a)]\overline{\frac{d\gamma}{dt}}(0), a\right\rangle + \Re \left\langle p(a), \frac{d\gamma}{dt}(0)\right\rangle$$

$$= \Re \langle [D_z p(a)]v + [D_{\bar z} p(a)]\bar v, a\rangle + \Re \langle p(a), v\rangle .$$

So

$$\Re \left\langle \left[\overline{D_z p(a)}\right]' a + [D_{\bar z} p(a)]'\bar a + p(a), v\right\rangle = 0.$$

Since the above relation holds for every arbitrary tangent vector v, we conclude that

$$\left[\overline{D_z p(a)}\right]' a + [D_{\bar z} p(a)]'\bar a + p(a)$$

is a normal vector to ∂B_r at the point a. On the other hand an outward normal vector to ∂B_r at the point a is also the vector a, hence, we can find a real number m such that the equality (3) holds. Now let us consider the following complex function

$$f(\zeta) = \zeta^{-1}\left\langle p(\zeta a\|a\|^{-1}), a\|a\|^{-1}\right\rangle, \quad \zeta \in U - \{0\}.$$

Since $p \in \mathcal{C}^1(B - \{0\})$, so $f \in \mathcal{C}^1(U - \{0\})$ and f satisfies, at $\zeta_0 = \|a\|$, the conditions

$$\Re f(\zeta_0) = 0, \Re f(\zeta) = |\zeta|^{-2}\Re \left\langle p(\zeta a\|a\|^{-1}), \zeta a\|a\|^{-1}\right\rangle,$$

so using the relation (2), we deduce that $\Re f(\zeta) \geq 0$ for $0 < |\zeta| \leq |\zeta_0|$. Therefore f fulfils all assumptions of Lemma 1, so the equality (1) holds. If is not difficult to see that

$$\zeta_0 \frac{\partial f}{\partial \zeta}(\zeta_0) = \left\langle [D_z p(a)]a, a\|a\|^{-2}\right\rangle - \left\langle p(a)\ ,\ a\|a\|^{-2}\right\rangle$$

and

$$\bar\zeta_0 \frac{\partial f}{\partial \bar\zeta}(\zeta_0) = \left\langle [D_{\bar z} p(a)]\bar a, a\|a\|^{-2}\right\rangle .$$

The relation (4) follows from (1) and from the above relations. □

Remark 1. *The example of the mapping* $p(z) = z - \bar{z}$ *and the points* $a \in B - \{0\}$ *with* $a = \bar{a}$*, shows that the set of the mappings which fulfil all assumptions of the above theorem, is nonempty (in this case* $m = 0$*).*

An immediate application of Theorem 1 is given in the following result:

Corollary 1. *Let* p *be a mapping from* $\mathcal{C}^1(B - \{0\})$*. Assume that there exists and is positive the limit*

$$\lim_{z \to 0} \|z\|^{-2} \, \Re \langle p(z), z \rangle = d.$$

If for all $z \in B - \{0\}$

$$\Re \left\langle [D_z p(z)] p(z) + [D_{\bar{z}} p(z)] \overline{p(z)}, z \right\rangle + \|p(z)\|^2 \neq 0, \tag{5}$$

then for all $z \in B - \{0\}$

$$\Re \langle p(z), z \rangle > 0.$$

Proof. Suppose that the thesis does not hold. Then the real value function

$$h(z) = \begin{cases} \|z\|^{-2} \, \Re \langle p(z), z \rangle & \text{for } z \in B - \{0\} \\ d & \text{for } z = 0 \end{cases}$$

is continuous on B and there exists a point $a \in B - \{0\}$ such that

$$h(a) = 0 = \min\{h(z) : \|z\| \leq \|a\|\} = \min\{h(z) : 0 < \|z\| \leq \|a\|\}.$$

Therefore

$$0 = \|a\|^2 h(a) = \min\{\|z\|^2 h(z) : 0 < \|z\| \leq \|a\|\},$$

because the relation

$$0 < \|z\|^2 h(z) \leq h(z) \ \ , \ \ 0 < \|z\| \leq \|a\|,$$

imply the relation

$$0 \leq \min\{\|z\|^2 h(z) : 0 < \|z\| \leq \|a\|\} \leq \min\{h(z) : 0 < \|z\| \leq \|a\|\}.$$

Consequently (2) is fulfilled. Theorem 1 gives that the equality (3) holds. Let us create the inner product of the vector $p(a)$ and both sides of (3).

$$\left\langle \left[\overline{D_z p(a)} \right]' a + [D_{\bar{z}} p(a)]' \bar{a} + p(a), p(a) \right\rangle = m \langle a, p(a) \rangle .$$

Then, in view of the properties of inner product and adjoint operators, we obtain

$$\Re \left\langle [D_z p(a)] p(a) + [D_{\bar{z}} p(a)] \overline{p(a)}, a \right\rangle + \|p(a)\|^2 = m \Re \langle a, p(a) \rangle = 0.$$

This is a contradiction with (5), so our supposition is false and the thesis is true.

Corollary 2. *Let q be a mapping from the class $\mathcal{C}^1(B-\{0\})$ with the property $\|q(z)\| \leq \|z\|$ for $z \in B-\{0\}$. If an $a \in B-\{0\}, a = \bar{a}$, is the fixed point of q, then it is the zero of the mapping*

$$\left[\overline{D_z q(a)}\right]' + [D_{\bar{z}} q(a)]'$$

or the fixed point of the mapping

$$s\left(\left[\overline{D_z q(a)}\right]' + [D_{\bar{z}} q(a)]'\right)$$

with some $s \neq 0$.

Proof. Let us put

$$p(z) = z - q(z),\ z \in B-\{0\}.$$

Then $p(a) = 0$ and

$$\Re\langle p(z), z\rangle = \|z\|^2 - \Re\langle q(z), z\rangle \geq \|z\|^2 - |\langle q(z), z\rangle| \geq \|z\|^2 - \|q(z)\|\|z\| \geq 0.$$

Therefore (2) holds and by Theorem 1 we obtain that there exists a real number m such that (3) is fulfilled. From this we have

$$\left[\overline{D_z q(a)}\right]' a + [D_{\bar{z}} q(a)]' a = (1-m)a$$

because $\bar{a} = a$ and $p(a) = 0$. From this there follows the thesis.

Remark 2. *If q is holomorphic, then $D_{\bar{z}} q = 0$ and the assumption $a = \bar{a}$ is not necessary.*

Theorem 2. *Let $p \in \mathcal{C}^1(B-\{0\})$. If in a point $a \in B-\{0\}$*

$$|\langle p(a), a\rangle| = M\|a\|^2 = \max\{|\langle p(z), z\rangle| : \|z\| \leq \|a\|\} > 0, \tag{6}$$

then there exist a real number m such that

$$e^{2i\theta}\left[\overline{D_z p(a)}\right]' a + [D_{\bar{z}} p(a)]' \bar{a} + p(a) = mae^{i\theta}, \tag{7}$$

where $\theta = \arg\langle p(a), a\rangle$.

Proof. Let $r = \|a\| \in (0,1)$ and v be an arbitrary vector of $T_a(\partial B_r)$. Then there exists an $\varepsilon > 0$ and an at least differentiable function $\gamma : (-\varepsilon, \varepsilon) \to \partial B_r$ such that $\gamma(0) = a$ and $\frac{d\gamma}{dt}(0) = v$. If we put

$$\beta(t) = |\langle p(\gamma(t)), \gamma(t)\rangle|^2, \quad t \in (-\varepsilon, \varepsilon),$$

then, from (6) we have

$$\beta(0) = M^2\|a\|^4 = \max\{\beta(t) : t \in (-\varepsilon, \varepsilon)\},$$

hence $\frac{d\beta}{dt}(0) = 0$.

An easily computation yields

$$\frac{d\beta}{dt}(0) = 2\Re\left\{\left\{\langle [D_z p(a)]v + [D_{\bar z} p(a)]\bar v, a\rangle + \langle p(a), v\rangle\right\}\overline{\langle p(a), a\rangle}\right\}, \tag{8}$$

so

$$0 = \Re\left\langle \lambda\left\{\left[\overline{D_z p(a)}\right]' a + [D_{\bar z} p(a)]'\bar a \;+\; p(a)\;\right\}, v\right\rangle,$$

where $\lambda = M\|a\|^2 e^{-i\theta}$.

From this we obtain (7) on the similar way as in the proof of Theorem 1. □

Remark 3. *The example of the mapping $p(z) = z + \bar z$ and the points $a \in B - \{0\}$ with $a = \bar a$ shows that the set of the mappings which fulfil all assumptions of the above theorem is nonempty.*

An application of the above result is given in the following:

Corollary 3. *Let p be a mapping from $\mathcal{C}^1(B - \{0\})$. Assume that there exists the limit*

$$\lim_{z\to 0} \|z\|^{-2} |\langle p(z), z\rangle| = d$$

and it belongs to the interval $[0, M)$. If for all $z \in B - \{0\}$ and all $\theta \in [0, 2\pi)$

$$\Im\left\langle e^{-2i\theta}[D_z p(z)]p(z) - [D_{\bar z} p(z)]\overline{p(z)}, z\right\rangle \neq 0, \tag{9}$$

then

$$|\langle p(z), z\rangle| < M\|z\|^2 \tag{10}$$

for all $z \in B - \{0\}$.

Proof. Suppose that the thesis does not hold. Then the real value function

$$h(z) = \begin{cases} \|z\|^{-2} |\langle p(z), z\rangle| & \text{for } z \in B - \{0\} \\ d & \text{for } z = 0 \end{cases}$$

is continuous on B and there exists a point $a \in B - \{0\}$ such that

$$h(a) = M = \max\{h(z) : \|z\| \le \|a\|\}.$$

From this there follows (6), hence we can find a real number m and such that the equality (7) is fulfilled.

Let us create the inner product of the vector $p(a)$ and of the both sides of (7).

$$\left\langle p(a), e^{2i\theta}\left[\overline{D_z p(a)}\right]' a + [D_{\bar z} p(a)]'\bar a + p(a)\right\rangle = m e^{i\theta}\left\langle p(a), a\right\rangle .$$

Therefore

$$\Im\left\langle p(a), e^{2i\theta}\left[\overline{D_z p(a)}\right]' a + [D_{\bar z} p(a)]'\bar a\right\rangle = 0,$$

because $\theta = \arg < p(a), a >$. From the above, in view of the properties of inner product and adjoint operators, we obtain

$$\Im\left\langle e^{-2i\theta}[D_z p(a)]p(a) - [D_{\bar z} p(a)]\overline{p(a)}, a\right\rangle = 0.$$

This is a contradiction with (9), so our supposition is false and the thesis is true. □

Remark 4. *If p is a holomorphic mapping, the results from Theorem 1 and Theorem 2 are similar to the results from* [1] *and* [2].

References

[1] G. Kohr, On some partial differential inequalities for holomorphic mappings in $\mathbb{C}^n$ (*to appear*).
[2] G. Kohr, Certain partial differential inequalities and applications for holomorphic mappings defined on the unit ball of $\mathbb{C}^n$ (*to appear*).
[3] P. T. Mocanu, Alpha-convex nonanalytic functions, *Mathematica* **29** (52) (1987), 49–55.

G. KOHR
FACULTY OF MATHEMATICS
BABES–BOLYAI UNIVERSITY
1.M. KOGALNICEANU STR.
3400 CLUJ–NAPOCA
ROMANIA
E-MAIL: GKohr@math.ubbcluj.ro

P. LICZBERSKI
INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF ŁÓDŹ
AV. POLITECHNIKI 11, 90 – 924, ŁÓDŹ
POLAND
E-MAIL: PiLiczb@Lodz1.p.Lodz.pl

(Received February 20, 1996; revised May 30, 1996)