Notes on tensorial connections

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To professor L. Tamássy on his 60th birthday

1. Introduction. Bompiani's tensorial connections have been discussed by many authors up to now. As for the original formulation of the question see BOMPIANI [1]. Earlier mainly Italian geometers investigated the problem (a good survey of their more important papers is accessible in the bibliography of [2]) but later many others joined in these investigations. Since the 60ˢ professor TAMÁSSY has published a lot of interesting papers in this field (see e.g. [7], [8], [9]).

In most of the indicated works the local foundations of the theory were treated by local means, of course. In these notes — which constitute a part of the author's thesis "*Horizontal maps and tensorial connections*" (Debrecen, 1981) — we are going to discuss the tensorial connections in a little more general setting and from vector bundle viewpoint. The "more generality" essentially means two things here. Firstly, the rank of the considered vector bundles differs from the dimension of the base manifold, in contrast with the classical theory. Secondly, we distinguish *general, homogeneous* and *linear tensorial connections.* (The exact meaning of these attributes will be clarified soon.)

The organization of our paper is the following. Section 2 is devoted to the necessary preparations. In Section 3 a local description of a tensor bundle is discussed. Section 4 contains the definitions of the different tensorial connections and the characterizations of their *decomposability*. The paper finishes with an observation on the lift of a vector field to tensor bundles (Section 5).

2. Preliminaries. Our basic work of reference is the monograph [4], we follow its notations, terminology and conventions as closely as feasible. So manifolds are always finite dimensional, Hausdorff, 2nd countable and smooth. A differentiable map — or simply a map — means a smooth map unless otherwise stated. If M is a manifold, then $C^\infty(M)$ is the ring of smooth functions $M \to \mathbf{R}$. A *vector bundle* over the *base manifold* B is denoted by $\xi=(E, \pi, B, F)$. The fixed vector space F is the *typical fiber* while the manifold E is the *total space* of ξ. $\pi\colon E \to B$ is the *projection map,* $F_x=\pi^{-1}(x)$ is the *fiber* at $x \in B$. In particular, the *tangent bundle* of a manifold M is written as $\tau_M=(TM, \pi_M, M, \mathbf{R}^m)$ ($m=\dim M$); its fibers are the tangent spaces $T_x(M)$ ($x \in M$). The $C^\infty(B)$-module of the *cross-sections* in ξ is denoted by $\operatorname{Sec}\xi$, in particular the module of vector fields on M is $\mathfrak{X}(M):=\operatorname{Sec}\tau_M$. $A^p(B;\xi)$ denotes the $C^\infty(B)$-module of the *ξ-valued p-forms on* B ($p \geqq 0$, $A^0(B;\xi):=\operatorname{Sec}\xi$);

$$i(X)\colon A^p(B;\xi) \to A^{p-1}(B;\xi) \quad (p \geqq 1,\ X \in \mathfrak{X}(B))$$

is the *substitution operator* ([4], Vol. 2, pp. 304—306).

Let the open set $U \subset B$ be a *trivializing neighbourhood* for ξ with the *trivializing map* $\psi_U: U \times F \to \pi^{-1}(U)$. Then the system of functions

$$\text{(1)} \qquad \begin{cases} x^i := u^i \circ \pi & (i = 1, \ldots, n) \\ y^\alpha := l^\alpha \circ pr_2 \circ \psi_U^{-1} & (\alpha = 1, \ldots, r) \end{cases}$$

— where $\{u^i\}$ is a local coordinate system on U and $\{l^\alpha\}$ is a basis of the conjugate space $L(F)$ — is a local coordinate system over $\pi^{-1}(U)$.

Consider the short exact sequence of vector bundles

$$\text{(2)} \qquad 0 \to V_\xi \xrightarrow{i} \tau_E \xrightarrow{\widetilde{d\pi}} \pi^*(\tau_B) \to 0$$

(cf. [4], Vol. 2, p. 335), where $V_\xi = (VE, \pi_V, E, F)$, $VE := \bigcup_{z \in E} \operatorname{Ker}(d\pi)_z$ is the *vertical subbundle* of τ_E, i is the inclusion map, $\pi^*(\tau_B)$ is the pull-back of τ_B over π and $\widetilde{d\pi}|_{T_zE} := (d\pi)_z$. If the strong bundle map $\mathbf{H}: \pi^*(\tau_B) \to \tau_E$ is a *splitting* of (2) then

$$\text{(3)} \qquad l^h := (\widetilde{d\pi}|_{\operatorname{Im}\mathbf{H}})^{-1}: \pi^*(\tau_B) \to \tau_E$$

is called the *horizontal lift* induced by $\mathbf{H}$, while the vector field

$$\text{(4)} \qquad X^h: z \in E \mapsto X^h(z) := l_z^h[X(\pi(z))], \quad X \in \mathfrak{X}(B)$$

is the so-called *horizontal lift* of X (evidently, $X^h \in \mathfrak{X}_H(E) := \operatorname{Sec} \operatorname{Im} \mathbf{H}$);

$$\text{(5)} \qquad \mathbf{h} := \mathbf{H} \circ \widetilde{d\pi} \quad \text{and} \quad \mathbf{v} := \iota - \mathbf{h} \quad (\iota := \text{identity map})$$

are the *horizontal* and the *vertical projection* belonging to $\mathbf{H}$, respectively;

$$\text{(6)} \qquad \mathbf{K} := \alpha \circ \mathbf{v}$$

is called *Dombrowski-map* (α denotes the *canonical* bundle map $V_\xi \to \xi$, see e.g. [4], Vol. 1, p. 291), finally we say that $\mathbf{H}$ satisfies the *homogeneity condition,* if

$$\text{(7)} \qquad \forall t \in \mathbf{R}: \ \mathbf{h} \circ d\mu_t = d\mu_t \circ \mathbf{h} \quad (\mu_t: E \to E, \ z \mapsto tz).$$

(For a detailed treatment of these important constructions see e.g. the author's Thesis.)

Now the *general connection* $\nabla: \operatorname{Sec} \xi \to A^1(B; \xi)$ induced by the splitting $\mathbf{H}$ is defined by

$$\text{(8)} \qquad \nabla: \sigma \in \operatorname{Sec} \xi \mapsto \nabla \sigma := \mathbf{K} \circ d\sigma \in A^1(B; \xi).$$

We speak about *linear connection* if ∇ is induced by such a splitting of (2) which satisfies the homogeneity condition. We get an "intermediate class" between the general connections and the linear ones assuming the following:

(i) $\mathbf{H}$ is only a continuous splitting of (2).

(ii) The horizontal projection given by (5) is smooth over $T\dot{E}$, $\dot{E} := \bigcup_{x \in B} (F_x \setminus \{0\})$, but it is *not* differentiable on the null section.

(iii) The homogeneity condition (7) is satisfied.

In this case the map (8) is called a *homogeneous connection* induced by **H**. — If $\nabla\colon \mathrm{Sec}\,\xi\to A^1(B;\xi)$ is a connection then the maps $\nabla_X := i(X)\circ\nabla\colon \mathrm{Sec}\,\xi\to\mathrm{Sec}\,\xi$ $(X\in\mathfrak{X}(B))$ are the *covariant derivatives* by X with respect to ∇.

Fixing the local coordinate system (1), a smooth splitting **H** of (2) can be described locally with the help of some (unique, smooth) functions $\Gamma_i^\alpha\colon \pi^{-1}(U)\to\mathbf{R}$, they are called *connection parameters.* It is known that in the linear case the the restricted functions $\Gamma_i^\alpha|_{F_x}$ are linear, namely

$$\Gamma_i^\alpha = y^\beta(\Gamma_{i\beta}^\alpha\circ\pi),\quad \Gamma_{i\beta}^\alpha\colon U\to\mathbf{R},\quad x\mapsto\frac{\partial\Gamma_i^\alpha}{\partial y^\beta}(z),\quad z=\pi(x);$$

while in the homogeneous case the $\Gamma_i^\alpha|_{F_x}$-s are only homogeneous.

3. Local coordinate systems on tensor bundles. From now on $\xi=(E,\pi,B,F)$ and $\eta=(E',\pi',B,H)$ will denote fixed vector bundles of rank r and s resp. over the n-dimensional base manifold B. $\xi\otimes\eta=(\bar E,\bar\pi,B,F\otimes H)$ is their *tensor product* and $\xi^*=(E^*,\pi^*,B,F^*)$ is the *dual bundle* of ξ. Throughout our following discussion we use indices as $i,j,k,\ldots=1,\ldots,n$; $\alpha,\beta,\ldots=1,\ldots,r$; $\lambda,\mu,\ldots=1,\ldots,s$; Einstein's summation convention is applied accordingly. We can choose such an open set $U\subset B$ which is a trivializing neighbourhood for ξ,η and $\xi\otimes\eta$ simultaneously. Denoting the corresponding trivializing maps by ψ_U^1,ψ_U^2 and ψ_U resp., consider the maps

$$\psi_{U,x}^1\colon F\to F_x,\quad a\mapsto\psi_{U,x}^1(a):=\psi_U^1(x,a);$$

$$\psi_{U,x}^2\colon H\to H_x,\quad b\mapsto\psi_{U,x}^2(b):=\psi_U^2(x,b);$$

$$\psi_{U,x}\colon F\otimes H\to F_x\otimes H_x,\quad a\otimes b\mapsto\psi_U(x,a\otimes b).$$

By the construction of $\xi\otimes\eta$ we can assume that $\psi_{U,x}$ is the tensor product of the linear maps $\psi_{U,x}^1$ and $\psi_{U,x}^2$, that is $\forall x\in U\colon \psi_{U,x}=\psi_{U,x}^1\otimes\psi_{U,x}^2$. On the analogy of (1), the system of functions

$$\bar x^i := u^i\circ\bar\pi,\quad z^{\alpha\lambda}:=l^\alpha\otimes f^\lambda\circ pr_2\circ\psi_U^{-1} \tag{9}$$

constitutes a local coordinate system over $\bar\pi^{-1}(U)$. (In (9) we kept the notations of (1) with the additional $\{f^\lambda\}$ which is a fixed basis of the conjugate space $L(H)$.) A simple but very useful observation is given in

Proposition 1. *The restrictions of the functions* $z^{\alpha\lambda}\colon\bar\pi^{-1}(U)\to\mathbf{R}$ *to the fibers* $\bar F_x := F_x\otimes H_x$ $(x\in U)$ *are decomposable elements of the conjugate space* $L(F_x\otimes H_x)$.

Proof. It is an elementary fact (see e.g. [3], p. 36) that there exists a canonical linear isomorphism between the conjugate spaces $L(F_x\otimes H_x)$ and $L(F_x)\otimes L(H_x)$ so that we can identify them. Thus it is not meaningless to speak about decomposability in the Proposition. Since the maps $\psi_{U,x}^2,\psi_{U,x}^1$ are linear isomorphisms their tensor product yields the linear isomorphism $\psi_{U,x}^1\otimes\psi_{U,x}^2\colon F\otimes H\xrightarrow{\cong}F_x\otimes H_x$. Dualizing, we obtain then isomorphism

$$(\psi_{U,x}^1\otimes\psi_{U,x}^2)^*\colon L(F_x\otimes H_x)\xrightarrow{\cong}L(F\otimes H)$$

or — what is the same — the isomorphism

$$(\psi^1_{U,x})^* \otimes (\psi^2_{U,x})^*: \ L(F_x)\otimes L(H_x) \xrightarrow{\cong} L(F)\otimes L(H).$$

Let $\{a_\alpha\}$ and $\{b_\lambda\}$ be the dual basis of $\{l^\alpha\}$ and $\{f^\lambda\}$ resp. Then $\{\psi_{U,x}(a_\alpha\otimes b_\lambda)=\psi^1_{U,x}(a_\alpha)\otimes\psi^2_{U,x}(b_\lambda)\}$ is a basis of $F_x\otimes H_x$ and this basis is constituted by *decomposable* tensors. If $z^{\beta\mu}_x := z^{\beta\mu}|^U_{F_x\otimes H_x}$, then

$$z^{\beta\mu}_x\big(\psi_{U,x}(a_\alpha\otimes b_\lambda)\big) := l^\beta\otimes f^\mu \circ pr_2\circ \psi_U^{-1}|_{F_x\otimes H_x}\big(\psi_U(x, a_\alpha\otimes b_\lambda)\big) =$$

$$= l^\beta\otimes f^\mu(a_\alpha\otimes b_\lambda) = l^\beta(a_\alpha) f^\mu(b_\lambda) = \delta^\beta_\alpha\delta^\mu_\lambda,$$

therefore the functions $z^{\beta\mu}_x \in L(F_x\otimes H_x)\equiv L(F_x)\otimes L(H_x)$ constitute the dual of the basis $\{\psi^1_{U,x}(a_\alpha)\otimes\psi^2_{U,x}(b_\lambda)\}$ from which the Proposition is immediate. ∎

Corollary. $\xi\otimes\eta$ *has such a* framing *over* U *which is constituted by* decomposable *sections, that is there are cross-sections* $e_\alpha\colon U\to E$, $g_\lambda\colon U\to E'$ *such that* $\forall x\in U\colon \{e_\alpha(x)\otimes g_\lambda(x)\}$ *is a basis of* $F_x\otimes H_x$.

PROOF. Indeed, consider those mappings $U\to\bar E$ which assign to each $x\in U$ the vectors of the dual basis of $z^{\alpha\lambda}_x$. ∎

Remarks.

1) The framing in question is called *induced* by the local coordinate system (9). In particular, we can consider the framing induced by (1).
2) Of course, a decomposable section of $\mathrm{Sec}(\xi\otimes\eta)$ has the form $\sigma\otimes\tau$ ($\sigma\in\mathrm{Sec}\,\xi$, $\tau\in\mathrm{Sec}\,\eta$). The set of these sections will be denoted by $\mathrm{Sec}_d(\xi\otimes\eta)$ in the next following considerations. We recall that the map $\mathcal{T}\colon \sigma\otimes\tau\mapsto\mathcal{T}(\sigma\otimes\tau)$, $\mathcal{T}(\sigma\otimes\tau)(x):=\sigma(x)\otimes\tau(x)$ is a module isomorphism between $\mathrm{Sec}\,\xi\otimes\mathrm{Sec}\,\eta$ and $\mathrm{Sec}(\xi\otimes\eta)$ ([4], Vol. 1, p. 80), in the sequel we identify these modules with the help of $\mathcal{T}$.
3) In the case of the dual bundle ξ^* — according to (1) — we fix the local coordinate system

$$\text{(1')}\qquad \overset{*}{x}{}^i := u^i\circ\pi^*,\quad y_\alpha := a_\alpha\circ pr_2\circ\overset{*}{\psi}{}_U^{-1}.$$

(1) and (1') yields on $\bar\pi^{-1}(U)$ the local coordinate system

$$\text{(9')}\qquad \bar x^i = u^i\circ\bar\pi,\quad z^\alpha_\beta = l^\alpha\otimes a_\beta\circ pr_2\circ\bar\psi_U^{-1}.$$

(The projection and the trivializing map for $\xi\otimes\xi^*$ are also denoted by $\bar\pi$ and $\bar\psi_U$ respectively. Of course, the Proposition holds for the functions z^α_β similarly.)

4. Tensorial connections and their decomposability.

Definition. A general, homogeneous, and linear connection over the tensor bundle $\xi\otimes\eta$ is called a general, homogeneous, and linear *tensorial connection,* respectively. The general tensorial connection $\nabla\colon \mathrm{Sec}(\xi\otimes\eta)\to A^1(B;\xi\otimes\eta)$ is called *decomposable* to the general connections $\overset{1}{\nabla}\colon \mathrm{Sec}\,\xi\to A^1(B;\xi)$, $\overset{2}{\nabla}\colon \mathrm{Sec}\,\eta\to A^1(B;\eta)$

over $\mathrm{Sec}_d(\xi \otimes \eta)$ if

$$\forall \sigma \in \mathrm{Sec}\,\xi,\ \tau \in \mathrm{Sec}\,\eta:\ \nabla(\sigma \otimes \tau) = \overset{1}{\nabla}\sigma \otimes \tau + \sigma \otimes \overset{2}{\nabla}\tau, \tag{10}$$

where e.g. $[(\overset{1}{\nabla}\sigma \otimes \tau)(x)](v) := (\overset{1}{\nabla}\sigma)_x(v) \otimes \tau(x)\ \ (x \in B,\ v \in T_x B)$.

Remarks.

1) It will be seen soon that these concepts of "tensorial connections" are immediate generalizations of Bompiani's ideas (in the classical theory $\xi = \eta = \tau_B$ or $\xi = \tau_B$, $\eta = \tau_B^*$). For the sake of simplicity we omit the several-factor tensor products.
2) The general construction of the operation $\otimes$ can be found in [4], Vol. II p. 314. — It is easy to see that the condition of decomposability (10) holds iff

$$\forall X \in \mathfrak{X}(B),\quad \sigma \in \mathrm{Sec}\,\xi,\quad \tau \in \mathrm{Sec}\,\eta:$$

$$\nabla_X(\sigma \otimes \tau) = \overset{1}{\nabla}_X \sigma \otimes \tau + \sigma \otimes \overset{2}{\nabla}_X \tau. \tag{11}$$

3) In the case of *linear* tensorial connections we can speak about decomposibility over the *whole* module $\mathrm{Sec}\,\xi \otimes \mathrm{Sec}\,\eta$.
Now, one can easily derive the next

Proposition 2.

(a) *Fixing the local coordinate system* (9), *for each splitting* $\overline{\mathbf{H}}: \bar{\pi}^*(\tau_B) \to \tau_{\bar{E}}$ *there are unique (smooth) functions* $\Gamma^{\alpha}{}_{i}{}^{\lambda}: \bar{\pi}^{-1}(U) \to \mathbf{R}$ *(they are the* connection parameters, *cf. section* 2) *such that at every point* $\bar{z} \in \bar{\pi}^{-1}(U)$ *the linear map* $\overline{\mathbf{H}}_{\bar{\pi}(\bar{z})} := \overline{\mathbf{H}}|_{T_{\bar{\pi}(\bar{z})}B}$ *is represented by the matrix*

$$\begin{pmatrix} E_n \\ \hline -\Gamma_i^{\alpha\lambda}(\bar{z}) \end{pmatrix}$$

of type $(n+rs) \times n$ *(where the block* E_n *is the unit matrix of type* $n \times n$*) with respect to the basis-pair*

$$\left\{\left(\frac{\partial}{\partial u^i}\right)_{\bar{\pi}(\bar{z})}\right\},\quad \left\{\left(\frac{\partial}{\partial \bar{x}^i}\right)_{\bar{z}},\quad \left(\frac{\partial}{\partial z^{\alpha\lambda}}\right)_{\bar{z}}\right\}.$$

(b) *The connection* ∇ *induced by* $\overline{\mathbf{H}}$ *is homogeneous or linear iff there are smooth functions*

$$\gamma_i^{\alpha\lambda}{}_{\beta\mu}:\ \bar{\pi}^{-1}(U) \cap \dot{\bar{E}} \to \mathbf{R} \quad \text{and} \quad \Gamma_i^{\alpha\lambda}{}_{\beta\mu}:\ U \to \mathbf{R}$$

resp. such that

$$\Gamma^{\alpha}{}_{i}{}^{\lambda} = z^{\beta\mu}\gamma_i^{\alpha\lambda}{}_{\beta\mu} \quad \text{or} \quad \Gamma^{\alpha}{}_{i}{}^{\lambda} = z^{\beta\mu}(\Gamma_i^{\alpha\lambda}{}_{\beta\mu} \circ \bar{\pi})$$

respectively. The local forms of the covariant derivatives by $X = X^i \dfrac{\partial}{\partial u^i}$ *with* *respect to* ∇ *are the following:*

$$(12a-b-c)\qquad \nabla_X t = \begin{cases} X^i\left(\dfrac{\partial t^{\alpha\lambda}}{\partial u^i} + \Gamma^{\alpha}{}_i{}^{\lambda}\circ t\right) e_\alpha\otimes g_\lambda & \text{in the general case,} \\ X^i\left(\dfrac{\partial t^{\alpha\lambda}}{\partial u^i} + t^{\beta\mu}(\gamma_i^{\alpha\lambda}{}_{\beta\mu}\circ t)\right) e_\alpha\otimes g_\lambda & \text{in the homogeneous case,} \\ X^i\left(\dfrac{\partial t^{\alpha\lambda}}{\partial u^i} + t^{\beta\mu}\Gamma_i^{\alpha\lambda}{}_{\beta\mu}\right) e_\alpha\otimes g_\lambda & \text{in the linear case} \end{cases}$$

$\big(t=t^{\alpha\lambda}e_\alpha\otimes g_\lambda,\ e_\alpha\otimes g_\lambda$ *is the framing induced by* (9)$\big)$. ∎

The starting point in the classical theory is just the local form of the covariant derivatives, see e.g. [9], (1.3) and [7], (11). Combining the formula (12a) with the formula (11) we have

Proposition 3. *Let* $\nabla, \overset{1}{\nabla}, \overset{2}{\nabla}$ *be general connections on* $\xi\otimes\eta$, ξ *and* η *respectively.* ∇ *is decomposable to* $\overset{1}{\nabla}$ *and* $\overset{2}{\nabla}$ *over* $\mathrm{Sec}_d(\xi\otimes\eta)$ *iff the relation between the corresponding connection parameters is the following: for each section*

$$\sigma = \sigma^\alpha e_\alpha\colon U\to E,\quad \tau=\tau^\lambda g_\lambda\colon U\to E'$$

$$(13)\qquad \Gamma^{\alpha}{}_i{}^{\lambda}\circ\sigma\otimes\tau = (\overset{1}{\Gamma}{}_i^{\alpha}\circ\sigma)\tau^\lambda + (\overset{2}{\Gamma}{}_i^{\lambda}\circ\tau)\sigma^\alpha. \ ∎$$

The criterion (13) contains as special cases the "decomposability results" of the classical theory. Namely:

1. *Homogeneous case* (cf. [8], Th. 2). Now $\Gamma^{\alpha}{}_i{}^{\lambda}=\gamma_i^{\alpha\lambda}{}_{\beta\mu}z^{\beta\mu}$, $\overset{1}{\Gamma}{}_i^{\alpha}=\overset{1}{\gamma}{}_i^{\alpha}{}_{\beta}y_1^\beta$, $\overset{2}{\Gamma}{}_i^{\lambda}=\overset{2}{\gamma}{}_i^{\lambda}{}_{\mu}y_2^\mu$ $(y_1^\beta:=y^\beta$, the definition of the functions $y_2^\mu:=(\pi')^{-1}(U)\to\mathbf{R}$ is the same as in (1)), so by (13)

$$\gamma_i^{\alpha\lambda}{}_{\beta\mu}z^{\beta\mu}\circ\sigma\otimes\tau = (\overset{1}{\gamma}{}_i^{\alpha}{}_{\beta}\, y_1^\beta\circ\sigma)\,\tau^\lambda + (\overset{2}{\gamma}{}_i^{\lambda}{}_{\mu}\, y_2^\mu\circ\tau)\,\sigma^\alpha.$$

From here a simple calculation yields the local criterion of decomposability

$$(14)\qquad \gamma_i^{\alpha\lambda}{}_{\beta\mu}\circ\sigma\otimes\tau = (\overset{1}{\gamma}{}_i^{\alpha}{}_{\beta}\circ\sigma)\delta_\mu^\lambda + (\overset{2}{\gamma}{}_i^{\lambda}{}_{\mu}\circ\tau)\delta_\beta^\alpha.$$

2. *Linear case* (cf. e.g. [7], Satz 2). Then $\Gamma^{\alpha}{}_i{}^{\lambda}=z^{\beta\mu}(\Gamma_i^{\alpha\lambda}{}_{\beta\mu}\circ\bar\pi)$, $\overset{1}{\Gamma}{}_i^{\alpha}=(\overset{1}{\Gamma}{}_i^{\alpha}{}_{\beta}\circ\pi)y_1^\beta$, $\overset{2}{\Gamma}{}_i^{\lambda}=(\overset{2}{\Gamma}{}_i^{\lambda}{}_{\mu}\circ\pi')y_2^\mu$, therefore by (13) the criterion of decomposability has the form $z^{\beta\mu}(\Gamma_i^{\alpha\lambda}{}_{\beta\mu}\circ\bar\pi)\circ\sigma\otimes\tau=\{[(\overset{1}{\Gamma}{}_i^{\alpha}{}_{\beta}\circ\pi)y_1^\beta]\circ\sigma\}\tau^\lambda+\{[(\overset{2}{\Gamma}{}_i^{\lambda}{}_{\mu}\circ\pi')y_2^\mu]\circ\tau\}\sigma^\alpha$. Hence after some further calculations we have:

$$(15)\qquad \Gamma_i^{\alpha\lambda}{}_{\beta\mu} = \overset{1}{\Gamma}{}_i^{\alpha}{}_{\beta}\delta_\mu^\lambda + \overset{2}{\Gamma}{}_i^{\lambda}{}_{\mu}\delta_\beta^\alpha.$$

— We note that formula (15) can be derived immediately from the observations

$$\overset{1}{\nabla}_{\frac{\partial}{\partial u^i}} e_\beta = \overset{1}{\Gamma}{}_i^{\alpha}{}_{\beta}e_\alpha,\quad \overset{2}{\nabla}_{\frac{\partial}{\partial u^i}} g_\mu = \overset{2}{\Gamma}{}_i^{\lambda}{}_{\mu}g_\lambda,$$

$$\nabla_{\frac{\partial}{\partial u^i}}(e_\beta\otimes g_\mu) = \Gamma_i^{\alpha\lambda}{}_{\beta\mu}e_\alpha\otimes g_\lambda.$$

3. *The case of the bundle* $\xi\otimes\xi^*$. Firstly, we remark the following. — Let the mapping

$$F_x\times L(F_x)\to\mathbf{R},\quad (z,\overset{*}{z})\mapsto\overset{*}{z}(z)\quad (x\in B)$$

be denoted by β_x.

Then for each $\Omega\in A^1(B;\xi)$, $\overset{*}{\sigma}\in\operatorname{Sec}\xi^*$ we get a 1-form $\beta_*(\Omega,\overset{*}{\sigma})$ on B by the definition

$$\beta_*(\Omega,\overset{*}{\sigma})(x,v):=\beta_x\big(\Omega_x(v),\overset{*}{\sigma}(x)\big)\quad (v\in T_xB).$$

Analogously, $\beta_*(\sigma,\overset{*}{\Omega})$ $\big(\sigma\in\operatorname{Sec}\xi,\ \overset{*}{\Omega}\in A^1(B,\xi^*)\big)$ is also a 1-form on B. Finally we can form the function

$$\beta_*(\sigma,\overset{*}{\sigma})\colon B\to\mathbf{R},\quad x\mapsto\beta_*(\sigma,\overset{*}{\sigma})(x):=\beta_x[\sigma(x),\overset{*}{\sigma}(x)].$$

Now let a general connection ∇ be given on ξ. ∇ induces a general connection $\overset{*}{\nabla}$ on ξ^* by the relation

$$\beta_*(\sigma,\overset{*}{\nabla}\overset{*}{\sigma})+\beta_*(\nabla\sigma,\overset{*}{\sigma}):=\delta\beta_*(\sigma,\overset{*}{\sigma})$$

where δ is the operator of the exterior derivative (cf. [4], Vol. 2, p. 320).

Proposition 4. *Let the connection parameters of the general connection* $\nabla\colon\operatorname{Sec}\xi\to A^1(B;\xi)$ *be the functions* Γ_i^α *with respect to* (1). *Then the connection parameters* $\overset{*}{\Gamma}_{i\alpha}$ *of the induced connection* $\overset{*}{\nabla}\colon\operatorname{Sec}\overset{*}{\xi}\to A^1(B;\xi^*)$ *are characterized by the relations*

$$\sigma^\alpha(\overset{*}{\Gamma}_{i\alpha}\circ\overset{*}{\sigma})=-\sigma_\alpha(\Gamma_i^\alpha\circ\sigma)\tag{16}$$

with respect to the local coordinate system (1′) $(\sigma\in\operatorname{Sec}\xi,\ \overset{*}{\sigma}\in\operatorname{Sec}\overset{*}{\xi},\ \sigma^\alpha$ *and* σ_α *are the component functions).*

Proof. Let $X=X^i\dfrac{\partial}{\partial u^i}$ be a vector field over the trivializing neighbourhood U. Then at each point $x\in B$

$$\beta_*(\sigma,\overset{*}{\nabla}\overset{*}{\sigma})\big(x,X(x)\big):=\beta_x\big[\sigma(x),(\overset{*}{\nabla}\overset{*}{\sigma})_x(X(x))\big]=$$
$$=\beta_x[\sigma(x),(\overset{*}{\nabla}_X\overset{*}{\sigma})(x)]:=[(\overset{*}{\nabla}_X\overset{*}{\sigma})(x)]\big(\sigma(x)\big)=$$
$$=\Big[X^i(x)\Big(\frac{\partial\sigma_\alpha}{\partial u^i}(x)+\overset{*}{\Gamma}_{i\alpha}(\overset{*}{\sigma}(x))\Big)\overset{*}{e}{}^\alpha(x)\Big]\big(\sigma^\beta(x)e_\beta(x)\big)=$$
$$=\Big[X^i\sigma^\alpha\Big(\frac{\partial\sigma_\alpha}{\partial u^i}+\overset{*}{\Gamma}_{i\alpha}\circ\overset{*}{\sigma}\Big)\Big](x).$$

In the same way,

$$\beta_*(\nabla\sigma,\overset{*}{\sigma})\big(x,\ X(x)\big)=\Big[X^i\sigma_\alpha\Big(\frac{\partial\sigma^\alpha}{\partial u^i}+\Gamma_i^\alpha\circ\sigma\Big)\Big](x).$$

Finally $\delta\big(\beta_*(\sigma,\ \overset{*}{\sigma})\big)X=X\big(\beta_*(\sigma,\ \overset{*}{\sigma})\big)=X(\sigma_\alpha\sigma^\alpha)=X^i\Big(\dfrac{\partial\sigma_\alpha}{\partial u^i}\sigma^\alpha+\dfrac{\partial\sigma^\alpha}{\partial u^i}\sigma_\alpha\Big)$. Combining these results we get the relation (16). ∎

Corollary 1. *The relation between the "original" connection parameters and the induced ones is given by*

$$\sigma_\alpha \sigma^\beta (\overset{*}{\Gamma}{}^{\alpha}_{i\,\beta} \circ \overset{*}{\sigma}) = -\sigma_\alpha \sigma^\beta (\Gamma^{\alpha}_{i\,\beta} \circ \sigma)$$

and

$$\overset{*}{\Gamma}{}^{\alpha}_{i\,\beta} = -\Gamma^{\alpha}_{i\,\beta}$$

in the homogeneous and the linear case, respectively. ∎

Corollary 2. *The general tensorial connection* $\overline{\nabla}\colon \mathrm{Sec}\,(\xi \otimes \overset{*}{\xi}) \to A^1(B;\, \xi \otimes \xi^*)$ *is decomposable to the connection* $\nabla\colon \mathrm{Sec}\,\xi \to A^1(B;\, \xi)$ *and the induced one* $\overset{*}{\nabla}\colon \mathrm{Sec}\,\overset{*}{\xi} \to A^1(B;\, \xi^*)$ *over* $\mathrm{Sec}_d(\xi \otimes \overset{*}{\xi})$ *iff locally the relations*

$$\Gamma^{\alpha}_{i\,\beta} \circ (\sigma \otimes \sigma^*) = (\Gamma^{\alpha}_{i} \circ \sigma)\sigma_\beta + (\overset{*}{\Gamma}_{i\beta} \circ \sigma)\sigma^\alpha$$

hold. In particular, if $\overline{\nabla}$ *and* ∇ *are linear connections then the criterion of decomposability can be given as*

$$\Gamma^{\alpha\alpha_1}_{i\ \ \beta\beta_1} = \Gamma^{\alpha}_{i\,\beta} \delta^{\alpha_1}_{\beta_1} - \Gamma^{\alpha_1}_{i\ \beta_1} \delta^{\alpha}_{\beta}. \quad ∎$$

5. Horizontal lift of vector fields to tensor bundles. The horizontal lift of a vector field to a tensor bundle was constructed by LEDGER and YANO [5] and in a different manner (from a principal bundle viewpoint) by MOK [6]. Now we show that using tensorial connection these lifts can be gained as special cases. — Let $X = X^i \frac{\partial}{\partial u^i}$ be again a vector field over the trivializing neighbourhood $U \subset B$ and $\overline{\nabla}$ be a tensorial connection on $\xi \otimes \eta$ with the connection parameters $\Gamma^{\alpha}{}_{i}{}^{\lambda}\colon \bar{\pi}^{-1}(U) \to \mathbf{R}$ induced by a splitting $\overline{\mathbf{H}}$. An easy calculation shows that the local form of the horizontal lift of X (which was defined by (4)) can be written as follows:

$$X^h = (X^i \circ \bar{\pi}) \left(\frac{\partial}{\partial \bar{x}^i} - \Gamma^{\alpha}{}_{i}{}^{\lambda} \frac{\partial}{\partial z^{\alpha\lambda}} \right).$$

If $\overline{\nabla}$ is a *linear* tensorial connection then

$$X^h = (X^i \circ \bar{\pi}) \left(\frac{\partial}{\partial \bar{x}^i} - z^{\beta\mu} (\Gamma^{\alpha\lambda}_{i\ \ \beta\mu} \circ \bar{\pi}) \right) \frac{\partial}{\partial z^{\alpha\lambda}}$$

and if in addition $\overline{\nabla}$ is *decomposable* then (according to (15))

$$X^h = (X^i \circ \bar{\pi}) \left(\frac{\partial}{\partial \bar{x}^i} - z^{\beta\lambda} (\overset{1}{\Gamma}{}^{\alpha}_{i\,\beta} \circ \bar{\pi}) - z^{\alpha\mu} (\overset{2}{\Gamma}{}^{\lambda}_{i\,\mu} \circ \bar{\pi}) \right) \frac{\partial}{\partial z^{\alpha\lambda}}. \tag{17}$$

If $\xi = \eta$, $\overset{1}{\nabla} = \overset{2}{\nabla} := \nabla$ then (17) yields

$$X^h = (X^i \circ \bar{\pi}) \left(\frac{\partial}{\partial \bar{x}^i} - z^{\beta\alpha_1} (\Gamma^{\alpha}_{i\,\beta} \circ \bar{\pi}) - z^{\alpha\beta_1} (\Gamma^{\alpha_1}_{i\ \beta_1} \circ \bar{\pi}) \right) \frac{\partial}{\partial z^{\alpha\alpha_1}}, \tag{18}$$

while in the case of $\eta=\overset{*}{\xi}$, $\overset{2}{\nabla}=\overset{*}{\nabla}$ we get the formula

$$X^h = (X^i \circ \bar{\pi}) \left(\frac{\partial}{\partial \bar{x}^i} - z_\beta^{\beta_1} (\Gamma^{\alpha}_{i\,\beta_1} \circ \bar{\pi}) + z^{\alpha}_{\alpha_1} (\Gamma^{\alpha_1}_{i\,\beta} \circ \bar{\pi}) \right) \frac{\partial}{\partial z^{\alpha}_{\beta}}. \tag{19}$$

If an addition $\xi=\tau_B$, then (18) and (19) reduce to the formula (4.4) of [6].

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