# On inequalities concerning the permanents of generalized doubly stochastic matrices

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## 1. Introduction

a) Let $R_n$ denote the $n$-dimensional real vector space with column vectors as its elements. Let $M$ denote the set of $n\times n$ matrices with real elements. Let $A^*$ denote the transpose of $A\in M$. $E\in M$ is the unit matrix.

Let $K\subset M$ denote the set of matrices, where all row and column sums are 1. Let $H\subset K$ be the set of matrices with non-negative elements, i.e. the set of so-called doubly stochastic matrices. Let $S_0\in H$ be the matrix where all the entries are $1/n$.

The elements of the set $K$ are said to be generalized doubly stochastic matrices.

If $A=(a_{jk})\in M$ then the permanent of $A$, denoted by $\operatorname{Per} A$, is defined as follows:

$$\operatorname{Per} A = \sum_{(i_1, \dots, i_n)} a_{1i_1} \dots a_{ni_n},$$

where $(i_1, \dots, i_n)$ runs over the full symmetric group.

Let $\Gamma$ be the set of vectors $(\beta_1, \dots, \beta_n)$, where the components are non-negative integers satisfying the conditions

$$0 \leqq \beta_k \leqq n \quad (k=1, \dots, n), \quad \beta_1+\dots+\beta_n = n.$$

Let $C_{\beta_1\dots\beta_n}(A)\in M$ denote the matrix, which consists of certain elements of $A\in M$. Namely the $k$-th column of $A$ appears $\beta_k$-times $(k=1, \dots, n)$ in $C_{\beta_1\dots\beta_n}(A)$, where $(\beta_1, \dots, \beta_n)$ runs over $\Gamma$.

Let $A=U\wedge V^*$ be the polar representation of $A\in M$, where $U\in M$, $V\in M$, $UU^*=VV^*=E$, and $\wedge$ is a diagonal matrix with diagonal elements $\lambda_j\geqq 0$ $(j=1, \dots, n)$. Using the Cauchy—Binet expansion theorem ([3], 579), we get

$$\operatorname{Per} A = \sum_{(\beta_1, \dots, \beta_n)\in\Gamma} \frac{\lambda_1^{\beta_1}\dots\lambda_n^{\beta_n}}{\beta_1!\dots\beta_n!} \operatorname{Per} C_{\beta_1\dots\beta_n}(U) \operatorname{Per} C_{\beta_1\dots\beta_n}(V).$$

b) The following Conjecture is due to VAN DER WAERDEN ([3], 586, Conjecture 1): If $A\in H$, then $\operatorname{Per} A\geqq n!/n^n$, with equality if and only if $A=S_0$. This conjecture was proved by G. P. JEGORITSEV in his paper [2].

In his paper [1] the author proved the following two theorems of Van der Waerden type, which are not a consequence of the Van der Waerden—Jegoritsev theorem because $H\subset K$.

If $(AA^*)^{1/2}$ denotes the only non-negative definite square root of $AA^*$, then we have the following theorem:

**Theorem 1.1.** *If* $A \in K$ *and if* $x \in R_1$, $y \in R_1$, $x+y=1$, *then*

$$x^2 \operatorname{Per} (AA^*)^{1/2} + y^2 \operatorname{Per} (A^*A)^{1/2} + 2xy \operatorname{Per} A \geqq n!/n^n$$

*with equality if and only if* $A = S_0$.

**Theorem 1.2.** *If* $A \in K$ *with polar representation* $A = U \wedge V^*$, *and if*

$$\operatorname{Per} C_{\beta_1 \ldots \beta_n}(U) \operatorname{Per} C_{\beta_1 \ldots \beta_n}(V) \geqq 0$$

$$(\beta_1, \ldots, \beta_n) \in \Gamma,$$

*then* $\operatorname{Per} A \geqq n!/n^n$ *with equality if and only if* $A = S_0$.

In this paper we give extensions of these two theorems.

## 2. The extension of the first Theorem

Let $A = U \wedge V^*$ be the polar representation of the matrix $A \in K$. Let $\lambda_k \geqq 0$ $(k=1, \ldots, n)$ be the elements of the diagonal matrix $\wedge$. Denote by $u_k$ and $v_k$ $(k=1, \ldots, n)$ the $k$-th column of the matrix $U$ and $V$, respectively.

As it is known $A \in K$ if and only if one of $\lambda_k$ $(k=1, \ldots, n)$ is equal to 1 and simultaneously all elements of the corresponding column of $U$ and $V$ are equal to $1/\sqrt{n}$. In the following, without loss of the generality, let us suppose that $\lambda_1 = 1$ and thus all elements of $u_1$ and $v_1$ are $1/\sqrt{n}$.

Let $1 \leqq r + \alpha \leqq n$, where $\alpha$ and $r$ are non-negative integers. Let $i_1, \ldots, i_r$ a combination of order $r$ of the elements $\alpha+1, \ldots, n$, without repetition and without permutation, i.e.

$$\alpha + 1 \leqq i_1 < \ldots < i_r \leqq n.$$

Let

$$U^{(\alpha)}_{i_1 \ldots i_r} = (u_1 \ldots u_\alpha u_{i_1} \ldots u_{i_r}),$$

$$V^{(\alpha)}_{i_1 \ldots i_r} = (v_1 \ldots v_\alpha v_{i_1} \ldots v_{i_r}),$$

$$\Lambda^{(\alpha)}_{i_1 \ldots i_r} = \begin{pmatrix} \lambda_1 & & & & & (0) \\ & \ddots & & & & \\ & & \lambda_\alpha & & & \\ & & & \lambda_{i_1} & & \\ & (0) & & & \ddots & \\ & & & & & \lambda_{i_r} \end{pmatrix}.$$

Then obviously

$$A^{(\alpha)}_{i_1 \ldots i_r} = U^{(1)}_{i_1 \ldots i_r} \Lambda^{(\alpha)}_{i_1 \ldots i_r} V^{(\alpha)*}_{i_1 \ldots i_r} \in M.$$

Let

$$G_r^{(\alpha)}(x, y) = \frac{1}{\binom{n-\alpha}{r}} \sum_{\alpha+1 \leqq i_1 < \dots < i_r \leqq n} \{x^2 \operatorname{Per}(A^{(\alpha)}_{i_1 \dots i_r} A^{(\alpha)*}_{i_1 \dots i_r})^{1/2} +$$

$$+ y^2 \operatorname{Per}(A^{(\alpha)*}_{i_1 \dots i_r} A^{(\alpha)}_{i_1 \dots i_r})^{1/2} + 2xy \operatorname{Per} A^{(\alpha)}_{i_1 \dots i_r}\},$$

where $x \in R_1$, $y \in R_1$.

**Theorem 2.1.** I. *If* $1 \leqq \alpha \leqq n$ *then*

$$A^{(\alpha)}_{i_1 \dots i_r} \in K. \tag{2.1}$$

II. *If* $x \in R_1$, $y \in R_1$, $x + y = 1$ *then*

$$G^{(\alpha)}_{r+1}(x, y) \geqq G^{(\alpha)}_r(x, y)$$

$$(r = 0, 1, \dots, n-\alpha-1;\ \alpha = 0, 1, \dots, n-1;\ 1 \leqq \alpha + r < n)$$

*with equality if and only if* $\lambda_{\alpha+1} = \dots = \lambda_n = 0$.

Proof. ad I. As a consequence of the condition we get that the first element of the diagonal matrix $A^{(\alpha)}_{i_1 \dots i_r}$ is equal to 1, moreover all elements of the first column of the matrices $U^{(\alpha)}_{i_1 \dots i_r}$ and $V^{(\alpha)}_{i_1 \dots i_r}$ are $1/\sqrt{n}$, i.e. (2,1) turned out to be right ([1], 108).

ad II. Taking into account that by the Cauchy—Binet expansion theorem

$$x^2 \operatorname{Per}(A^{(\alpha)}_{i_1 \dots i_{r+1}} A^{(\alpha)*}_{i_1 \dots i_{r+1}})^{1/2} + y^2 \operatorname{Per}(A^{(\alpha)*}_{i_1 \dots i_{r+1}} A^{(\alpha)}_{i_1 \dots i_{r+1}})^{1/2} + 2xy \operatorname{Per} A^{(\alpha)}_{i_1 \dots i_{r+1}} =$$

$$= \sum_{\beta_1 + \dots + \beta_{\alpha+r} = n} \frac{\lambda_1^{\beta_1} \dots \lambda_\alpha^{\beta_\alpha} \lambda_{i_r}^{\beta_{\alpha+1}} \dots \lambda_{i_r}^{\beta_{\alpha+r}}}{\beta_1! \dots \beta_\alpha!\, \beta_{\alpha+1}! \dots \beta_{\alpha+r}!} \times$$

$$\times [x \operatorname{Per} C_{\beta_1 \dots \beta_{\alpha+r}}(U^{(\alpha)}_{i_1 \dots i_r}) + y \operatorname{Per} C_{\beta_1 \dots \beta_{\alpha+r}}(V^{(\alpha)}_{i_1 \dots i_r})]^2 +$$

$$+ \sum_{\substack{\beta_1 + \dots + \beta_{\alpha+r+1} = n \\ \beta_{\alpha+r+1} \geqq 1}} \frac{\lambda_1^{\beta_1} \dots \lambda_\alpha^{\beta_\alpha} \lambda_{i_1}^{\beta_{\alpha+1}} \dots \lambda_{i_{r+1}}^{\beta_{\alpha+r+1}}}{\beta_1! \dots \beta_\alpha!\, \beta_{\alpha+1}! \dots \beta_{\alpha+r+1}!} \times$$

$$\times [x \operatorname{Per} C_{\beta_1 \dots \beta_{\alpha+r+1}}(U^{(\alpha)}_{i_1 \dots i_{r+1}}) + y \operatorname{Per} C_{\beta_1 \dots \beta_{r+\alpha+1}}(V^{(\alpha)}_{i_1 \dots i_{r+1}})]^2,$$

we obtain

$$x^2 \operatorname{Per}(A^{(\alpha)}_{i_1 \dots i_{r+1}} A^{(\alpha)*}_{i_1 \dots i_{r+1}})^{1/2} + y^2 \operatorname{Per}(A^{(\alpha)*}_{i_1 \dots i_{r+1}} A^{(\alpha)}_{i_1 \dots i_{r+1}})^{1/2} + \tag{2.2}$$

$$+ 2xy \operatorname{Per} A^{(\alpha)}_{i_1 \dots i_{r+1}} \geqq x^2 \operatorname{Per}(A^{(\alpha)}_{i_1 \dots i_r} A^{(\alpha)*}_{i_1 \dots i_r})^{1/2} +$$

$$+ y^2 \operatorname{Per}(A^{(\alpha)*}_{i_1 \dots i_r} A^{(\alpha)}_{i_1 \dots i_r})^{1/2} + 2xy \operatorname{Per} A^{(\alpha)}_{i_1 \dots i_r}$$

with equality if and only if $\lambda_{i_{r+1}} = 0$. Namely if $\lambda_{i_{r+1}} > 0$ and since $\lambda_1 = 1$, we get equality if and only if

$$x \operatorname{Per} C_{\beta_1 = i, \beta_{i_{r+1}} = n-i}(U^{(\alpha)}_{i_1 \dots i_{r+1}}) + y \operatorname{Per} C_{\beta_1 = i, \beta_{i_{r+1}} = n-i}(V^{(\alpha)}_{i_1 \dots i_{r+1}}) = 0 \tag{2.3}$$

$$(i = 0, \dots, n-1).$$

Let $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ the components of the vectors $u_{i_{r+1}}$ and $v_{i_{r+1}}$, respectively. Using the notation

$$G_i(x_1, \ldots, x_n) = \sum_{1 \leqq \alpha_1 < \ldots < \alpha_i \leqq n} x_{\alpha_1} \ldots x_{\alpha_i}$$

$$(i = 1, \ldots, n),$$

we get from (2,3) the system of equations

$$xG_i(r_1, \ldots, r_n) + yG_i(s_1, \ldots, s_n) = 0 \tag{2.4}$$

$$(i = 1, \ldots, n).$$

Since $x+y=1$ we may suppose, without loss of the generality, that $x \neq 0$. Thus from (2,4)

$$G_i(r_1, \ldots, r_n) = cG_i(s_1, \ldots, s_n) \tag{2.5}$$

$$(i = 1, \ldots, n),$$

where $c = -\frac{x}{y}$. Let

$$f(x) = (x-r_1)\ldots(x-r_n), \quad g(x) = c(x-s_1)\ldots(x-s_n).$$

In consequence of (2,5) $f(x) \equiv g(x)$. Thus $c=1$, i.e. $x+y=0$, what is impossible because $x+y=1$. Therefore $\lambda_{i_{r+1}}=0$.

If $i_{r+1}$ runs over all the numbers $\alpha+1, \ldots, n$ different from $i_1, \ldots, i_r$ then on the basis (2,2) we get that

$$\frac{1}{n-\alpha-r} \sum_{(i_{r+1})} x^2 \operatorname{Per} (A^{(\alpha)}_{i_1 \ldots i_{r+1}} A^{(\alpha)*}_{i_1 \ldots i_{r+1}})^{1/2} + \tag{2.6}$$

$$+ y^2 \operatorname{Per} (A^{(\alpha)*}_{i_1 \ldots i_{r+1}} A^{(\alpha)}_{i_1 \ldots i_{r+1}})^{1/2} + 2xy \operatorname{Per} A^{(\alpha)}_{i_1 \ldots i_{r+1}} \geqq$$

$$\geqq x^2 \operatorname{Per}(A^{(\alpha)}_{i_1 \ldots i_r} A^{(\sigma)*}_{i_1 \ldots i_r})^{1/2} + y^2 \operatorname{Per} (A^{(\alpha)*}_{i_1 \ldots i_r} A^{(\alpha)}_{i_1 \ldots i_r})^{1/2} + 2xy \operatorname{Per} A^{(\alpha)}_{i_1 \ldots i_r}$$

with equality if and only if the numbers $\lambda_1, \ldots, \lambda_n$ different from $\lambda_1, \ldots, \lambda_\alpha, \lambda_{i_1}, \ldots, \lambda_{i_r}$, are zero.

Using formula (2,6) for all combinations $i_1, \ldots, i_r$ of order $r$ of the elements $\alpha+1, \ldots, n$, without repetition and without permutation, it can be shown that the expression of the square brackets on the left hand side of (2,6) appears $r+1$ times among these inequalities. Calculating the arithmetic mean of these inequalities, i. e. dividing both side of the sum of these inequalities by $\binom{n-\alpha}{r}$, after all we devide the left hand side and the right hand side of the sum of these inequalities by $\frac{n-\alpha-r}{r+1}\binom{n-\alpha}{r} = \binom{n-\alpha}{r+1}$ and by $\binom{n-\alpha}{r}$, respectively. Thus we obtain the inequality of our Theorem 2.1. with equality if and only if $\lambda_{\alpha+1} = \ldots = \lambda_n = 0$.

**Corollary 2.1.** *If* $A\in K$, $x\in R_1$, $y\in R_1$, $x+y=1$, *then*

$$G_{r+1}^{(1)}(x,y) \geqq G_r^{(1)}(x,y) \geqq \frac{n!}{n^n} \quad (r=0,1,\ldots,n-2)$$

*with equality if and only if* $A=S_0$.

PROOF. In this case all components of the vectors $U_1$ and $v_1$ are $1/\sqrt{n}$, thus $G_0^{(1)}(x,y)=n!/n^n$. Equality if and only if $\lambda_2=\ldots=\lambda_n=0$.

We get Theorem 1.1 from Corollary 2.1 in the case $r+1=n-1$.

**Corollary 2.2.** *If* $A=U\wedge V^*$ *is the polar representation of* $A\in K$ *and if* $x\in R_1$, $y\in R_1$, $x+y=1$ *then*

*I.* $A_{i_1\ldots i_r}^{(0)}\in K$ *if and only if* $i_1=1$.

*II.* $$G_{r+1}^{(0)}(x,y) > G_r^{(0)}(x,y) \quad (r=1,\ldots,n-1).$$

PROOF. $A_{i_1\ldots i_r}^{(0)}\in K$ if and only if one of $\lambda_1,\ldots,\lambda_n$ is equal to 1 and the corresponding column-elements of $U$ and $V$ are $1/\sqrt{n}$. $U$ and $V$ are orthogonal matrices, therefore these conditions are satisfied o ly if $j=1$.

In the case II. equality holds if and only if $\lambda_1=\ldots=\lambda_n=0$ contradicting to $\lambda_1=1$.

## 3. The extension of the second Theorem

Let

$$G_r^{(\alpha)}(A) = \frac{1}{\binom{n-\alpha}{r}} \sum_{\alpha+1\leqq i_1<\ldots<i_r\leqq n} \operatorname{Per} A_{i_1\ldots i_r}^{(\alpha)}$$

$$(r=0,1,\ldots,n-\alpha-1;\ \alpha=0,1,\ldots,n-1;\ r+\alpha\geqq 1).$$

**Theorem 3.1.** *If the matrix* $A\in K$ *has the polar representation* $A=U\wedge V^*$ *and if*

(3.1) $$\operatorname{Per} C_{\beta_1\ldots\beta_n}(U)\operatorname{Per} C_{\beta_1\ldots\beta_n}(V)\geqq 0.$$

$$(\beta_1,\ldots,\beta_n)\in\Gamma,$$

*then*

(3.2) $$G_{r+1}^{(\alpha)}(A)\geqq G_r^{(\alpha)}(A)$$

$$(\alpha=0,1,\ldots,n-1;\ r=0,1,\ldots,n-\alpha-1,\ r+\alpha\geqq 1)$$

*with equality if and only if* $\lambda_{\alpha+1}=\ldots=\lambda_n=0$.

PROOF. Applying again the Cauchy—Binet expansion theorem, we get

$$\operatorname{Per} A^{(\alpha)}_{i_1\dots i_{r+1}} = \sum_{\beta_1+\dots+\beta_{\alpha+r}=n} \frac{\lambda_1^{\beta_1}\dots\lambda_\alpha^{\beta_\alpha}\lambda_{i_1}^{\beta_{\alpha+1}}\dots\lambda_{i_r}^{\beta_{\alpha+r}}}{\beta_1!\dots\beta_\alpha!\,\beta_{\alpha+1}!\dots\beta_{\alpha+r}!}\times$$
$$\times \operatorname{Per} C_{\beta_1\dots\beta_{\alpha+r}}(U^{(\alpha)}_{i_1\dots i_r}) \operatorname{Per} C_{\beta_1\dots\beta_{\alpha+r}}(V^{(\alpha)}_{i_1\dots i_r})+$$
$$+\sum_{\substack{\beta_1+\dots+\beta_{\alpha+r+1}\\ \beta_{\alpha+r+1}\geqq 1}} \frac{\lambda_1^{\beta_1}\dots\lambda_\alpha^{\beta_\alpha}\lambda_{i_1}^{\beta_{\alpha+1}}\dots\lambda_{i_{r+1}}^{\beta_{\alpha+r+1}}}{\beta_1!\dots\beta_\alpha!\,\beta_{\alpha+1}!\dots\beta_{\alpha+r+1}!}\times$$
$$\times \operatorname{Per} C_{\beta_1\dots\beta_{\alpha+r+1}}(U^{(\alpha)}_{i_1\dots i_{r+1}}) \operatorname{Per} C_{\beta_1\dots\beta_{\alpha+r+1}}(V^{(\alpha)}_{i_1\dots i_{r+1}}).$$

Thus

$$\operatorname{Per} A^{(\alpha)}_{i_1\dots i_{r+1}} \geqq \operatorname{Per} A^{(\alpha)}_{i_1\dots i_r}$$

with equality if and only if $\lambda_{i_{r+1}}=0$. Namely let us suppose that $\lambda_{i_{r+1}}>0$. Since $\lambda_1=1$ according to condition (3.1) we have

$$\operatorname{Per} C_{\beta_1=i,\,\beta_{i_{r+1}}=n-i}(U^{(\alpha)}_{i_1\dots i_{r+1}}) \operatorname{Per} C_{\beta_1=i,\,\beta_{i_{r+1}}=n-i}(V^{(\alpha)}_{i_1\dots i_{r+1}}) = 0$$
$$(i=0,1,\dots,n-1)$$

that is

(3.3) $$G_i(r_1,\dots,r_n)G_i(s_1,\dots,s_n)=0$$
$$(i=1,\dots,n).$$

Since

$$r_1+\dots+r_n=0,\quad r_1^2+\dots+r_n^2=1,$$

we get

$$G_2(r_1,\dots,r_n)=-\frac{1}{2},$$

and similarly

$$G_2(s_1,\dots,s_n)=-\frac{1}{2},$$

i.e. condition (3,3) is not satisfied in case $i=2$. Thus $\lambda_{i_{r+1}}=0$.

From this the proof is similar then the proof of Theorem 2.1. If $i_{r+1}$ runs over all the numbers $\alpha+1,\dots,n$ different from $i_1,\dots,i_r$ then

(3.4) $$\frac{1}{n-\alpha-r}\sum_{(i_{r+1})}\operatorname{Per} A^{(\alpha)}_{i_1\dots i_{r+1}} \geqq \operatorname{Per} A^{(\alpha)}_{i_1\dots i_r}$$

with equality if and only if the numbers $\lambda_1,\dots,\lambda_n$ different from $\lambda_1,\dots,\lambda_\alpha,\lambda_{i_1},\dots,\lambda_{i_r}$ are zero.

Using formula (3.4) for all combinations $i_1,\dots,i_r$ of order $r$ of the elements $\alpha+1,\dots,n$, without repetition, and without permutation, it can be shown that $\operatorname{Per} A^{(\alpha)}_{i_\alpha\dots i_{r+1}}$ appears $r+1$ times among the left hand side of these inequalities. Taking the arithmetic mean of these inequalities, i.e. dividing both side of the sum of these inequalities by $\binom{n-\alpha}{r}$, after all we devide the left hand side and the right hand side of the sum of these inequalities by $\frac{n-\alpha-r}{r+1}\binom{n-\alpha}{r}-\binom{n-\alpha}{r+1}$ and by $\binom{n-\alpha}{r}$, respectively. Thus inequality (3.2) holds with equality if and only if $\lambda_{\alpha+1}=\ldots=\lambda_n=0$.

**Corollary 3.1.** *Let* $A=U\wedge V^*$ *be the polar representation of* $A\in K$. *If condition* (3.1) *is satisfied then*

$$G(A_{r+1}^{(1)})\geqq G(A_r^{(1)})\geqq\frac{n!}{n^n}$$

$$(r=0,1,\ldots,n-2)$$

*with equality if and only if* $A=S_0$.

PROOF. In this case $G_0^{(1)}(A)=u_1v_1^*=S_0$ with equality if and only if $\lambda_\alpha=\ldots=$ $=\lambda_n=0$, i.e. $A=S_0$.

**Corollary 3.2.** *If the matrix* $A\in K$ *satisfies condition* (3.1) *then*

$$G_{r+1}^{(0)}(A)>G_r^{(0)}(A)\quad(r=0,1,\ldots,n-1).$$

PROOF. Equality holds if and only if $\lambda_1=\ldots=\lambda_n=0$, contradicting to $\lambda_1=1$.

**Corollary 3.3.** *If* $A\in K$ *is a symmetric positive semidefinite matrix then*

$$G_{r+1}^{(\alpha)}(A)\geqq G_r^{(\alpha)}(A)$$

$$(\alpha=0,1,\ldots,n-1;\ r=0,1,\ldots,n-\alpha-1;\ r+\alpha\geqq 1).$$

*In particular*

$$G_{r+1}^{(1)}(A)\geqq G_r^{(1)}(A)\geqq\frac{n!}{n^n}\quad(r=0,1,\ldots,n-2)$$

*with equality if and only if* $A=S_0$. *Moreover*

$$G_{r+1}^{(0)}(A)>G_r^{(0)}(A)\quad(r=0,1,\ldots,n-2).$$

PROOF. In this case condition (3.1) is satisfied trivially.

Since $G_{n-1}^{(1)}(A)=A$, Corollary 3.3. contains Theorem 1.2. too.

Finally the importance of Theorem 3.1. is expresses by the following theorem:

**Theorem 3.2.** *Condition* (3.1) *is satisfied by the elements of an infinite subset of* $K$, *containing non-symmetric non-positive semidefinite matrices.*

PROOF. Let us consider the set of the $n\times n$ orthogonal matrices, which have $1/\sqrt{n}$ as the elements of their first column. The power of these orthogonal matrices is infinite. We decompose now this set into mutually disjoint subsets. Two orthogonal matrices $U$ and $V$ belong to the same subset if condition (3.1) is satisfied by them. Since we have finitely many subsets $\left(\text{their numbers is equal to } \exp_2\binom{2n-1}{2}\right)$, at least one subset should contain infinitely many elements. This complets the proof of Theorem 3.2.

## References

[1] B. Gyires, On inequalities concerning the permanent of matrices. *JCISS* **2** (1977), 107—113.
[2] G. P. Jegoritsev, Solution of the Van der Waerden's permanent conjecture. Preprint 13, *Physical Research Institute, Krasnojarsk,* 1980. (In Russian)
[3] M. Marcus and H. Mink, Permanents. *Amer. Math. Monthly* **72** (1965), 577—591.

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